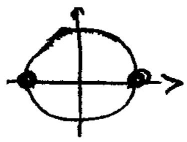
Aufgabe 5.68

Ermitteln Sie die komplexen

a) Quadrat-, vierten und sechsten Wurzeln aus 1,
b) Quadratwurzeln aus $2(-1+\sqrt{3}\,\mathrm{i})$,
c) Quadrat- und dritten Wurzeln aus i !

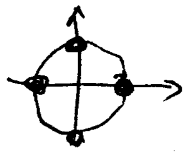
Lösung:

a)



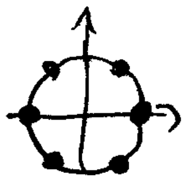
$$\begin{array}{lcl}
1=\cos 0+\mathrm{i}\sin 0 & \longrightarrow & w_1=\sqrt{1}=\cos 0+\mathrm{i}\sin 0=1\\
1=\cos 2\pi+\mathrm{i}\sin 2\pi & & w_2=\sqrt{1}=\cos \pi+\mathrm{i}\sin \pi=-1\\
(1=\cos 4\pi+\mathrm{i}\sin 4\pi & & w_1=\sqrt{1}=\cos 2\pi+\mathrm{i}\sin 2\pi=1 \text{ usw.})
\end{array}$$

$$\sqrt{1}=\pm 1$$



$$\begin{array}{lcl}
1=\cos 0+\mathrm{i}\sin 0 & \longrightarrow & w_1=\sqrt[4]{1}=\cos 0+\mathrm{i}\sin 0=1\\
1=\cos 2\pi+\mathrm{i}\sin 2\pi & & w_2=\sqrt[4]{1}=\cos \frac{\pi}{2}+\mathrm{i}\sin \frac{\pi}{2}=\mathrm{i}\\
1=\cos 4\pi+\mathrm{i}\sin 4\pi & & w_3=\sqrt[4]{1}=\cos \pi+\mathrm{i}\sin \pi=-1\\
1=\cos 6\pi+\mathrm{i}\sin 6\pi & & w_4=\sqrt[4]{1}=\cos \frac{3\pi}{2}+\mathrm{i}\sin \frac{3\pi}{2}=-\mathrm{i}\\
(1=\cos 8\pi+\mathrm{i}\sin 8\pi & & w_1=\sqrt[4]{1}=\cos 2\pi+\mathrm{i}\sin 2\pi=1 \text{ usw.})
\end{array}$$

$$\sqrt[4]{1}=\pm 1,\ \pm\mathrm{i}$$



$$\begin{array}{lcl}
1=\cos 0+\mathrm{i}\sin 0 & \longrightarrow & w_1=\sqrt[6]{1}=\cos 0+\mathrm{i}\sin 0=1\\
1=\cos 2\pi+\mathrm{i}\sin 2\pi & & w_2=\sqrt[6]{1}=\cos \frac{\pi}{3}+\mathrm{i}\sin \frac{\pi}{3}=\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\\
1=\cos 4\pi+\mathrm{i}\sin 4\pi & & w_3=\sqrt[6]{1}=\cos \frac{2\pi}{3}+\mathrm{i}\sin \frac{2\pi}{3}=-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\\
1=\cos 6\pi+\mathrm{i}\sin 6\pi & & w_4=\sqrt[6]{1}=\cos \pi+\mathrm{i}\sin \pi=-1\\
1=\cos 8\pi+\mathrm{i}\sin 8\pi & & w_5=\sqrt[6]{1}=\cos \frac{4\pi}{3}+\mathrm{i}\sin \frac{4\pi}{3}=-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\\
1=\cos 10\pi+\mathrm{i}\sin 10\pi & & w_6=\sqrt[6]{1}=\cos \frac{5\pi}{3}+\mathrm{i}\sin \frac{5\pi}{3}=\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}
\end{array}$$

$$\sqrt[6]{1}=\pm 1,\quad \pm\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}$$

b)
$$2(-1+\sqrt{3}\mathrm{i})=4\left(\cos\tfrac{2\pi}{3}+\mathrm{i}\sin\tfrac{2\pi}{3}\right)\begin{cases}=4(\cos 120^\circ+\mathrm{i}\sin 120^\circ)\\=4(\cos 480^\circ+\mathrm{i}\sin 480^\circ),\end{cases}$$

$$\sqrt{2(-1+\sqrt{3}\mathrm{i})}\begin{cases}=2(\cos 60^\circ+\mathrm{i}\sin 60^\circ)=1+\sqrt{3}\mathrm{i}\\=2(\cos 240^\circ+\mathrm{i}\sin 240^\circ)=-1-\sqrt{3}\mathrm{i}\end{cases}$$

Probe: $\left(\pm(1+\sqrt{3}\mathrm{i})\right)^2=(\pm(1+\sqrt{3}\mathrm{i})^2=1+2\sqrt{3}\mathrm{i}-3=-2+2\sqrt{3}\mathrm{i}=2(-1+\sqrt{3}\mathrm{i})$

c)
$$\mathrm{i}=\cos\tfrac{\pi}{2}+\sin\tfrac{\pi}{2}=\cos\tfrac{5\pi}{2}+\sin\tfrac{5\pi}{2},$$

$$\sqrt{\mathrm{i}}\begin{cases}=\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathrm{i}\\=\cos\frac{5\pi}{4}+\mathrm{i}\sin\frac{5\pi}{4}=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\mathrm{i}\end{cases},\quad \text{d.h. } \sqrt{\mathrm{i}}=\pm\frac{1}{\sqrt{2}}(1+\mathrm{i})$$

$$\mathrm{i}=\cos\tfrac{\pi}{2}+\sin\tfrac{\pi}{2}=\cos\tfrac{5\pi}{2}+\sin\tfrac{5\pi}{2}=\cos\tfrac{9\pi}{2}+\sin\tfrac{9\pi}{2}$$

$$\sqrt[3]{\mathrm{i}}\begin{cases}=\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\\=\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\\=\cos\frac{3\pi}{2}+\mathrm{i}\sin\frac{3\pi}{2}=-\mathrm{i}\end{cases},\quad \text{d.h. } \sqrt[3]{\mathrm{i}}=\pm\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i},\quad -\mathrm{i}$$