Cordial Volterra integral operators

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Let $\varphi \in L^1(0,1)$. Consider the Volterra integral operator

$$(V_\varphi u)(t) = \int_0^t t^{-1}\varphi(t^{-1}s)u(s)ds = \int_0^1 \varphi(x)u(tx)dx, \;\; 0 \le t \le T.$$

We call V_φ a *cordial operator* and φ its core. The class of cordial operators corresponding to all $\varphi \in L^1(0,1)$ form a commutative Banach algebra with $V_\varphi V_\psi = V_{\varphi\star\psi}$,

$$(\varphi \star \psi)(s) = \int_s^1 t^{-1}\varphi(t)\psi(t^{-1}s)dt, \;\; 0 < s < 1,$$

$$\varphi \star \psi = \psi \star \varphi, \;\; \| \varphi \star \psi \|_{L^1(0,1)} \le \| \varphi \|_{L^1(0,1)} \| \psi \|_{L^1(0,1)} .$$

This enables to establish the formulae for the spectrum $\sigma_m(V_\varphi)$ of V_φ as a bounded (but noncompact) operator in the space $C^m[0,T]$:

$$\sigma_0(V_\varphi) = \{0\} \cup \{\hat{\varphi}(\lambda) : \; \lambda \in \mathbb{C}, \, \mathrm{Re}\lambda \ge 0\}, \;\; \hat{\varphi}(\lambda) := \int_0^1 \varphi(s)s^\lambda ds,$$

$$\sigma_m(V_\varphi) = \{0\} \cup \{\hat{\varphi}(k) : \; k = 0, ..., m-1\} \cup \{\hat{\varphi}(\lambda) : \; \mathrm{Re}\lambda \ge m\}, \;\; m \ge 1.$$

As examples, we localise the spectra of Diogo's, Lighthill's and of some other noncompact Volterra integral operators occuring in literature.

We also consider Volterra integral operators of a more general form

$$(V_{\varphi,a} u)(t) = \int_0^t t^{-1}\varphi(t^{-1}s)a(t,s)u(s)ds, \;\; 0 \le t \le T,$$

where $\varphi \in L^1(0,1)$, $a \in C^m(\Delta_T)$, $\Delta_T = \{(t,s) : 0 \le s \le t \le T\}$, $m \ge 0$. It occurs that $V_{\varphi,a} : C^m[0,T] \to C^m[0,T]$ is bounded and $V_{\varphi,a-a(0,0)} : C^m[0,T] \to C^m[0,T]$ is compact. This enables to establish the formula

$$\sigma_m(V_{\varphi,a}) = a(0,0)\sigma_m(V_\varphi), \;\; m \ge 0.$$

We examine the convergence of the polynomial collocation method and its discrete versions for the Volterra integral equation $\mu u = V_{\varphi,a}u + f$ assuming that $\mu \notin \sigma_0(V_{\varphi,a})$, i.e., $\mu \ne 0$, $\mu \ne a(0,0)\hat{\varphi}(\lambda)$ for $\mathrm{Re}\lambda \ge 0$. Then for $\varphi \in L^1(0,1)$, $a \in C^m(\Delta_T)$, $f \in C^m[0,T]$, also the solution of the equation belongs to $C^m[0,T]$. In the algorithms and convergence analysis, we make use of the fact that $V_\varphi u_k = \hat{\varphi}(k)u_k$ for $u_k(t) = t^k$, $k = 0, 1, ...$ (the cordial operators have even the property that $V_\varphi u_\lambda = \hat{\varphi}(\lambda)u_\lambda$ for $u_\lambda(t) = t^\lambda$ and $\mathrm{Re}\lambda \ge 0$).