Computing matrix geometric means

Dario A. Bini
University of Pisa, Italy

Different definitions have been given of the matrix geometric mean

$$G_k = G_k(A_1, \dots, A_k)$$

of k positive definite $n \times n$ matrices A_i, $i = 1, \dots, k$. Ando, Li and Mathias [1] stated a set of properties which a "good" mean should satisfy and gave a recursive definition of mean as the common limit $G_k(A_1, \dots, A_k)$ for $\nu \to \infty$ of the k matrix sequences defined by

$$A_i^{(\nu+1)} = G_{k-1}(A_1^{(\nu)}, \dots, A_{i-1}^{(\nu)}, A_{i+1}^{(\nu)}, \dots, A_k^{(\nu)}), \quad i = 1, \dots, k \tag{1}$$

where $A_i^{(0)} = A_i$ and $G_2(A_1, A_2) = A_1(A_1^{-1}A_2)^{1/2}$. We refer to this mean as the ALM-mean. The linear convergence with rate 1/2 of the sequences (1), proven in [1], makes the computation of G_k quite expensive.

We provide a new definition of matrix geometric mean which satisfies all the properties stated by Ando, Li and Mathias and, likewise the ALM-mean, is expressed in terms of the common limit of k matrix sequences. We prove that our sequences have a cubic convergence. This property leads to a dramatic acceleration, in terms of cpu time, in the concrete applications described in [4].

We provide a geometric interpretation of our definition in terms of Riemannian geometry [2] together with a generalization which leads to a class of means of which the ALM-mean and our mean are specific instances. We address the problem of computing a structured geometric mean in the case where the input matrices belong to a given linear space of structured matrices like that of Toeplitz matrices.

References

[1] T. Ando, C.-K. Li, and R. Mathias. Geometric means. Linear Algebra Appl., 385:305–334, 2004.

[2] R. Bhatia and J. Holbrook. Riemannian geometry and matrix geometric means. Linear Algebra Appl., 413:594–618, 2006.

[3] D. Bini, B. Meini, F. Poloni, An effective matrix geometric mean satisfying the Ando–Li–Mathias properties, Math. Comp. (to appear)

[4] M. Moakher. On the averaging of symmetric positive-definite tensors. J. Elasticity, 82:273–296, 2006.