Aufgabe 9.42

Berechnen Sie folgende Summen bzw. Reihen:

a) $\sum_{n=4}^{\infty} \frac{1}{n(n-1)}$, b) $\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+5)}$, c) $\sum_{n=1}^{100} 1.01^n$, d) $\sum_{n=10}^{\infty} (-0.04)^n$!

Lösung:

a) $$S_N = \sum_{n=4}^{N} \frac{1}{n(n-1)} = \sum_{n=4}^{N} \left(\frac{1}{n-1} - \frac{1}{n} \right)$$
$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} \pm \dots + \frac{1}{N-2} - \frac{1}{N-1} + \frac{1}{N-1} - \frac{1}{N} = \frac{1}{3} - \frac{1}{N} \xrightarrow[N\to\infty]{} \frac{1}{3},$$
also gilt $\sum_{n=4}^{\infty} \frac{1}{n(n-1)} = \lim_{N\to\infty} S_N = \frac{1}{3}$.

b) $$\frac{1}{2n+1} - \frac{1}{2n+5} = \frac{(2n+5)-(2n+1)}{(2n+1)(2n+5)} = \frac{4}{(2n+1)(2n+5)}$$
$$S_N = \sum_{n=0}^{N} \frac{1}{(2n+1)(2n+5)} = \frac{1}{4} \sum_{n=0}^{N} \left(\frac{1}{2n+1} - \frac{1}{2n+5} \right) =$$
$$\frac{1}{4}\left(1 - \frac{1}{5} + \frac{1}{3} - \frac{1}{7} + \frac{1}{5} - \frac{1}{9} + \frac{1}{7} - \frac{1}{11} \pm \dots + \frac{1}{2N-3} - \frac{1}{2N+1} + \frac{1}{2N-1} - \frac{1}{2N+3} + \frac{1}{2N+1} - \frac{1}{2N+5}\right)$$
$$= \frac{1}{4}\left(1 + \frac{1}{3} - \frac{1}{2N+3} - \frac{1}{2N+5}\right) \xrightarrow[N\to\infty]{} \frac{1}{4}\left(1 + \frac{1}{3}\right) = \frac{1}{3},$$
also gilt $\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+5)} = \lim_{N\to\infty} S_N = \frac{1}{3}$.

c) $$\sum_{n=1}^{100} 1.01^n = \sum_{n=0}^{100} 1.01^n - 1.01^0 = \frac{1-1.01^{101}}{1-1.01} - 1 = 100\,(1.01^{101}-1) - 1 \approx 172.186$$

d) $$\sum_{n=10}^{\infty} (-0.04)^n = (-0.04)^{10} \sum_{n=0}^{\infty} (-0.04)^n = (-0.04)^{10} \frac{1}{1-(-0.04)} \approx 1.008246 \cdot 10^{-14}$$