An introduction to pseudospectra

The analysis of eigenvalues and spectra of matrices and operators has been one of the most fruitful fields of mathematics for about 100 years. Although the limits of eigenvalue analysis were realized sporadically by several researchers, it took nearly 90 years, until 1990, before Trefethen came to a deeper understanding of important phenomena caused by nonnormal operators and matrices and in this connection developed the idea of the pseudospectrum to a powerful tool of present-day analysis. The lectures are intended as an introduction to pieces of the fascinating mathematics around pseudospectra and their applications.

The ε-pseudospectrum of a matrix or a linear operator A is defined as the plane set

$$\sigma_\varepsilon(A) = \{\lambda \in \mathbf{C} : \|(A - \lambda I)^{-1}\| > 1/\varepsilon\}$$

or, alternatively, as

$$\sigma_\varepsilon(A) = \bigcup_{\|E\|<\varepsilon} \sigma(A+E),$$

that is, as the union of all possible spectra that can be achieved by perturbations to A of norm less than ε. We demonstrate by various examples, many of which are taken from Trefethen and Embree's recent monograph on the topic, that it is just the pseudospectra of an operator that allow us to understand a series of problems governed by nonnormal operators. These problems include the analysis of numerical algorithms, fluid dynamics, localization and delocalization phenomena for random matrices, and card shuffling. Notice that if A is a normal Hilbert space operator, then $\sigma_\varepsilon(A) = \{\lambda \in \mathbf{C} : \operatorname{dist}(\lambda, \sigma(A)) < \varepsilon\}$ and hence all information about the pseudospectrum is already captured in the spectrum. This explains the success of sole eigenvalue analysis in the case of normal (and in particular Hermitian) operators and matrices.

We also embark on theoretical aspects of the matter such as rigorous convergence results for pseudospectra of ordinary differential operators or Toeplitz matrices and the use of C^*-algebra techniques in the pseudospectral business, and thus on areas that have been actively enriched by work of mathematicians from Chemnitz. In addition, we consider so-called structured pseudospectra, which can be used to tackle questions of control theory or of the Hatano-Nelson model.

Pseudospectra encode information about an operator in a visual manner and are hence easily comprehensible to our eyes. It is therefore no question that the lectures will also be feast for the eyes, with numerous beautiful and partially even bizzare pictures.

Amount: 3 lectures of 45 minutes (A. Böttcher) and probably 2 tutorials of 45 minutes (so far n.n.)