Aufgabe 5.57

Bestimmen Sie die Polardarstellungen der komplexen Zahlen $z_1 = 4 - \sqrt{48}\mathrm{i}$, $z_2 = -\sqrt{75} - 5\,\mathrm{i}$ und $z_3 = -4 + 4\mathrm{i}$ und ermitteln Sie mit ihrer Hilfe $z_1 z_2$, $\frac{z_1}{z_2}$, z_1^4, $\frac{z_1^3}{z_2^7}$, z_3^6, $\overline{z_3}^6$ und $\overline{z_3^6}$! Geben Sie die Ergebnisse auch in algebraischer Darstellung an!

Lösung:

$$|z_1| = \sqrt{16+48} = 8,\ \varphi = \arctan\left(-\frac{\sqrt{48}}{4}\right) = \arctan\left(-\frac{\sqrt{3\cdot 16}}{4}\right) = \arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

(da IV. Quadrant)

$$z_1 = 8\,\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$$

$$|z_2| = \sqrt{75+25} = 10,\ \varphi = \arctan\frac{-5}{-\sqrt{75}} + \pi = \arctan\frac{5}{\sqrt{3\cdot 25}} + \pi = \arctan\frac{1}{\sqrt{3}} + \pi = \frac{\pi}{6} + \pi = \frac{7\pi}{6}$$

(da III. Quadrant)

$$z_2 = 10\,\left(\cos\frac{7\pi}{6} + \mathrm{i}\sin\frac{7\pi}{6}\right)$$

$$z_1 z_2 = 8\,\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)\cdot 10\,\left(\cos\frac{7\pi}{6} + \mathrm{i}\sin\frac{7\pi}{6}\right) = 80\,\left(\cos\frac{-2+7}{6}\pi + \mathrm{i}\sin\frac{-2+7}{6}\pi\right)$$

$$= 80\,\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right) = 80\,\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) = -40\sqrt{3} + 40\,\mathrm{i}$$

$$z_3 = 4\sqrt{2}\,\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)$$

$$\frac{z_1}{z_2} = \frac{8\,\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)}{10\,\left(\cos\frac{7\pi}{6} + \mathrm{i}\sin\frac{7\pi}{6}\right)} = \frac{8}{10}\left(\cos\frac{-2-7}{6}\pi + \mathrm{i}\sin\frac{-2-7}{6}\pi\right)$$

$$= \frac{4}{5}\left(\cos\left(-\frac{3\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{2}\right)\right) = \frac{4}{5}\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right) = \frac{4}{5}\mathrm{i}$$

$$z_1^4 = 8\,\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)^4 = 8^4\,\left(\cos\left(-\frac{4\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{3}\right)\right) = 2^{12}\,\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$$

$$= 4096\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) = -2048 + 2048\sqrt{3}\mathrm{i}$$

$$\frac{z_1^3}{z_2^7} = \frac{\left(8\,\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)\right)^3}{\left(10\,\left(\cos\frac{7\pi}{6} + \mathrm{i}\sin\frac{7\pi}{6}\right)\right)^7} = \frac{8^3}{10^7}\left(\cos\frac{-3\cdot 2 - 7\cdot 7}{6}\pi + \mathrm{i}\sin\frac{-3\cdot 2 - 7\cdot 7}{6}\pi\right)$$

$$= \frac{2^9}{10^7}\left(\cos\left(-\frac{55\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{55\pi}{6}\right)\right) = \frac{512}{10^7}\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right) = \frac{512}{10^7}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$$

$$= -2.56\sqrt{3}\cdot 10^{-5} + 2.56\cdot 10^{-5}\,\mathrm{i}$$

$$z_3^6 = \left(4\sqrt{2}\,\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)\right)^6 = \left(2^{\frac{5}{2}}\right)^6\left(\cos\frac{18\pi}{4} + \mathrm{i}\sin\frac{18\pi}{4}\right) = 2^{15}\,\left(\cos\frac{9\pi}{2} + \mathrm{i}\sin\frac{9\pi}{2}\right)$$

$$= 2^{15}\,\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right) = 32768\,\mathrm{i}$$

$$\overline{z_3} = 4\sqrt{2}\,\left(\cos\frac{3\pi}{4} - \mathrm{i}\sin\frac{3\pi}{4}\right) = 4\sqrt{2}\,\left(\cos\left(-\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right)$$

$$\overline{z_3}^6 = \left(4\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right)\right)^6 = \left(2^{\frac{5}{2}}\right)^6\left(\cos\left(-\frac{9\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{9\pi}{2}\right)\right)$$

$$= 2^{15}\left(\cos\frac{3\pi}{2}+\mathrm{i}\sin\frac{3\pi}{2}\right) = -32768\,\mathrm{i}$$

$$\overline{z_3^6} = \overline{32768\,\mathrm{i}} = -32768\,\mathrm{i} = \overline{z_3}^6$$