Aufgabe 9.45

Berechnen Sie a) $\sum\limits_{n=0}^{\infty}\left(2^{-3n}+3^{-2n}\right)$, b) $\sum\limits_{n=0}^{\infty}\frac{1}{(n+1)(n+3)}$!

Lösung:

a) $$\sum_{n=0}^{\infty}\left(2^{-3n}+3^{-2n}\right)=\sum_{n=0}^{\infty}\left(\frac{1}{2^{3n}}+\frac{1}{3^{2n}}\right)=\sum_{n=0}^{\infty}\left(\frac{1}{8^n}+\frac{1}{9^n}\right)=\frac{1}{1-\frac{1}{8}}+\frac{1}{1-\frac{1}{9}}=\frac{1}{\frac{7}{8}}+\frac{1}{\frac{8}{9}}$$
$$=\frac{8}{7}+\frac{9}{8}=\frac{64+63}{56}=\frac{127}{56}$$

b) $$\frac{1}{n+1}-\frac{1}{n+3}=\frac{(n+3)-(n+1)}{(n+1)(n+3)}=\frac{2}{(n+1)(n+3)}$$
$$S_N=\sum_{n=0}^{N}\frac{1}{(n+1)(n+3)}=\frac{1}{2}\sum_{n=0}^{N}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)$$
$$=\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}\pm\cdots+\frac{1}{N}-\frac{1}{N+2}+\frac{1}{N+1}-\frac{1}{N+3}\right)$$
$$=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{N+2}-\frac{1}{N+3}\right)\xrightarrow[n\to\infty]{}\frac{1}{2}\left(1+\frac{1}{2}\right)=\frac{3}{4},$$
also gilt $$\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+3)}=\lim_{N\to\infty}S_N=\frac{3}{4}.$$