Fast Fourier transform for nonequispaced data with applications in MRI

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Abstract

In this talk we give an introduction in the fast Fourier transform for nonequispaced data (NFFT) [4, 3, 2], show the commons with the gridding reconstruction in MRI and extend this approach to field inhomogeneity correction. This talk base on a joint paper with H. Eggers and T. Knopp [1].

1 NFFT and Gridding

In the following we summarise for the one dimensional case the relationship between the NFFT and gridding reconstruction. Let a function $\varphi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, the so-called window function, be given. Its one periodisation $\tilde{\varphi}(k) := \sum_{p=-\infty}^{\infty} \varphi(k+p)$ is assumed to have a uniformly convergent Fourier series. Hence, it may be written as $\tilde{\varphi}(k) = \sum_{x=-\infty}^{\infty} c_x(\tilde{\varphi}) \mathrm{e}^{2\pi \mathrm{i} kx}$, with Fourier coefficients

$$c_x(\tilde{\varphi}) := \int_{-1/2}^{1/2} \tilde{\varphi}(k) \mathrm{e}^{-2\pi \mathrm{i} kx} \, \mathrm{d}k \,, \qquad (1)$$

where $x \in \mathbb{Z}$. Substituting k by $k - k'$ in (1) yields

$$c_x(\tilde{\varphi}) = \int_{-1/2}^{1/2} \tilde{\varphi}(k-k') \mathrm{e}^{-2\pi \mathrm{i}(k-k')x} \, \mathrm{d}k' \,, \qquad (2)$$

which may be approximated by

$$c_x(\tilde{\varphi}) \approx \frac{1}{\alpha N} \sum_{l=-\alpha N/2}^{\alpha N/2-1} \tilde{\varphi}(k - \frac{l}{\alpha N}) \mathrm{e}^{-2\pi \mathrm{i}(k - \frac{l}{\alpha N})x} \qquad (3)$$

for $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $x = -\frac{N}{2}, \ldots, \frac{N}{2}$. The factor $\alpha > 1$ is commonly referred to as oversampling factor. For the sake of simplicity, N and αN are assumed to be even. Provided that all $c_x(\tilde{\varphi})$ are unequal zero, (3) may be rewritten as

$$\mathrm{e}^{2\pi \mathrm{i} kx} \approx \frac{1}{\alpha N c_x(\tilde{\varphi})} \sum_{l=-\alpha N/2}^{\alpha N/2-1} \tilde{\psi}(k - \frac{l}{\alpha N}) \mathrm{e}^{2\pi \mathrm{i} \frac{lx}{\alpha N}} \,, \qquad (4)$$

where $\tilde{\varphi}$ has been replaced by $\tilde{\psi}$. The latter is the one periodisation of a truncation of φ defined by

$$\psi(k) := \begin{cases} \varphi(k) & k \in [-\frac{m}{\alpha N}, \frac{m}{\alpha N}] \,, \\ 0 & k \notin [-\frac{m}{\alpha N}, \frac{m}{\alpha N}] \,. \end{cases}$$

The support of ψ is determined by $2m$, the so-called kernel size. Typically, $m \in \mathbb{N}$ is chosen such that $m \ll N$. The standard NFFT evaluates the trigonometric polynomial

$$f(k) := \sum_{x=-N/2}^{N/2-1} \hat{f}_x \mathrm{e}^{2\pi \mathrm{i} kx} \qquad (5)$$

for N given equispaced samples $\hat{f}_x$ at M given non-equispaced positions $k_j \in [-\frac{1}{2}, \frac{1}{2}]$. It uses the approximation (4), which yields

$$f_j \approx \sum_{l=-\alpha N/2}^{\alpha N/2-1} \tilde{\psi}(k_j - \frac{l}{\alpha N}) \sum_{x=-N/2}^{N/2-1} \frac{\hat{f}_x}{\alpha N c_x(\tilde{\varphi})} \times \mathrm{e}^{2\pi \mathrm{i} \frac{lx}{\alpha N}} \,, \qquad (6)$$

where $f_j := f(k_j)$. In matrix-vector notation, (5) then reads $\boldsymbol{f} = \boldsymbol{A}\hat{\boldsymbol{f}}$, with $\boldsymbol{f} := (f_j)_{j=0}^{M-1}$, $\hat{\boldsymbol{f}} := \left(\hat{f}_x\right)_{x=-N/2}^{N/2-1}$, $\boldsymbol{A} := \left(\mathrm{e}^{2\pi \mathrm{i} k_j x}\right)_{j=0,x=-N/2}^{M-1,N/2-1}$. According to (6), $\boldsymbol{A}$ may be approximated by $\boldsymbol{BFD}$, where $\boldsymbol{D}$ is a diagonal matrix with entries $d_{x,x} = 1/c_x(\tilde{\varphi})$, $\boldsymbol{F}$ an oversampled Fourier matrix, which includes the factor $1/(\alpha N)$, and $\boldsymbol{B}$ a sparse matrix with entries $b_{j,l} = \tilde{\psi}(k_j - l/(\alpha N))$. Similarly, the evaluation of the adjoint to (5), i.e. of the sum

$$\sum_{j=0}^{M-1} f_j \mathrm{e}^{-2\pi \mathrm{i} k_j x}$$

for M given non-equispaced samples f_j at N given equispaced positions $x = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$, may be performed by a matrix-vector multiplication with $\boldsymbol{A}^{\mathsf{H}} \approx \boldsymbol{D}^{\mathsf{H}}\boldsymbol{F}^{\mathsf{H}}\boldsymbol{B}^{\mathsf{H}}$. As pointed out in [3], gridding reconstruction is simply a fast algorithm for the application of $\boldsymbol{D}^{\mathsf{H}}\boldsymbol{F}^{\mathsf{H}}\boldsymbol{B}^{\mathsf{H}}$ to a vector of non-equispaced samples. Unified approaches to the efficient computation of (5) with the NFFT were suggested in [4, 3]. Obviously, the NFFT and gridding reconstruction not only involve closely related processing, but also exploit the same approximation. To address field inhomogeneity correction similarly, we conclude this subsection with the derivation of an approximation that lifts the restriction on x to be an integer in (4). Starting from

$\hat{\varphi}(x) := \int\limits_{-\infty}^{\infty} \varphi(k) \mathrm{e}^{-2\pi \mathrm{i} k x} \, \mathrm{d}k$ instead of (1) leads to

$$\hat{\varphi}(x) = \int_{-1/2}^{1/2} \sum_{p=-\infty}^{\infty} \varphi(k+p) \mathrm{e}^{-2\pi \mathrm{i}(k+p)x} \, \mathrm{d}k$$

and, with the same steps as from (2) to (4), to

$$\mathrm{e}^{2\pi \mathrm{i} k x} \approx \frac{1}{\alpha N \hat{\varphi}(x)} \sum_{l=-\alpha N/2}^{\alpha N/2-1} \sum_{p=-\infty}^{\infty} \psi(k - \frac{l}{\alpha N} + p) \times \mathrm{e}^{2\pi \mathrm{i} (\frac{l}{\alpha N} + p) x}$$

for $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $x \in [-\frac{N}{2}, \frac{N}{2}]$. Like (4), this approximation may be reduced to

$$\mathrm{e}^{2\pi \mathrm{i} k x} \approx \frac{1}{\alpha N \hat{\varphi}(x)} \sum_{l=-\alpha N/2}^{\alpha N/2-1} \psi(k - \frac{l}{\alpha N}) \mathrm{e}^{2\pi \mathrm{i} \frac{l x}{\alpha N}} \quad (7)$$

for $k \in [-\frac{1}{2} + \frac{m}{\alpha N}, \frac{1}{2} - \frac{m}{\alpha N}]$, since the support of ψ is $[-\frac{m}{\alpha N}, \frac{m}{\alpha N}]$. Consequently, (7) is a good approximation if $kx \in [-\frac{N}{4} + \frac{m}{2\alpha}, \frac{N}{4} - \frac{m}{2\alpha}]$.

2 Theory

In MRI, the demodulated signal $s(t)$ received from an object with a magnetisation $m(\boldsymbol{r})$ at a reference time point $t = 0$ is ideally given by

$$s(t) = \int_{\mathbb{R}^3} m(\boldsymbol{r}) \, \mathrm{e}^{\mathrm{i} \boldsymbol{k}(t) \boldsymbol{r}} \, \mathrm{d}\boldsymbol{r} \, . \quad (8)$$

$\boldsymbol{k}(t)$ denotes the trajectory, along which samples are acquired in the spatial frequency domain, the so-called k-space. It is determined by the time variant gradient field applied during the measurement.

Any inhomogeneity of the main field distorts the Fourier encoding that (8) describes. Taking this imperfection into account, $s(t)$ is more accurately modelled by

$$s(t) = \int_{\mathbb{R}^3} m(\boldsymbol{r}) \, \mathrm{e}^{\mathrm{i} \omega(\boldsymbol{r}) t} \mathrm{e}^{\mathrm{i} \boldsymbol{k}(t) \boldsymbol{r}} \, \mathrm{d}\boldsymbol{r} \, . \quad (9)$$

$\omega(\boldsymbol{r})$ denotes the angular off-resonance frequency, which is proportional to the local deviation of the main field from its nominal strength. Other imperfections, such as relaxation, are not considered in this work. From now on we restrict ourselves to 2D imaging. The sampled area of k-space is then confined to $\boldsymbol{k} \in [-\pi, \pi]^2$, and the covered field of view to $\boldsymbol{r} \in [-\frac{N_1}{2}, \frac{N_1}{2}] \times [-\frac{N_2}{2}, \frac{N_2}{2}]$. Discretising the integral in (9) on $N_1 N_2$ equispaced voxel positions $\boldsymbol{r}_\rho$ and the signal $s(t)$ on M time points t_κ yields

$$s_\kappa \approx \tilde{s}_\kappa := \sum_{\rho=0}^{N_1 N_2 - 1} m_\rho \mathrm{e}^{\mathrm{i} \omega_\rho t_\kappa} \mathrm{e}^{\mathrm{i} \boldsymbol{k}_\kappa \boldsymbol{r}_\rho}, \quad (10)$$

where $s_\kappa := s(t_\kappa)$, $\tilde{s}_\kappa := \tilde{s}(t_\kappa)$, $m_\rho := m(\boldsymbol{r}_\rho)$, $\omega_\rho := \omega(\boldsymbol{r}_\rho)$, and $\boldsymbol{k}_\kappa := \boldsymbol{k}(t_\kappa)$. Using the vectors $\boldsymbol{s} := (s_\kappa)_{\kappa=0}^{M-1}$, $\boldsymbol{m} := (m_\rho)_{\rho=0}^{N_1 N_2 - 1}$, and the matrix

$$\boldsymbol{H} := \left(\mathrm{e}^{\mathrm{i} \omega_\rho t_\kappa} \mathrm{e}^{\mathrm{i} \boldsymbol{k}_\kappa \boldsymbol{r}_\rho} \right)_{\kappa=0, \rho=0}^{M-1, N_1 N_2 - 1} \, ,$$

this may be rewritten as $\boldsymbol{s} \approx \boldsymbol{H}\boldsymbol{m}$.

We propose determining $\boldsymbol{m}$ by a weighted least squares approach

$$\|\boldsymbol{s} - \boldsymbol{H}\boldsymbol{m}\|_{\boldsymbol{W}} = \sqrt{\sum_{\kappa=0}^{M-1} w_\kappa |s_\kappa - \tilde{s}_\kappa|^2} \xrightarrow{\boldsymbol{m}} \min \, , \quad (11)$$

with factors $w_\kappa > 0$ that compensate for variations in the local sampling density. It leads to the weighted normal equation of first kind

$$\boldsymbol{H}^{\mathsf{H}} \boldsymbol{W} \boldsymbol{H} \boldsymbol{m} = \boldsymbol{H}^{\mathsf{H}} \boldsymbol{W} \boldsymbol{s} \, , \quad (12)$$

where $\boldsymbol{W} := \mathrm{diag}(w_\kappa)_{\kappa=0}^{M-1}$. Due to the size of this linear system, we suggest to solve it iteratively with a suitable variant of the conjugate gradient method, such as the Conjugate Gradient Normal Equation Residual (CGNR) method.

Embedding the data in a higher dimensional space and using the approximation (7) we obtain different fast methods for the matrix times vector multiplication with the matrix $\boldsymbol{H}$. See [1] for details.

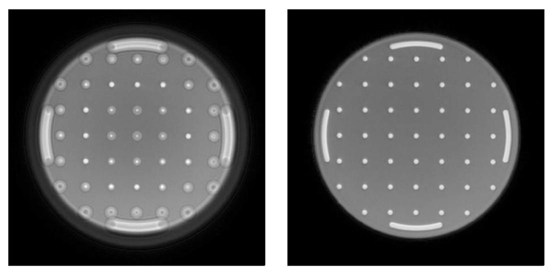

Figure 1: Results of phantom experiments. Uncorrected and corrected images.

References

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