A Proof of the Marcinkiewicz-Zygmund Inequality on the Bi-Sphere

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Abstract. In this technical report we present a complete proof of the Marcinkiewicz-Zygmund inequality on the bi-Sphere from [1, Theorem 4.5].

We use the same notation as in [1]. The Theorem 4.5 of [1] reads as follows:

Theorem 1. *Let $\boldsymbol{N} \in \mathbb{N}^2$ and $\mathcal{R}$ be a admissible decomposition of $\mathbb{S}^2 \times \mathbb{S}^2$ according to a sampling set $\mathcal{X}$ such that*

$$\|\mathcal{R}\| \le \frac{\eta}{21\,\tilde{N}}, \tag{1}$$

where $\eta \in (0,1)$ is arbitrarily fixed. Then for $r=1$ or $r=\infty$ and any $p \in \Pi_{\boldsymbol{N}}$, we have

$$(1-\eta)\,\|p\|_{\mathbb{S}^2\times\mathbb{S}^2,r} \le \|p\|_{\mathcal{X},r} \le (1+\eta)\,\|p\|_{\mathbb{S}^2\times\mathbb{S}^2,r}. \tag{2}$$

We apply subsequently the following inequalities:

Lemma 2. *The following estimates are valid,*

(i) $\max\limits_{t\in[0,\pi]} |v_N(\cos t)| \le \dfrac{1}{\pi}\, N^2$,

(ii) $\displaystyle\int_{\xi_j^-}^{\xi} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_N(\cos t)\big|\,\mathrm{d}t \le \frac{2}{\pi} N^3\,\|\mathcal{R}\|$,

(iii) $\displaystyle\int_{\|\mathcal{R}\|}^{\pi-\|\mathcal{R}\|} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_N(\cos t)\big|\sin t\,\mathrm{d}t \le \frac{3}{2\pi}(5N+1)$.

Proof of Lemma 2: From [2, equations (3.13), (3.14)], we obtain immediately the inequality (i). Using the Bernstein inequality $\|T'\|_\infty \le 2N\|T\|_\infty$ for $v_N(\cos t)$ and (i) we get (ii). The integral in (iii) can be estimated as in the proof of Theorem 4.2 in [2]. ■

Proof of Theorem 1: We start with the case $r=\infty$. For proving the left-hand side let $(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{S}^2\times\mathbb{S}^2$ an arbitrary fixed point and assume that $(\boldsymbol{u},\boldsymbol{v}) \in \mathcal{X}$ is a point with $d((\boldsymbol{x},\boldsymbol{y}),(\boldsymbol{u},\boldsymbol{v})) = d((\boldsymbol{x},\boldsymbol{y}),\mathcal{X}) = \delta_{\mathcal{X}}$. We conclude by [1, Lemma 4.4]

$$\begin{aligned} |p(\boldsymbol{x},\boldsymbol{y})| &\le |p(\boldsymbol{x},\boldsymbol{y}) - p(\boldsymbol{u},\boldsymbol{v})| + |p(\boldsymbol{u},\boldsymbol{v})| \\ &\le 2\tilde{N}\,d\left((\boldsymbol{x},\boldsymbol{y}),(\boldsymbol{u},\boldsymbol{v})\right)\ \|p\|_{\mathbb{S}^2\times\mathbb{S}^2,\infty} + \max_{j=1,\dots,M} |p(\boldsymbol{x}_j,\boldsymbol{y}_j)|\,. \end{aligned}$$

Taking the maximum over $(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{S}^2\times\mathbb{S}^2$ we obtain by using the fact $d((\boldsymbol{x},\boldsymbol{y}),(\boldsymbol{u},\boldsymbol{v})) = d((\boldsymbol{x},\boldsymbol{y}),\mathcal{X}) \le \|\mathcal{R}\|$ the L^∞-MZ inequality

$$\left(1-2\tilde{N}\|\mathcal{R}\|\right)\|p\|_{\mathbb{S}^2\times\mathbb{S}^2,\infty} \le \max_{j=1,\dots,M}\left|p(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\right| \le \left(1+2\tilde{N}\|\mathcal{R}\|\right)\|p\|_{\mathbb{S}^2\times\mathbb{S}^2,\infty}. \quad (3)$$

This provides in combination with (1) the inequality (2) for $r=\infty$. For $r=1$ we are going to show both inequalities of (2) simultaneously by proving

$$\left|\|p\|_{\mathbb{S}^2\times\mathbb{S}^2,1} - \|p\|_{\mathcal{X},1}\right| \le \eta\,\|p\|_{\mathbb{S}^2\times\mathbb{S}^2,1}.$$

Using [1, Theorem 4.3 (i)] we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{S}^2\times\mathbb{S}^2}|p(\boldsymbol{x},\boldsymbol{y})|\,\mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) - \sum_{j=1}^M |p(\boldsymbol{x}_j,\boldsymbol{y}_j)|\,\tau(R_j)\right| \\
= &\left|\sum_{j=1}^M\int_{R_j}|p(\boldsymbol{x},\boldsymbol{y})| - |p(\boldsymbol{x}_j,\boldsymbol{y}_j)|\,\mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y})\right| \\
\le &\sum_{j=1}^M\int_{R_j}|p(\boldsymbol{x},\boldsymbol{y}) - p(\boldsymbol{x}_j,\boldsymbol{y}_j)|\,\mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
\le &\sum_{j=1}^M\int_{R_j}|\mathcal{V}_{\boldsymbol{N}}p(\boldsymbol{x},\boldsymbol{y}) - \mathcal{V}_{\boldsymbol{N}}p(\boldsymbol{x}_j,\boldsymbol{y}_j)|\,\mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
\le &\sup_{(\boldsymbol{u},\boldsymbol{v})\in\mathbb{S}^2\times\mathbb{S}^2}\left(\sum_{j=1}^M\int_{R_j}|v_{\boldsymbol{N}}(\boldsymbol{x}\cdot\boldsymbol{u},\boldsymbol{y}\cdot\boldsymbol{v}) - v_{\boldsymbol{N}}(\boldsymbol{x}_j\cdot\boldsymbol{u},\boldsymbol{y}_j\cdot\boldsymbol{v})|\,\mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y})\right) \\
&\times\int_{\mathbb{S}^2\times\mathbb{S}^2}|p(\boldsymbol{u},\boldsymbol{v})|\,\mathrm{d}\tau(\boldsymbol{u},\boldsymbol{v}).
\end{aligned}$$

It suffices to show

$$\sup_{(\boldsymbol{u},\boldsymbol{v})\in\mathbb{S}^2\times\mathbb{S}^2}\left(\sum_{j=1}^M\int_{R_j}|v_{\boldsymbol{N}}(\boldsymbol{x}\cdot\boldsymbol{u},\boldsymbol{y}\cdot\boldsymbol{v}) - v_{\boldsymbol{N}}(\boldsymbol{x}_j\cdot\boldsymbol{u},\boldsymbol{y}_j\cdot\boldsymbol{v})|\,\mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y})\right) \le \eta. \quad (4)$$

To this end we use spherical coordinates for the spheres with respect to the point $(\boldsymbol{u},\boldsymbol{v}) \in$

$\mathbb{S}^2 \times \mathbb{S}^2$. This leads to $\boldsymbol{x}\cdot\boldsymbol{u} = \cos\xi,\ \boldsymbol{y}\cdot\boldsymbol{v} = \cos\eta$. We obtain

$$\begin{aligned}
\Sigma := & \sum_{j=1}^{M} \int_{R_j} |v_{\boldsymbol{N}}(\boldsymbol{x}\cdot\boldsymbol{u}, \boldsymbol{y}\cdot\boldsymbol{v}) - v_{\boldsymbol{N}}(\boldsymbol{x}_j\cdot\boldsymbol{u}, \boldsymbol{y}_j\cdot\boldsymbol{v})| \,\mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
= & \sum_{j=1}^{M} \int_{R_j} |v_{N_1}(\cos\xi)\, v_{N_2}(\cos\eta) - v_{N_1}(\cos\xi_j)\, v_{N_2}(\cos\eta_j)| \,\mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
= & \sum_{j=1}^{M} \int_{R_j} |v_{N_2}(\cos\eta)| \int_{\xi_j}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t \ \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
& + \sum_{j=1}^{M} \int_{R_j} v_{N_1}(\cos\xi_j)| \int_{\eta_j}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}).
\end{aligned}$$

We denote the first sum in the expression above by Σ_1 and the second sum by Σ_2. For further estimation of the above expression we introduce a partition of $\mathbb{S}^2 \times \mathbb{S}^2$ in the following way. Let $L = \left\lfloor \frac{\pi}{\|\mathcal{R}\|} \right\rfloor$ and define $\mathcal{I}_\nu = [(\nu-1)\pi L^{-1}, (\nu+1)\pi L^{-1}],\ \nu = 1,\ldots,L-1$. For $\ell_1,\ell_2 \in \{1,\ldots,L-1\}$ let

$$B_{\ell_1,\ell_2} := \left\{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{S}^2\times\mathbb{S}^2 : (\xi,\eta) \in \mathcal{I}_{\ell_1}\times\mathcal{I}_{\ell_2}\right\},$$

where $(\boldsymbol{x},\boldsymbol{y}) = (\sin\xi\sin\varphi, \sin\xi\cos\varphi, \cos\xi, \sin\eta\sin\phi, \sin\eta\cos\phi, \cos\eta),\ \xi,\eta\in[0,\pi],\ \varphi,\phi \in [-\pi,\pi[$. Let further $B_\nu = \{\boldsymbol{x}\in\mathbb{S}^2 : \xi\in\mathcal{I}_\nu\}$. Then we have

(i) $B_{\ell_1,\ell_2} = B_{\ell_1}\times B_{\ell_2}$, where $B_{\ell_\nu} = \{\boldsymbol{x}\in\mathbb{S}^2; \xi\in I_\nu\}$,

(ii) $\displaystyle\bigcup_{\ell_1,\ell_2=1}^{L-1} B_{\ell_1,\ell_2} = \mathbb{S}^2\times\mathbb{S}^2$,

(iii) $B_{\ell_1,\ell_2}\cap B_{\ell_1',\ell_2'} = \emptyset$ for $(\ell_1 > \ell_1'+2 \vee \ell_1 < \ell_1'-2) \wedge (\ell_2 > \ell_2'+2 \vee \ell_2 < \ell_2'-2)$.

This provides a decomposition of $\mathbb{S}^2\times\mathbb{S}^2$ into overlapping regions. Let us denote by ξ_j^-,η_j^- and ξ_j^+,η_j^+ the minimal resp. maximal value of the coordinates ξ resp. η of the region R_j. Since

$$\max\{\xi_j^+ - \xi_j^-, \eta_j^+ - \eta_j^-\} \le \mathsf{diam}(R_j) \le \|\mathcal{R}\| \le \pi L^{-1},$$

the region R_j is contained completely in at least one B_{ℓ_1,ℓ_2} with $\ell_1,\ell_2\in\{1,\ldots,L-1\}$. Accordingly we have

$$\begin{aligned}
\Sigma_1 \le & \sum_{\ell_1=1}^{L-1}\sum_{\ell_2=1}^{L-1}\sum_{R_j\subset B_{\ell_1,\ell_2}} \int_{R_j} |v_{N_2}(\cos\eta)| \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t\ \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}), \\
\Sigma_2 \le & \sum_{\ell_1=1}^{L-1}\sum_{\ell_2=1}^{L-1}\sum_{R_j\subset B_{\ell_1,\ell_2}} \int_{R_j} |v_{N_1}(\cos\xi_j)| \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\right| \mathrm{d}t\ \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}).
\end{aligned}$$

Now we have to distinguish between different cases. These cases are

$$(1.1):\ \ell_1 = 1 \wedge \{\ell_2 = 1 \vee \ell_2 = L-1\}, \quad (1.2):\ \ell_1 = L-1 \wedge (\ell_2 = 1 \vee \ell_2 = L-1,$$

$$(1.3):\ \ell_2 = 1 \wedge (\ell_1 = 1 \vee \ell_1 = L-1), \quad (1.4):\ \ell_2 = L-1 \wedge (\ell_1 = 1 \vee \ell_1 = L-1),$$

$$(2.1):\ \ell_1 = 1 \wedge \ell_2 \in \{2,\ldots,L-2\}, \qquad (2.2):\ \ell_1 = L-1 \wedge \ell_2 \in \{2,\ldots,L-2\},$$

$$(2.3):\ \ell_2 = 1 \wedge \ell_1 \in \{2,\ldots,L-2\}, \qquad (2.4):\ \ell_2 = L-1 \wedge \ell_1 \in \{2,\ldots,L-2\},$$

$$(3):\ \ell_1 \in \{2,\ldots,L-2\} \wedge \ell_2 \in \{2,\ldots,L-2\}.$$

We now treat the various cases separately.

Case (1.1): Using Lemma 2 (i),(ii) we get

$$\begin{aligned}
S_{1,1}^{(1)} &= \sum_{R_j \subset B_{1,1} \cup B_{1,L-1}} \int_{R_j} \left|v_{N_2}(\cos\eta)\right| \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\overset{\text{Lemma 2.(ii)}}{\leq} \frac{2}{\pi} N_1^3\, \|\mathcal{R}\|\, \frac{1}{\pi} N_2^2\, \{\tau(B_{1,1}) + \tau(B_{1,L-1})\} \\
&\leq \frac{2}{\pi} N_1^3\, \|\mathcal{R}\|\, \frac{1}{\pi} N_2^2 \left\{ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_0^{2\pi L^{-1}} \int_0^{2\pi L^{-1}} \right. \\
&\quad \left. + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{\pi - 2\pi L^{-1}}^{\pi} \int_{\pi - 2\pi L^{-1}}^{\pi} \right\} \sin\xi\, \sin\eta\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\varphi\, \mathrm{d}\phi \\
&\leq \frac{2}{\pi^2} N_1^3\, \|\mathcal{R}\|\, N_2^2\, 4\pi^2\; 2 \left(\int_0^{2\pi L^{-1}} \sin t\, \mathrm{d}t \right)^2 \\
&\leq 16\, N_2^2\, N_1^3\, \|\mathcal{R}\| (\pi L^{-1})^4 \\
&\overset{\pi L^{-1} \leq 2\|\mathcal{R}\|}{\leq} 1024\, N_2^2\, N_1^3\, \|\mathcal{R}\|^5.
\end{aligned}$$

It is obvious that due to symmetry we have $S_{1,k}^{(1)} \leq 1024 N_1^3\, N_2^2\, \|\mathcal{R}\|$ for $k = 2,3,4$.

Case (2.1):

$$\begin{aligned}
S_{2,1}^{(1)} &:= \sum_{\ell_2=2}^{L-2} \sum_{R_j \subset B_{1,\ell_2}} \int_{R_j} \left|v_{N_2}(\cos\eta)\right| \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\leq \sum_{\ell_2=2}^{L-2} \int_{B_{1,\ell_2}} \left|v_{N_2}(\cos\eta)\right| \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y})
\end{aligned}$$

$$
\begin{aligned}
&\le && \int_{B_1}\int_{\xi_j^-}^{\xi} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\big| \,\mathrm{d}t\,\mathrm{d}\sigma(\boldsymbol{x}) \sum_{\ell_2=2}^{L-2}\int_{B_{\ell_2}} \big|v_{N_2}(\cos\eta)\big| \,\mathrm{d}\sigma(\boldsymbol{y})\\
&\overset{\text{Lemma 2.(ii)}}{\le} && \frac{2}{\pi} N_1^3 \|\mathcal{R}\| \sigma(B_1) \sum_{\ell_2=2}^{L-2} \int_{-\pi}^{\pi}\int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2+1)\pi L^{-1}} \big|v_{N_2}(\cos\eta)\big| \,\sin\eta\,\mathrm{d}\eta\,\mathrm{d}\phi\\
&\le && 4N_1^3 \|\mathcal{R}\| \sigma(B_1) \sum_{\ell_2=2}^{L-2} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2+1)\pi L^{-1}} \big|v_{N_2}(\cos\eta)\big| \,\sin\eta\,\mathrm{d}\eta\,\mathrm{d}\phi\\
&\le && 8N_1^3 \|\mathcal{R}\| \sigma(B_1) \int_{\pi L^{-1}}^{(L-1)\pi L^{-1}} \big|v_{N_2}(\cos\eta)\big| \,\sin\eta\,\mathrm{d}\eta\\
&\le && 8N_1^3 \|\mathcal{R}\| \sigma(B_1) \int_{\pi L^{-1}}^{\pi} \big|v_{N_2}(\cos\eta)\big| \,\sin\eta\,\mathrm{d}\eta\\
&\le && 8N_1^3 \|\mathcal{R}\|\, \|v_{N_2}\|_\infty \int_{-\pi}^{\pi}\int_0^{2\pi L^{-1}} \sin\xi\,\mathrm{d}\xi\,\mathrm{d}\varphi \int_{\pi L^{-1}}^{\pi} \sin\eta\,\mathrm{d}\eta\\
&\le && 8N_1^3 \|\mathcal{R}\|\, \frac{N_2^2}{\pi}\, 2\pi \int_0^{2\pi L^{-1}} \sin\xi\,\mathrm{d}\xi \int_{\pi L^{-1}}^{\pi} \sin\eta\,\mathrm{d}\eta\\
&\le && 16N_1^3 N_2^2 \|\mathcal{R}\| \left[\frac{1}{2}\xi^2\right]_0^{2\pi L^{-1}} \left[-\cos\eta\right]_{\pi L^{-1}}^{\pi}\\
&\le && 32N_1^3 N_2^2 \|\mathcal{R}\|\, (\pi L^{-1})^2\, (1-\cos(\pi L^{-1}))\\
&\le && 32N_1^3 N_2^2 \|\mathcal{R}\|\, (\pi L^{-1})^2\, 2\sin^2(\frac{\pi L^{-1}}{2})\\
&\le && 64N_1^3 N_2^2 \|\mathcal{R}\|\, (\pi L^{-1})^2 \left(\frac{\pi L^{-1}}{2}\right)^2\\
&\overset{\pi L^{-1}\le 2\|\mathcal{R}\|}{\le} && 256\; N_1^3\; N_2^2\; \|\mathcal{R}\|^5
\end{aligned}
$$

Case (2.2):

$$
\begin{aligned}
S_{2,2}^{(1)} &:= && \sum_{\ell_2=2}^{L-2}\sum_{R_j\subset B_{L-1,\ell_2}} \int_{R_j} \big|v_{N_2}(\cos\eta)\big| \int_{\xi_j^-}^{\xi} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\big| \,\mathrm{d}t\,\mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y})\\
&\le && \sum_{\ell_2=2}^{L-2}\int_{B_{1,\ell_2}} \big|v_{N_2}(\cos\eta)\big| \int_{\xi_j^-}^{\xi} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\big| \,\mathrm{d}t\,\mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y})\\
&\le && \int_{B_{L-1}}\int_{\xi_j^-}^{\xi} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\big| \,\mathrm{d}t\,\mathrm{d}\sigma(\boldsymbol{x}) \sum_{\ell_2=2}^{L-2}\int_{B_{\ell_2}} \big|v_{N_2}(\cos\eta)\big| \,\mathrm{d}\sigma(\boldsymbol{y})
\end{aligned}
$$

$$\overset{\text{Lemma 2.(ii)}}{\leq} \quad \frac{2}{\pi}\, N_1^3\, \|\mathcal{R}\|\, \sigma(B_{L-1}) \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_2}} \left|v_{N_2}(\cos\eta)\right| \,\mathrm{d}\sigma(\boldsymbol{y})$$

$$\leq \quad 256\, N_1^3\, N_2^2\, \|\mathcal{R}\|^5$$

Case (2.3):

$$S_{2,3}^{(1)} := \sum_{\ell_1=2}^{L-2} \sum_{R_j \subset B_{\ell_1,1}} \int_{R_j} \left|v_{N_2}(\cos\eta)\right| \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y})$$

$$\leq \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1,1}} \left|v_{N_2}(\cos\eta)\right| \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y})$$

$$\leq \int_{B_1} \left|v_{N_2}(\cos\eta)\right| \,\mathrm{d}\sigma(\boldsymbol{y}) \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1}} \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\sigma(\boldsymbol{x})$$

$$\leq \int_{B_1} \left|v_{N_2}(\cos\eta)\right| \,\mathrm{d}\sigma(\boldsymbol{y}) \sum_{\ell_1=2}^{L-2} \int_{-\pi}^{\pi} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t\, \sin\xi\, \mathrm{d}\xi\, \mathrm{d}\varphi$$

$$\overset{\sin\xi\leq 2\sin t}{\leq} \quad 4\pi \int_{B_1} \left|v_{N_2}(\cos\eta)\right| \,\mathrm{d}\sigma(\boldsymbol{y}) \sum_{\ell_2=2}^{L-2} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \sin t\, \mathrm{d}t\, \mathrm{d}\xi$$

$$\leq \quad 4\pi(\pi L^{-1}) \int_{B_1} \left|v_{N_2}(\cos\eta)\right| \,\mathrm{d}\sigma(\boldsymbol{y}) \sum_{\ell_2=2}^{L-2} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \sin t\mathrm{d}t$$

$$\leq \quad 4\pi(\pi L^{-1}) \int_{B_1} \left|v_{N_2}(\cos\eta)\right| \,\mathrm{d}\sigma(\boldsymbol{y}) \int_{\|\mathcal{R}\|}^{\pi-\|\mathcal{R}\|} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \sin t\, \mathrm{d}t$$

$$\overset{\text{Lemma 2.(iii)}}{\leq} \quad 4\pi(\pi L^{-1}) \int_{-\pi}^{\pi} \int_0^{2\pi L^{-1}} \left|v_{N_2}(\cos\eta)\right| \sin\eta\, \mathrm{d}\eta\, \mathrm{d}\phi \quad \frac{3}{2\pi}(5N_1+1)$$

$$\leq \quad 6(5N_1+1)(\pi L^{-1})\ 2\pi\, \frac{1}{\pi} N_2^2 \int_0^{2\pi L^{-1}} \sin\eta\, \mathrm{d}\eta$$

$$\leq \quad 12(5N_1+1)N_2^2\, (\pi L^{-1}) \left[\frac{1}{2}\eta^2\right]_0^{2\pi L^{-1}}$$

$$\leq \quad 24(5N_1+1)N_2^2\, (\pi L^{-1})^3$$

$$\overset{\pi L^{-1}\leq 2\|\mathcal{R}\|}{\leq} \quad 1152\, N_1\, N_2^2\, \|\mathcal{R}\|^3$$

Case (2.4):

$$\begin{aligned}
S_{2,4}^{(1)} &:= \sum_{\ell_1=2}^{L-2} \sum_{R_j \subset B_{\ell_1,L-1}} \int_{R_j} \left|v_{N_2}(\cos\eta)\right| \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t \, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\leq \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1,L-1}} \left|v_{N_2}(\cos\eta)\right| \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t \, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\leq \int_{B_{L-1}} \left|v_{N_2}(\cos\eta)\right| \, \mathrm{d}\sigma(\boldsymbol{y}) \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1}} \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t \, \mathrm{d}\sigma(\boldsymbol{x}) \\
&\leq 1152 \, N_1 \, N_2^2 \, \|\mathcal{R}\|^3
\end{aligned}$$

Case (3):

$$\begin{aligned}
S_3^{(1)} &:= \sum_{\ell_1=2}^{L-2} \sum_{\ell_2=2}^{L-2} \sum_{R_j \subset B_{\ell_1,\ell_2}} \int_{R_j} \left|v_{N_2}(\cos\eta)\right| \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t \, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\leq \sum_{\ell_1=2}^{L-2} \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_1,\ell_2}} \left|v_{N_2}(\cos\eta)\right| \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t \, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\leq \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1}} \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t \, \mathrm{d}\sigma(\boldsymbol{x}) \; \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_2}} \left|v_{N_2}(\cos\eta)\right| \mathrm{d}\sigma(\boldsymbol{y}) \\
&\leq \sum_{\ell_1=2}^{L-2} \int_{-\pi}^{\pi} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t \, \sin\xi \, \mathrm{d}\xi \, \mathrm{d}\varphi \\
&\quad \times \sum_{\ell_2=2}^{L-2} \int_{-\pi}^{\pi} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \left|v_{N_2}(\cos\eta)\right| \, \sin\eta \, \mathrm{d}\eta \, \mathrm{d}\phi \\
&\overset{\sin\xi \leq 2\sin t}{\leq} 4\pi^2 \, 2 \sum_{\ell_1=2}^{L-2} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \, \sin t \, \mathrm{d}t \, \mathrm{d}\xi \\
&\quad \times 2 \int_{\pi L^{-1}}^{(L-1)\pi L^{-1}} \left|v_{N_2}(\cos\eta)\right| \, \sin\eta \, \mathrm{d}\eta \\
&\leq 16\pi^2 \sum_{\ell_1=2}^{L-2} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \int_{\xi_j^-}^{\xi} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \, \sin t \, \mathrm{d}t \, \mathrm{d}\xi \\
&\quad \times \|v_{N_2}\|_\infty \int_{\pi L^{-1}}^{\pi} \sin\eta \, \mathrm{d}\eta
\end{aligned}$$

$$\begin{aligned}
&\leq \quad 16\pi^2\, 2(\pi L^{-1}) \sum_{\ell_1=2}^{L-2} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\big| \,\sin t\,\mathrm{d}t \\
&\qquad \times \|v_{N_2}\|_\infty \int_{\pi L^{-1}}^{\pi} \sin\eta\,\mathrm{d}\eta \\
&\leq \quad 32\pi^2\,(\pi L^{-1}) \int_{\|\mathcal{R}\|}^{\pi-\|\mathcal{R}\|} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\big| \,\sin t\,\mathrm{d}t \\
&\qquad \times \frac{1}{\pi} N_2^2\,[-\cos\eta]_{\pi L^{-1}}^{\pi} \\
&\leq \quad 32\pi^2\,(\pi L^{-1})\,\frac{3}{2\pi}(5N_1+1)\frac{1}{\pi}N_2^2\,2\sin^2(\frac{\pi L^{-1}}{2}) \\
&\leq \quad 96\,(\pi L^{-1})\,(5N_1+1)N_2^2\,\left(\frac{\pi L^{-1}}{2}\right)^2 \\
&\leq \quad 24\,(5+\frac{1}{N_1})\,N_1\,N_2^2\,(\pi L^{-1})^3 \\
&\overset{\pi L^{-1}\leq 2\|\mathcal{R}\|}{\leq} \quad 1152\,N_1\,N_2^2\,\|\mathcal{R}\|^3.
\end{aligned}$$

All the estimates considered so far provide an estimate for the sum Σ_1. For the second sum Σ_2 we obtain similarly

Case (1.1): Using Lemma 2 (i),(ii) we get

$$\begin{aligned}
S_{1,1}^{(2)} &= \sum_{R_j\subset B_{1,1}\cup B_{1,L-1}} \int_{R_j} \big|v_{N_1}(\cos\xi_j)\big| \int_{\eta_j^-}^{\eta} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\big|\,\mathrm{d}t\,\mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\overset{\text{Lemma 2.(ii)}}{\leq} \frac{2}{\pi}N_2^3\,\|\mathcal{R}\|\,\frac{1}{\pi}N_1^2\,\{\tau(B_{1,1})+\tau(B_{1,L-1})\} \\
&\leq \frac{2}{\pi}N_2^3\,\|\mathcal{R}\|\,\frac{1}{\pi}N_1^2 \\
&\qquad \times \left\{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^{2\pi L^{-1}}\int_0^{2\pi L^{-1}} + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{\pi-2\pi L^{-1}}^{\pi}\int_{\pi-2\pi L^{-1}}^{\pi}\right\}\sin\xi\,\sin\eta\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\varphi\,\mathrm{d}\phi \\
&\leq \frac{2}{\pi^2}N_2^3\,\|\mathcal{R}\|\,N_1^2\,4\pi^2\,2\left(\int_0^{2\pi L^{-1}}\sin t\,\mathrm{d}t\right)^2 \\
&\leq 16\,N_1^2\,N_2^3\,\|\mathcal{R}\|(\pi L^{-1})^4 \\
&\overset{\pi L^{-1}\leq 2\|\mathcal{R}\|}{\leq} 1024\,N_1^2\,N_2^3\,\|\mathcal{R}\|^5.
\end{aligned}$$

Due to symmetry we have $S_{1,k}^{(2)} \leq 1024\,N_1^2\,N_2^3\,\|\mathcal{R}\|^5,\ k=2,3,4.$

Case (2.1):

$$
\begin{aligned}
S^{(2)}_{2,1} \quad &:= \quad \sum_{\ell_2=2}^{L-2} \sum_{R_j \subset B_{1,\ell_2}} \int_{R_j} \big|v_{N_1}(\cos\xi_j)\big| \int_{\eta_j^-}^{\eta} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\big| \,\mathrm{d}t\,\mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\leq \quad \sum_{\ell_2=2}^{L-2} \int_{B_{1,\ell_2}} \big|v_{N_1}(\cos\xi_j)\big| \int_{\eta_j^-}^{\eta} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\big| \,\mathrm{d}t\,\mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\leq \quad \int_{B_1} \big|v_{N_1}(\cos\xi_j)\big| \;\mathrm{d}\sigma(\boldsymbol{x}) \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_2}} \int_{\eta_j^-}^{\eta} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\big| \,\mathrm{d}t\,\mathrm{d}\sigma(\boldsymbol{y}) \\
&\overset{\text{Lemma 2.(i)}}{\leq} \quad \|v_{N_1}\|_\infty \int_{-\pi}^{\pi} \int_0^{2\pi L^{-1}} \sin\xi \,\mathrm{d}\xi\,\mathrm{d}\varphi \\
&\qquad \times \sum_{\ell_2=2}^{L-2} \int_{-\pi}^{\pi} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2+1)\pi L^{-1}} \int_{\eta_j^-}^{\eta} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\big| \;\sin\eta \,\mathrm{d}t\,\mathrm{d}\eta\,\mathrm{d}\phi \\
&\leq \quad 4\pi N_1^2 \left[\frac{1}{2}\xi^2\right]_0^{2\pi L^{-1}} \sum_{\ell_2=2}^{L-2} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2+1)\pi L^{-1}} \int_{\eta_j^-}^{\eta} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\big| \;\sin\eta \,\mathrm{d}t\,\mathrm{d}\eta \\
&\leq \quad 8\pi\, N_1^2\, (\pi L^{-1})^2 \sum_{\ell_2=2}^{L-2} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2+1)\pi L^{-1}} \int_{\eta_j^-}^{\eta} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\big| \,2\sin t \,\mathrm{d}t\,\mathrm{d}\eta \\
&\leq \quad 16\pi\, N_1^2\, (\pi L^{-1})^2\, 2(\pi L^{-1}) \sum_{\ell_2=2}^{L-2} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2+1)\pi L^{-1}} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\big| \;\sin t \,\mathrm{d}t \\
&\overset{\pi L^{-1} \leq 2\|\mathcal{R}\|}{\leq} \quad 256\pi\, N_1^2\, \|\mathcal{R}\|^3 \int_{\|\mathcal{R}\|}^{\pi-\|\mathcal{R}\|} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\big| \;\sin t \,\mathrm{d}t \\
&\overset{\text{Lemma 2.(iii)}}{\leq} \quad 256\pi\, N_1^2\, \|\mathcal{R}\|^3\; \frac{3}{2\pi}(5N_2+1) \\
&\leq \quad 256\, N_1^2\, N_2\, \|\mathcal{R}\|^3\; \frac{3}{2}(5+\frac{1}{N_2}) \\
&\leq \quad 2304\, N_1^2\, N_2\, \|\mathcal{R}\|^3
\end{aligned}
$$

Case (2.2): This case is similar to the case (2.1) and therefore we can shorten the calculation.

$$
S^{(2)}_{2,2} \;:=\; \sum_{\ell_2=2}^{L-2} \sum_{R_j \subset B_{L-1,\ell_2}} \int_{R_j} \big|v_{N_1}(\cos\xi_j)\big| \int_{\eta_j^-}^{\eta} \big|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\big| \,\mathrm{d}t\,\mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y})
$$

$$
\begin{aligned}
&\le \sum_{\ell_2=2}^{L-2} \int_{B_{L-1,\ell_2}} \left|v_{N_1}(\cos\xi_j)\right| \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\le \int_{B_{L-1}} \left|v_{N_2}(\cos\xi_j)\right| \,\mathrm{d}\sigma(\boldsymbol{x}) \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_2}} \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\sigma(\boldsymbol{y}) \\
&\le 2304\, N_1^2\, N_2\, \|\mathcal{R}\|^3
\end{aligned}
$$

Case (2.3):

$$
\begin{aligned}
S_{2,3}^{(2)} \quad &:= \quad \sum_{\ell_1=2}^{L-2} \sum_{R_j \subset B_{\ell_1,1}} \int_{R_j} \left|v_{N_1}(\cos\xi_j)\right| \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\le \quad \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1,1}} \left|v_{N_1}(\cos\xi_j)\right| \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\le \quad \int_{B_1} \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\sigma(\boldsymbol{x}) \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1}} \left|v_{N_2}(\cos\xi_j)\right| \,\mathrm{d}\sigma(\boldsymbol{y}) \\
&\le \quad 4\pi^2 \int_0^{2\pi L^{-1}} \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \sin\eta\, \mathrm{d}t\, \mathrm{d}\eta \\
&\qquad \times \sum_{\ell_2=2}^{L-2} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \left|v_{N_2}(\cos\xi_j)\right| \,\sin\xi\, \mathrm{d}\xi \\
&\overset{\text{Lemma 2.(ii)}}{\le} \quad 4\pi^2\, \frac{2}{\pi}\, N_1^3 \|\mathcal{R}\| \int_0^{2\pi L^{-1}} \sin\eta\, \mathrm{d}\eta \quad \|v_{N_1}\|_\infty\; 2 \int_{\pi L^{-1}}^{\pi} \sin\xi\, \mathrm{d}\xi \\
&\overset{\text{Lemma 2.(i)}}{\le} \quad 16\pi\; N_1^3 \|\mathcal{R}\|\; \frac{1}{\pi} N_2^2 \;\left[\frac{1}{2}\eta^2\right]_0^{2\pi L^{-1}} \quad [-\cos\xi]_{\pi L^{-1}}^{\pi} \\
&\le \quad 32\, N_1^3 \|\mathcal{R}\|\, N_2^2\; (\pi L^{-1})^2\; 2\sin^2(\frac{\pi L^{-1}}{2}) \\
&\le \quad 16\, N_1^3\, N_2^2\, \|\mathcal{R}\|\, (\pi L^{-1})^4 \\
&\overset{\pi L^{-1} \le 2\|\mathcal{R}\|}{\le} \quad 256\, N_1^3\, N_2^2\, \|\mathcal{R}\|^5
\end{aligned}
$$

Case (2.4):

$$
S_{2,4}^{(2)} \quad := \quad \sum_{\ell_1=2}^{L-2} \sum_{R_j \subset B_{\ell_1,L-1}} \int_{R_j} \left|v_{N_1}(\cos\xi_j)\right| \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y})
$$

$$
\begin{aligned}
&\leq && \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1,L-1}} \left|v_{N_1}(\cos\xi_j)\right| \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\leq && \int_{B_{L-1}} \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_1}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\sigma(\boldsymbol{x}) \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1}} \left|v_{N_2}(\cos\xi_j)\right|\, \mathrm{d}\sigma(\boldsymbol{y}) \\
&\leq && 4\pi^2 \int_{\pi-2\pi L^{-1}}^{\pi} \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\right| \sin\eta\, \mathrm{d}t\, \mathrm{d}\eta \sum_{\ell_2=2}^{L-2} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \left|v_{N_2}(\cos\xi_j)\right| \sin\xi\, \mathrm{d}\xi \\
&\overset{\text{Lemma 2.(ii)}}{\leq} && 4\pi^2\, \frac{2}{\pi}\, N_1^3 \|\mathcal{R}\| \int_{\pi-2\pi L^{-1}}^{\pi} \sin\eta\, \mathrm{d}\eta \quad \|v_{N_1}\|_\infty\ 2 \int_{\pi L^{-1}}^{\pi} \sin\xi\, \mathrm{d}\xi \\
&\leq && 4\pi^2\, \frac{2}{\pi}\, N_1^3 \|\mathcal{R}\| \int_{0}^{2\pi L^{-1}} \sin\eta\, \mathrm{d}\eta \quad \|v_{N_1}\|_\infty\ 2 \int_{\pi L^{-1}}^{\pi} \sin\xi\, \mathrm{d}\xi \\
&\overset{\text{Lemma 2.(i)}}{\leq} && 16\pi\ N_1^3 \|\mathcal{R}\|\, \frac{1}{\pi} N_2^2\ \left[\frac{1}{2}\eta^2\right]_0^{2\pi L^{-1}} \quad [-\cos\xi]_{\pi L^{-1}}^{\pi} \\
&\leq && 32\, N_1^3 \|\mathcal{R}\|\, \frac{1}{\pi} N_2^2\ (\pi L^{-1})^2\ 2\sin^2\!\left(\frac{\pi L^{-1}}{2}\right) \\
&\leq && 16\, N_1^3\, N_2^2\, \|\mathcal{R}\|\, (\pi L^{-1})^4 \\
&\overset{\pi L^{-1} \leq 2\|\mathcal{R}\|}{\leq} && 256\, N_1^3\, N_2^2\, \|\mathcal{R}\|^5
\end{aligned}
$$

Case (3):

$$
\begin{aligned}
S_3^{(2)} &:= && \sum_{\ell_1=2}^{L-2} \sum_{\ell_2=2}^{L-2} \sum_{R_j \subset B_{\ell_1,\ell_2}} \int_{R_j} \left|v_{N_1}(\cos\xi_j)\right| \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\leq && \sum_{\ell_1=2}^{L-2} \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_1,\ell_2}} \left|v_{N_1}(\cos\xi_j)\right| \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\tau(\boldsymbol{x},\boldsymbol{y}) \\
&\leq && \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1}} \left|v_{N_1}(\cos\xi_j)\right| \mathrm{d}\sigma(\boldsymbol{x}) \ \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_2}} \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\right| \mathrm{d}t\, \mathrm{d}\sigma(\boldsymbol{y}) \\
&\leq && \sum_{\ell_1=1}^{L-2} \int_{-\pi}^{\pi} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1-1)\pi L^{-1}} \left|v_{N_1}(\cos\xi_j)\right|\, \sin\xi\, \mathrm{d}\xi\, \mathrm{d}\varphi \\
& && \times \sum_{\ell_2=2}^{L-2} \int_{-\pi}^{\pi} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2-1)\pi L^{-1}} \int_{\eta_j^-}^{\eta} \left|\frac{\mathrm{d}}{\mathrm{d}t} v_{N_2}(\cos t)\right| \mathrm{d}t\ \sin\eta\, \mathrm{d}\eta\, \mathrm{d}\phi
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\sin\xi\le 2\sin t}{\le} && 4\pi^2\, 2\int_{\pi L^{-1}}^{(L-1)\pi L^{-1}} \big|v_{N_2}(\cos\xi_j)\big|\,\sin\xi\,\mathrm{d}\xi \\
& && \times 2\sum_{\ell_2=2}^{L-2}\int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2-1)\pi L^{-1}}\int_{\eta_j^-}^{\eta}\big|\frac{\mathrm{d}}{\mathrm{d}t}v_{N_2}(\cos t)\big|\,\sin t\,\mathrm{d}t\,\mathrm{d}\eta \\
&\le && 16\pi^2\,\|v_{N_1}\|_\infty\int_{\pi L^{-1}}^{\pi}\sin\xi\,\mathrm{d}\xi \\
& && \times\sum_{\ell_2=2}^{L-2}\int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2-1)\pi L^{-1}}\int_{\eta_j^-}^{\eta}\big|\frac{\mathrm{d}}{\mathrm{d}t}v_{N_2}(\cos t)\big|\,\sin t\,\mathrm{d}t\,\mathrm{d}\eta \\
&\le && 16\pi^2\,\frac{1}{\pi}\,N_1^2\,\left[-\cos\xi\right]_{\pi L^{-1}}^{\pi}\;2(\pi L^{-1})\sum_{\ell_2=2}^{L-2}\int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2-1)\pi L^{-1}}\big|\frac{\mathrm{d}}{\mathrm{d}t}v_{N_2}(\cos t)\big|\,\sin t\,\mathrm{d}t\,\mathrm{d}\eta \\
&\le && 16\pi^2\,\frac{1}{\pi}\,N_1^2\,2\sin^2(\frac{\pi L^{-1}}{2})\,2(\pi L^{-1})\,2\int_{\|\mathcal{R}\|}^{\pi\|\mathcal{R}\|}\big|\frac{\mathrm{d}}{\mathrm{d}t}v_{N_2}(\cos t)\big|\,\sin t\,\mathrm{d}t \\
&\le && 128\pi\,N_1^2\left(\frac{\pi L^{-1}}{2}\right)^2(\pi L^{-1})\,\frac{3}{2\pi}(5N_2+1) \\
&\le && 16\,(5+\frac{1}{N_2})\,N_1^2\,N_2\,(\pi L^{-1})^3 \\
&\overset{\pi L^{-1}\le 2\|\mathcal{R}\|}{\le} && 768\,N_1^2\,N_2\,\|\mathcal{R}\|^3\,.
\end{aligned}
$$

Putting all the estimates for $\Sigma^{(1)}$ and $\Sigma^{(2)}$ together we obtain the following result .

$$\Sigma \le 8832\widetilde{N}^3\,\|\mathcal{R}\|^3 + 9216\widetilde{N}^5\,\|\mathcal{R}\|^5 \tag{5}$$

Assume $\|\mathcal{R}\| \le \frac{\eta}{C\widetilde{N}}$. Then we obtain

$$\Sigma \le \left(8832\frac{1}{C^3} + 9216\frac{1}{C^5}\right)\eta \tag{6}$$

A short computation shows that for $C = 21$ the first factor in (6) is less than 1. This shows the desired result. ■

References

[1] F. Filbir and D. Potts. Scattered data approximation on the bi-sphere and application to texture analysis. *Preprint, Helmholtz Zentrum München*, 2009.

[2] F. Filbir and W. Themistoclakis. Polynomial approximation on the sphere using scattered data. *Math. Nachr.*, 281:650 – 668, 2008.