# Spectral approximations and spectral inclusion sets for non-self-adjoint tridiagonal and band-dominated operators

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Let $A : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be a tridiagonal operator given by

$$(Ax)_j = \alpha_{j-1}x_{j-1} + \beta_j x_j + \gamma_{j+1}x_{j+1}, \quad j \in \mathbb{Z},$$

where $\alpha = (\alpha_i)$, $\beta = (\beta_i)$, and $\gamma = (\gamma_i)$ are bounded sequences of complex numbers. Let $P_{n,k} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the projection operator given by

$$(P_{n,k}x)_j = \begin{cases} x_j, & j = k+1, k+2, \dots, k+n, \\ 0, & \text{otherwise}, \end{cases}$$

and let $X_{n,k} := P_{n,k}(\ell^2(\mathbb{Z}))$ be the $n$-dimensional range of $P_{n,k}$.

We first construct new inclusion sets for the spectrum and pseudospectrum of $A$ from those of the finite section operators $A_{n,k} : X_{n,k} \to X_{n,k}$ defined by $A_{n,k} := P_{n,k}A|_{X_{n,k}}$, obtaining results reminiscent of the Giršgorin theorem and its generalisations [1]. Where spec denotes the spectrum and $\text{spec}_\epsilon$ the $\epsilon$-pseudospectrum, and defining

$$\Sigma_{n,\epsilon} := \bigcup_{k \in \mathbb{Z}} \text{spec}_\epsilon A_{n,k},$$

we show that, for $n \in \mathbb{N}$ and $\epsilon > 0$,

$$\text{spec}\, A \subset \overline{\Sigma_{n,\epsilon_n}}, \quad \text{spec}_\epsilon A \subset \Sigma_{n,\epsilon+\epsilon_n},$$

where $\epsilon_n$ is given explicitly as the solution of a nonlinear equation, with $\epsilon_n < \eta_n := 2(\|\alpha\|_\infty + \|\gamma\|_\infty)\sin(\pi/(2n+2))$. In general $\overline{\Sigma_{n,\epsilon_n}}$ may be much larger than $\text{spec}\, A$ and does not converge to $\text{spec}\, A$ as $n \to \infty$, but in some cases $\overline{\Sigma_{n,\epsilon_n}} = \text{spec}\, A$ for all $n$.

Our second result modifies these constructions, with something of the flavours of [2, 3]. For $n \in \mathbb{N}$, $\eta > 0$, and $\lambda \in \mathbb{C}$ let

$$B^+_{n,k}(\lambda) := P_{n,k}(A - \lambda I)^*(A - \lambda I)\big|_{X_{n,k}}, \quad B^-_{n,k}(\lambda) := P_{n,k}(A - \lambda I)(A - \lambda I)^*\big|_{X_{n,k}},$$

and let

$$\Gamma_{n,\eta} := \bigcup_{k \in \mathbb{Z}} \left\{ \lambda \in \mathbb{C} : \min\left(\min \text{spec}\, B^+_{n,k}(\lambda), \min \text{spec}\, B^-_{n,k}(\lambda)\right) < \eta^2 \right\}.$$

Then we show that, for $n \in \mathbb{N}$ and $\epsilon > 0$,

$$\text{spec}\, A \subset \overline{\Gamma_{n,\eta_n}}, \quad \Gamma_{n,\epsilon} \subset \text{spec}_\epsilon A \subset \Gamma_{n,\epsilon+\eta_n},$$

so that $\overline{\Gamma_{n,\eta_n}} \to \text{spec}\, A$ in the Haussdorf metric as $n \to \infty$.

Finally we indicate how these results extend to the cases where $A$ is a band or band-dominated operator on $\ell^2(\mathbb{Z})$ or on $L^2(\mathbb{Z}, Y)$, for some Banach space $Y$.

The talk is based on joint work with Ratchanikorn Chonchaiya (Reading) and Marko Lindner (Chemnitz).

## References

[1] R. S. Varga, *Geršgorin and his Circles*, Springer, 2004.

[2] E. B. Davies, Spectral enclosures and complex resonances for general self-adjoint operators. *LMS J. Comput. Math.* **1** (1998) 42–74.

[3] A. C. Hansen, On the approximation of spectra of linear operators on Hilbert spaces. *J. Funct. Anal.* **254** (2008) 2092–2126.