Explizite m-stufige Runge-Kutta-Verfahren

$$
\begin{aligned}
u_0 &= y_0, \\
u_{j+1} &= u_j + h_j\left(\gamma_1 k_1(t_j, u_j) + \ldots + \gamma_m k_m(t_j, u_j)\right),
\end{aligned}
$$

wobei $k_r(t_j, y_j) \approx y'(s_r) = f(s_r, y(s_r))$ mit

$$
s_1 = t_j, \qquad s_r = t_j + \alpha_r h_j, \quad \alpha_r = \sum_{l=1}^{r-1} \beta_{rl}
$$

$$
\begin{aligned}
k_1(t_j, u_j) &= f(t_j, u_j) \\
k_2(t_j, u_j) &= f(t_j + \alpha_2 h_j, u_j + h_j \beta_{21} k_1(t_j, u_j)) \\
k_3(t_j, u_j) &= f(t_j + \alpha_3 h_j, u_j + h_j(\beta_{31} k_1(t_j, u_j) + \beta_{32} k_2(t_j, u_j)) \\
&\vdots \\
k_m(t_j, u_j) &= f(t_j + \alpha_m h_j, u_j + h_j(\beta_{m1} k_1(t_j, u_j) + \ldots \\
&\qquad \ldots + \beta_{m,m-1} k_{m-1}(t_j, u_j)))
\end{aligned}
$$

Butcher-Diagramm

$$
\begin{array}{c|cccccc}
0 & & & & & & \\
\alpha_2 & \beta_{21} & & & & & \\
\vdots & \vdots & \ddots & & & & \\
\alpha_r & \beta_{r1} & \cdots & \beta_{r,r-1} & & & \\
\vdots & \vdots & & & \ddots & & \\
\alpha_m & \beta_{m1} & \cdots & \cdots & \cdots & \beta_{m,m-1} & \\
\hline
 & \gamma_1 & \cdots & \cdots & \cdots & \gamma_{m-1} & \gamma_m
\end{array}
$$

$\alpha_r \rightsquigarrow$ Quadratur-Stützstellen, $\gamma_r =$ Quadratur-Gewichte.

Einige explizite Runge-Kutta Verfahren

1-stufig

— *Eulerverfahren*

$\gamma_1 = 1$

$u_{j+1} = u_j + h_j f(t_j, u_j)$

Ordnung 1

$$\begin{array}{c|c} 0 & \\ \hline & 1 \end{array}$$

2-stufig

— *modifiziertes Eulerverfahren*

$\gamma_1 = 0 \quad \gamma_2 = 1$

$\beta_{21} = \alpha_2 = \frac{1}{2}$

$u_{j+1} = u_j + h_j f(t_j + \frac{1}{2}h_j, u_j + \frac{1}{2}h_j f(t_j, u_j))$

Ordnung 2

$$\begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array}$$

— *Verfahren von Heun*

$\gamma_1 = \gamma_2 = \frac{1}{2}$

$\beta_{21} = \alpha_2 = 1$

$$u_{j+1} = u_j + \frac{h_j}{2}(f(t_j, u_j) + f(t_{j+1}, u_j + h_j f(t_j, u_j)))$$

Ordnung 2

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

3-stufig

— *Methode von Heun 3.Ordnung*

Ordnung 3

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{3} & \frac{1}{3} & & \\ \frac{2}{3} & 0 & \frac{2}{3} & \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array}$$

— *Methode von Kutta 3.Ordnung* (Simpsonregel)

Ordnung 3

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{array}$$

4-stufig

— *Klassisches Runge-Kutta-Verfahren*

Ordnung 4

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

— (3/8–Regel der Quadratur)

Ordnung 4

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{3} & \frac{1}{3} & & & \\ \frac{2}{3} & -\frac{1}{3} & 1 & & \\ 1 & 1 & -1 & 1 & \\ \hline & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$$

Implizite Runge-Kutta-Verfahren

$$k_r(t_j, u_j) = f(t_j + \alpha_r h_j, u_j + h_j(\beta_{r1} k_1 + \ldots + \beta_{rm} k_m))$$

und damit

$$u_{j+1} = u_j + h_j(\gamma_1 k_1(t_j, u_j) + \ldots + \gamma_m k_m(t_j, u_j))$$

Butcher-Diagramm

$$\begin{array}{c|ccc}
\alpha_1 & \beta_{11} & \ldots & \beta_{1m} \\
\alpha_2 & \beta_{21} & \ldots & \beta_{2m} \\
\vdots & \vdots & & \vdots \\
\alpha_m & \beta_{m1} & \ldots & \beta_{mm} \\
\hline
 & \gamma_1 & \ldots & \gamma_m
\end{array}$$

a) *Gauß-Form*: $\alpha_j, \beta_{jl}, \gamma_j$ beliebig wählbar

b) *Radau-Form*: $\alpha_1 = \beta_{11} = \beta_{12} = \ldots = \beta_{1m} = 0$
oder $\alpha_m = 1$ und $\beta_{1m} = \beta_{2m} = \ldots = \beta_{mm} = 0$.

c) *Lobatto-Form*: $\alpha_1 = \beta_{11} = \beta_{12} = \ldots = \beta_{1m} = \beta_{2m} = \ldots = \beta_{mm} = 0, \alpha_m = 1$.

Satz von Butcher (1963/1965)

a) Erreichbare Konsistenzordnung p eines expliziten m-stufigen Runge-Kutta-Verfahrens:

m	1	2	3	4	5	6	7	8	9	$m \geq 9$
$p(m)$	1	2	3	4	4	5	6	6	7	$\leq m-2$

b) Erreichbare Konsistenzordnung p eines impliziten m-stufigen Runge-Kutta-Verfahrens mit $m(m+1)$ freien Parametern: $2m$.

Einige implizite Runge-Kutta-Verfahren

Gauß-Form

— $m = 1, p = 2$

$$\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$$

$$\begin{aligned} k_1 &= f(t_j + \frac{h_j}{2}, u_j + \frac{h_j}{2} k_1) \\ u_{j+1} &= u_j + h_j k_1 \end{aligned}$$

— $m = 2, p = 4$

$$\begin{array}{c|cc} \frac{(3-\sqrt{3})}{6} & \frac{1}{4} & \frac{(3-2\sqrt{3})}{12} \\ \frac{(3+\sqrt{3})}{6} & \frac{(3+2\sqrt{3})}{12} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Radau-Form

— *Euler-Verfahren*: $m = 1, p = 1$

$$\begin{array}{c|c} 0 & \\ \hline & 1 \end{array}$$

— $m = 2, p = 3$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array} \qquad \begin{array}{c|cc} \frac{1}{3} & \frac{1}{3} & 0 \\ 1 & 1 & 0 \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array}$$

Lobatto-Form

— *Verfahren von Heun*

$m = 2, p = 2$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$u_{j+1} = u_j + \frac{h_j}{2}(f(t_j, u_j) + f(t_j + h_j, u_j + f(t_j, u_j))).$$