Aufgabe 5.58

Bestimmen Sie die Polardarstellungen der komplexen Zahlen $z_1 = -\sqrt{12}+2\mathrm{i}$ und $z_2 = -3-\sqrt{27}\,\mathrm{i}$ und ermitteln Sie mit ihrer Hilfe z_1z_2, $\frac{z_1}{z_2}$, z_1^5 und $\frac{\overline{z_2}^2}{z_1^{10}}$! Geben Sie die Ergebnisse auch in algebraischer Darstellung an!

Lösung:

$$|z_1|=\sqrt{12+4}=4,\ \varphi=\arctan\left(-\frac{2}{\sqrt{12}}\right)+\pi=\arctan\left(-\frac{2}{\sqrt{3\cdot 4}}\right)+\pi=\arctan\left(-\frac{1}{\sqrt{3}}\right)+\pi=-\frac{\pi}{6}+\pi=\frac{5\pi}{6}$$

(da II. Quadrant)

$$z_1 = 4\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)$$

$$|z_2|=\sqrt{9+27}=6,\ \varphi=\arctan\frac{\sqrt{27}}{3}+\pi=\arctan\frac{\sqrt{3\cdot 9}}{3}+\pi=\arctan\sqrt{3}+\pi=\frac{\pi}{3}+\pi=\frac{4\pi}{3}$$

(da III. Quadrant)

$$z_2 = 6\left(\cos\frac{4\pi}{3}+\mathrm{i}\sin\frac{4\pi}{3}\right)$$

$$z_1z_2 = 4\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)\cdot 6\left(\cos\frac{4\pi}{3}+\mathrm{i}\sin\frac{4\pi}{3}\right) = 24\left(\cos\frac{5+8}{6}\pi+\mathrm{i}\sin\frac{5+8}{6}\pi\right)$$

$$= 24\left(\cos\frac{13\pi}{6}+\mathrm{i}\sin\frac{13\pi}{6}\right) = 24\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right) = 24\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right) = 12\sqrt{3}+12\mathrm{i}$$

$$\frac{z_1}{z_2} = \frac{4\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)}{6\left(\cos\frac{4\pi}{3}+\mathrm{i}\sin\frac{4\pi}{3}\right)} = \frac{2}{3}\left(\cos\frac{5-8}{6}\pi+\mathrm{i}\sin\frac{5-8}{6}\pi\right) = \frac{2}{3}\left(\cos\left(-\frac{\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right) = -\frac{2}{3}\mathrm{i}$$

$$z_1^5 = \left(4\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)\right)^5 = 4^5\left(\cos\frac{25\pi}{6}+\mathrm{i}\sin\frac{25\pi}{6}\right) = 1024\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)$$

$$= 1024\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right) = 512\sqrt{3}+512\mathrm{i}$$

$$\overline{z_2} = 6\left(\cos\frac{4\pi}{3}-\mathrm{i}\sin\frac{4\pi}{3}\right) = 6\left(\cos\left(-\frac{4\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{4\pi}{3}\right)\right) = 6\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)$$

$$\frac{\overline{z_2}^2}{z_1^{10}} = \frac{\left(6\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)\right)^2}{\left(4\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)\right)^{10}} = \frac{6^2\left(\cos\frac{4\pi}{3}+\mathrm{i}\sin\frac{4\pi}{3}\right)}{4^{10}\left(\cos\frac{50\pi}{6}+\mathrm{i}\sin\frac{50\pi}{6}\right)} = \frac{2^2\cdot 3^2}{2^{20}}\left(\cos\frac{4-25}{3}\pi+\mathrm{i}\sin\frac{4-25}{3}\pi\right)$$

$$= \frac{9}{262144}(\cos(-7\pi)+\mathrm{i}\sin(-7\pi)) = \frac{9}{262144}(\cos\pi+\mathrm{i}\sin\pi) = -\frac{9}{262144}$$