Fast Fourier transform at nonequispaced knots and applications

In this series of lectures we give a unified approach to the *fast Fourier transform at nonequispaced knots* (NFFT).

Let $I_N^d := \left\{ \boldsymbol{k} \in \mathbb{Z}^d : -\frac{N}{2} \leq \boldsymbol{k} < \frac{N}{2} \right\}$ and $\mathbb{T}^d := \left\{ \boldsymbol{v} \in \mathbb{R}^d : -\frac{1}{2} \leq \boldsymbol{v} < \frac{1}{2} \right\}$, where the inequalities hold componentwise. Our concern is the fast evaluation of

$$f_j = f\left(\boldsymbol{v}_j\right) := \sum_{\boldsymbol{k} \in I_N^d} \hat{f}_{\boldsymbol{k}} \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{v}_j} \quad \left(j \in I_M^1\right) \tag{1}$$

and

$$\hat{h}_{\boldsymbol{k}} := \sum_{j \in I_M^1} f_j \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{v}_j} \quad \left(\boldsymbol{k} \in I_N^d\right) \tag{2}$$

for arbitrary knots $\boldsymbol{v}_j \in \mathbb{T}^d$. Direct computation of (1) or (2) has computational complexity $\mathcal{O}\left(N^d M\right)$. In matrix–vector notation (2) is the *adjoint* version of (1). For $N^d = M$ and equispaced knots $\boldsymbol{v}_{\boldsymbol{j}} = \frac{1}{N}\boldsymbol{j}$ $\left(\boldsymbol{j} \in I_N^d\right)$ the computation of (1) is known as (multivariate) discrete Fourier transform (DFT). In this special case, the input data $\hat{f}_{\boldsymbol{k}}$ are called discrete Fourier coefficients and the samples f_j can be computed by the well–known fast Fourier transform (FFT) with only $\mathcal{O}\left(N^d \log N\right)$ arithmetic operations. Furthermore, we have $\hat{h}_{\boldsymbol{k}} = N^d \hat{f}_{\boldsymbol{k}}$ $\left(\boldsymbol{k} \in I_N^d\right)$.

Our generalisation of the FFT is an approximative algorithm and has computational complexity $\mathcal{O}\left(N^d \log N + \log^d\left(1/\varepsilon\right) M\right)$, where ε denotes the desired accuracy. The main idea is to use standard FFTs and a *window function* φ which is well localised in frequency and time domain. In particular, we are interested in the approximation error as function of the arithmetic complexity of the algorithm. We discuss different window functions φ such as *Gaussian*, *B–splines*, *powers of Sinc functions* and *Kaiser–Bessel functions.*

In the application part we discuss
– the iterative reconstruction of band-limited functions,
– the fast summation of radial functions and
– the inversion of the 2D-Radon-transform
– applications in computerised tomographie and in magnetic resonance imaging.

The exercises is focused on the application of the NFFT-Software library. This is a C subroutine library for computing the Nonequispaced Discrete Fourier Transform in one, two or three dimensions, of complex data, and of arbitrary input size. Implementation details and the programming interface will be discussed. Our library is free software and based on the FFTW.
Amount: ?? lectures (D. Potts),
?? exercises (S. Kunis and D. Potts) (each 1.5 hour)