Lecture Notes

Orthogonal Polynomials

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Remark: The present lecture notes contain only a framework of the contents of the lectures. The lectures themselves present detailed expositions, proofs and examples.

Contents

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Chapter 1

Introduction

1.1 Orthogonality

By $\mathbb{K}[x]$ we denote the linear space of all polynomials of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, \tag{1.1}$$

where $a_k \in \mathbb{K}$ are from a field (for example, $\mathbb{C}$ or $\mathbb{R}$) and $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. hence, the elements of $\mathbb{C}[x]$ are polynomials with complex coefficients, which we also consider as maps $p : \mathbb{R} \longrightarrow \mathbb{C}$, $x \mapsto p(x)$ from the set of real numbers into the set of complex numbers. Thus, the independent variable x sei also eine beliebige reelle Zahl. If a_n in (1.1) is different from zero, then $\deg p = n$ is called **degree of the polynomial** $p(x)$ and a_n the **leading coefficient** of this polynomial. The linear subspace of $\mathbb{C}[x]$ of all polynomials of degree less than n will be denoted by $\mathbb{C}_n[x]$, $n \in \mathbb{N} = \{1, 2, \ldots\}$.

For any $n \in \mathbb{N}$, the system $M_k(x) = x^k$, $k = 0, 1, \ldots, n-1$, forms a basis in $\mathbb{C}_n[x]$, which means that the representation of $p(x)$ from $\mathbb{C}_{n+1}[x]$ in (1.1) is unique. This also means that each finite subsystem of the system $(M_n)_{n=0}^\infty$ is linearly independent, because of which we also call this infinite system linearly independent. As an example, let us define on $\mathbb{C}[x]$ a **scalar product**, also called **inner product** ,

$$\langle p, q \rangle = \int_{-1}^{1} p(x)\overline{q(x)}\, dx, \quad p, q \in \mathbb{C}[x]. \tag{1.2}$$

I we apply the well known Schmidt's orthogonalization method to the system $(M_n)_{n=0}^\infty$, then we get a system $(P_n)_{n=0}^\infty$ of polynomials $P_n(x)$ with the properties $\deg P_n = n$,

$$\text{span}\,\{P_0, \ldots, P_{n-1}\} = \mathbb{C}_n[x], \quad n \in \mathbb{N},$$

and

$$\langle P_k, P_j \rangle = \delta_{jk} = \begin{cases} 0 & : \quad j \neq k \\ 1 & : \quad j = k \end{cases}, \quad j, k \in \mathbb{N}_0. \tag{1.3}$$

With span $\{P_0, \ldots, P_{n-1}\}$ we refer to the set of all linear combinations of the polynomials P_j , $j = 0, 1 \ldots, n-1$. Since this set is equal to span $\{M_0, \ldots, M_{n-1}\}$, it is easily seen that, for $k \neq j$, the orthogonality relations (1.3) are equivalent to

$$\langle M_k, P_n \rangle = 0, \quad k = 0, \ldots, n-1, \; n \in \mathbb{N}. \tag{1.4}$$

Consider the polynomials $f_n(x) = (1-x^2)^n$ und $Q_n(x) = f_n^{(n)}(x)$. Using partial integration leads to

$$\langle M_k, Q_n\rangle = \int_{-1}^{1} x^k \frac{d^n}{dx^n}(1-x^2)^n\,dx = -k\int_{-1}^{1} x^{k-1}\frac{d^{n-1}}{dx^{n-1}}(1-x^2)^n\,dx\,,$$

where $n \in \mathbb{N}$ und $0 < k \le n$. Repeating this yields

$$\langle M_k, Q_n\rangle = (-1)^k k!\int_{-1}^{1} \frac{d^{n-k}}{dx^{n-k}}(1-x^2)^n\,dx = \begin{cases} 0 & : \quad k = 0,\ldots,n-1, \\ (-1)^n n!\,\kappa_n & : \quad k = n, \end{cases}$$

with $\kappa_n = \displaystyle\int_{-1}^{1}(1-x^2)^n\,dx$. Again by partial integration we get, for $n \in \mathbb{N}$,

$$\kappa_n = \kappa_{n-1} - \int_{-1}^{1}\left[(1-x^2)^{n-1}x\right]x\,dx = \kappa_{n-1} - \frac{1}{2n}\,\kappa_n$$

and consequently, since $\kappa_0 = 2$, [1]

$$\kappa_n = \frac{2n}{2n+1}\,\kappa_{n-1} = \ldots = \frac{2^n n!}{(2n+1)!!}\,\kappa_0 = \frac{2^{n+1}n!}{(2n+1)!!}.$$

Now we normalize $Q_n(x)$ in order to satisfy the condition $\langle P_n, P_n\rangle = 1$. To this end, we make the ansatz $P_n(x) = \delta_n Q_n(x)$ and require (for definiteness), that the leading coefficient of $P_n(x)$ is positive. The leading coefficient of $Q_n(x) = \dfrac{d^n}{dx^n}\left[(-1)^n x^{2n} + \ldots + 1\right]$ equals $(-1)^n\,\dfrac{(2n)!}{n!}$. Because of

$$\delta_n^2\int_{-1}^{1}\left[Q_n(x)\right]^2\,dx = \delta_n^2(-1)^n\,\frac{(2n)!}{n!}\int_{-1}^{1} x^n Q_n(x)\,dx = \delta_n^2(2n)!\,\kappa_n = \delta_n^2\,\frac{2\,(2^n n!)^2}{2n+1}$$

we get that, for

$$\delta_n = c_n\,\frac{(-1)^n}{2^n n!} \quad \text{mit} \quad c_n = \sqrt{\frac{2n+1}{2}}\,,$$

the condition $\langle P_n, P_n\rangle = 1$ is fulfilled. Now, for the leading coefficient of the polynomial $P_n(x) = \gamma_n x^n + \ldots$ we get $\gamma_n = \dfrac{c_n}{2^n}\dbinom{2n}{n}$. The poynomial

$$L_n(x) = \frac{(-1)^n}{2^n n!}\,\frac{d^n}{dx^n}\,(1-x^2)^n \tag{1.5}$$

is called nth **Legendre polynomial**. Up to the constant c_n this polynomialis the nth polynomial in **orthonormal system of polynomials** $(P_n)_{n=0}^{\infty}$ w.r.t. the inner product defined in (1.2). The formula (1.5) is due to **Rodrigues**.

The sequence $(P_n)_{n=0}^{\infty}$ of polynomials is a solution of the following **moment problem**:

Given is a sequence $(\mu_n)_{n=0}^{\infty}$ of numbers $\mu_n = \displaystyle\int_{-1}^{1} x^n\,dx = \frac{1-(-1)^{n+1}}{n+1}$. On the linear set $\mathbb{C}[x]$ of polynomials of the form (1.1) define a linear functional $\mathcal{L}$ by

$$\mathcal{L}[p] = a_n\mu_n + a_{n-1}\mu_{n-1} + \ldots + a_1\mu_1 + a_0\mu_0.$$

[1] $(2n+1)!! = 1\cdot 3\cdot \ldots \cdot (2n+1)$

Find a system of polynomials $P_n(x)$ with $\deg P_n = n$, $n \in \mathbb{N}_0$, such that the orthogonality relations

$$\mathcal{L}[P_m P_n] = 0 \quad \text{f"ur} \quad m \neq n$$

are fulfilled. If the polynomials $P_n(x)$ also satisfy

$$\mathcal{L}[P_n^2] \neq 0, \quad n \in \mathbb{N}_0 ,$$

then the sequence $(P_n(x))_{n=0}^{\infty}$ is called **orthogonal polynomial system** (OPS) w.r.t. the **moment functional** $\mathcal{L}$. In the following chapter we will answer the question under which conditions on the moment sequence $(\mu_n)_{n=0}^{\infty}$ such an OPS exists.

Another OPS we can find in the following way:

Using the trigonometric relation

$$2 \cos m\theta \cos n\theta = \cos(m+n)\theta + \cos(m-n)\theta \tag{1.6}$$

one get

$$\int_0^{\pi} \cos m\theta \cos n\theta \, d\theta = \begin{cases} 0 & , \quad m \neq n , \\ \pi & , \quad m = n = 0 , \\ \dfrac{\pi}{2} & , \quad m = n \in \mathbb{N} . \end{cases} \tag{1.7}$$

With $x = \cos\theta$ and $T_n(x) = \cos n\theta$, $n \in \mathbb{N}_0$, from (1.6) for $m = 1$ we obtain

$$2\, x\, T_n(x) = T_{n+1}(x) + T_{n-1}(x) \tag{1.8}$$

or

$$T_{n+1}(x) = 2\, x\, T_n(x) - T_{n-1}(x) , \quad n \in \mathbb{N} \tag{1.9}$$

Since $T_0(x) = 1$ and $T_1(x) = x$ the recursive relation (1.9) shows, that $T_n(x)$ is a polynomial of degree n with leading coefficient 2^{n-1} for $n \in \mathbb{N}$. From (1.7) it follows

$$\int_{-1}^{1} T_m(x) T_n(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & , \quad m \neq n , \\ \pi & , \quad m = n = 0 , \\ \frac{\pi}{2} & , \quad m = n \in \mathbb{N} . \end{cases} \tag{1.10}$$

$T_n(x)$ is called **Chebyshev polynomial of first kind** and of degree n, the function $(1-x^2)^{-\frac{1}{2}}$ **Chebyshec weight of first kind** .

Considering the polynomials $xP_n(x) = M_1(x)P_n(x)$ and using the orthogonality relations (1.4) we can obtain a recursion formula for the polynomials $P_n(x)$ analogous to (1.8). Indeed, we have the representation

$$M_1 P_n = \sum_{k=0}^{n+1} \varepsilon_{nk} P_k ,$$

where $\varepsilon_{nk} = \langle M_1 P_n, P_k \rangle = \langle P_n, M_1 P_k \rangle = 0$ for $k = 0, \ldots, n-2$. Since $x[P_n(x)]^2$ is an odd function, we also get $\varepsilon_{nn} = 0$. For the remaining tw coefficients we obtain $\varepsilon_{n,n+1} = \gamma_n \langle M_{n+1}, P_{n+1} \rangle = \gamma_n \gamma_{n+1}^{-1} \langle P_{n+1}, P_{n+1} \rangle = \gamma_n \gamma_{n+1}^{-1}$ and $\varepsilon_{n,n-1} = \gamma_{n-1} \langle P_n, M_n \rangle = \gamma_{n-1} \gamma_n^{-1}$. Hence,

$$\rho_{n+1} P_{n+1}(x) = x P_n(x) - \rho_n P_{n-1}, \quad n \in \mathbb{N}_0, \tag{1.11}$$

with $\rho_n = \gamma_{n-1}\gamma_n^{-1}$ is proved, where w set $P_{-1}(x) \equiv 0$. We will see that such a three term recurrence relation is a typical property of orthogonal polynomial systems. Moreover, such a relation enables us to compute interesting numerical parameters of orthogonal polynomials. For example, the zeros of orthogonal polynomials are eigenvalues of tridiagonal matrices. Indeed, writing down the relation (1.11) for $n = 0, 1, \ldots, m-1$ in matrix language gives

$$\begin{bmatrix} 0 & \rho_1 & & & 0 \\ \rho_1 & 0 & \rho_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \rho_{m-2} & 0 & \rho_{m-1} \\ 0 & & & \rho_{m-1} & 0 \end{bmatrix} \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{m-2}(x) \\ P_{m-1}(x) \end{bmatrix} = x \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{m-2}(x) \\ P_{m-1}(x) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \rho_m P_m(x) \end{bmatrix}, \tag{1.12}$$

and it is easily seen that a zero of the mth orthogonal polynomial $P_m(x)$ is an eigenvalue of the matrix in (1.12).

Relation (1.8) written in matrix language gives

$$\begin{bmatrix} 0 & 2 & & & 0 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ 0 & & & 1 & 0 \end{bmatrix} \begin{bmatrix} T_0(x) \\ T_1(x) \\ \vdots \\ T_{m-2}(x) \\ T_{m-1}(x) \end{bmatrix} = 2x \begin{bmatrix} T_0(x) \\ T_1(x) \\ \vdots \\ T_{m-2}(x) \\ T_{m-1}(x) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ T_m(x) \end{bmatrix}.$$

hence, the eigenvalues of the $m \times m$ matrix on the left-hand side of this equation are just the numbers

$$2\cos\frac{(2k-1)\pi}{2m}, \quad k = 1, \ldots, m.$$

If we normalize the polynomials $T_n(x)$ by $\widehat{T}_n(x) = \sqrt{\frac{2}{\pi}}T_n(x)$, $n \in \mathbb{N}$ und $\widehat{T}_0(x) = \sqrt{\frac{1}{\pi}}$, relation (1.9) can be written as

$$\begin{aligned} \frac{1}{2}\widehat{T}_{n+1}(x) &= x\widehat{T}_n(x) - \frac{1}{2}\widehat{T}_{n-1}(x), \quad n = 2, 3, \ldots, \\ \frac{1}{2}\widehat{T}_2(x) &= x\widehat{T}_1(x) - \frac{1}{\sqrt{2}}\widehat{T}_0(x), \\ \frac{1}{\sqrt{2}}\widehat{T}_1(x) &= x\widehat{T}_0(x). \end{aligned}$$

Consequently, the numbers $\cos\frac{2k-1}{2m}$, $k = 1, \ldots, m$ are the eigenvalues of the symmetric tridiagonal matrix

$$\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & & & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & & & \frac{1}{2} & 0 \end{bmatrix}$$

associated to these last recursion formulas.

1.2 Generating function

Consider the following function of two variables ($a \neq 0$)

$$G(x,w) = e^{-aw}(1+w)^x = \sum_{m=0}^{\infty} \frac{(-a)^m w^m}{m!} \sum_{n=0}^{\infty} \binom{x}{n} w^n \,. \tag{1.13}$$

The Cauchy product of the two series in (1.13) is equal to

$$G(x,w) = \sum_{n=0}^{\infty} P_n(x) w^n \tag{1.14}$$

with

$$P_n(x) = \sum_{k=0}^{n} \binom{x}{k} \frac{(-a)^{n-k}}{(n-k)!} \,. \tag{1.15}$$

Due to $\binom{x}{k} = \dfrac{x(x-1)\cdots(x-k+1)}{k!}$ the function $P_n(x)$ is a polynomial of degree n. The function $G(x,w)$ is called **generating function** of the polynomials $P_n(x)$, the so-called **Charlier polynomials**. Relation (1.13) implies

$$a^x G(x,v) G(x,w) = e^{-a(v+w)} [a(1+v)(1+w)]^x$$

and, consequently,

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{a^k G(k,v) G(k,w)}{k!} &= e^{-a(v+w)} \sum_{k=0}^{\infty} \frac{[a(1+v)(1+w)]^k}{k!} \\ &= e^{-a(v+w)} e^{a(1+v)(1+w)} = e^a e^{avw} = \sum_{n=0}^{\infty} \frac{e^a a^n (vw)^n}{n!} \,. \end{aligned} \tag{1.16}$$

Using (1.14) we get

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{a^k G(k,v) G(k,w)}{k!} &= \sum_{k=0}^{\infty} \frac{a^k}{k!} \sum_{m,n=0}^{\infty} P_m(k) P_n(k) v^m w^n \\ &= \sum_{m,n=0}^{\infty} \left(\sum_{k=0}^{\infty} P_m(k) P_n(k) \frac{a^k}{k!} \right) v^m w^n \,. \end{aligned} \tag{1.17}$$

A comparison of the coefficients in (1.16) and (1.17) leads to

$$\sum_{k=0}^{\infty} P_m(k) P_n(k) \frac{a^k}{k!} = \begin{cases} 0 & , \quad m \neq n \,, \\ \dfrac{e^a a^n}{n!} & , \quad m = n \,. \end{cases} \tag{1.18}$$

If we define

$$\mu_n = \mathcal{L}[M_n] = \sum_{k=0}^{\infty} k^n \frac{a^k}{k!} \,,$$

then, due to (1.18), it follows

$$\mathcal{L}[P_m P_n] = \frac{e^a a^n}{n!} \delta_{mn} \,, \quad \delta_{mn} := \begin{cases} 0 & , \quad m \neq n \,, \\ 1 & , \quad m = n \,. \end{cases}$$

1.3 Birth and death process

We model a birth and death process (a special Markov process the state space of which is the set $\mathbb{N}_0$ of the nonnegative integers). By $p_{mn}(t)$, $m,n \in \mathbb{N}_0$, $t \geq 0$, we denote the so called transition probabilities. This means that $p_{mn}(t)$ is the probability that the system (for example, the size of a population) changes from the state m to the state n during the time period t. Thge matrix $P(t) = [p_{mn}(t)]_{m,n=0}^{\infty}$ is called transition matrix. Here, $p_{mn}(t)$ really depends only on m, n, t and does not depend on how the system reached the stae m (stationary process). This is equivalent to the relation

$$P(s+t) = P(s)P(t). \tag{1.19}$$

For $t \longrightarrow +0$ we assume that the transition probabilies satisfy [2]

$$p_{mn}(t) = \begin{cases} \lambda_m t + o(t) & : \quad n = m+1, \\ 1 - \lambda_m t - \eta_m t + o(t) & : \quad n = m, \\ \eta_m t + o(t) & : \quad n = m-1 \end{cases} \tag{1.20}$$

and

$$\sum_{n=0}^{m-2} p_{mn}(t) + \sum_{n=m+2}^{\infty} p_{mn}(t) = o(t). \tag{1.21}$$

The coefficients λ_m and η_m are called birth and death rate at state m, respectively, and are assumed to have the properties

$$\lambda_m > 0,\ \eta_{m+1} > 0,\ m \in \mathbb{N}_0,\ \eta_0 \geq 0. \tag{1.22}$$

Now, let $\Delta t > 0$. The conditon (1.19) leads to $P(t+\Delta t) = P(t)P(\Delta t)$, i.e., in view of (1.20),

$$\begin{aligned} p_{mn}(t+\Delta t) &= \sum_{k=0}^{\infty} p_{mk}(t)p_{kn}(\Delta t) \\ &= p_{m,n-1}(t)\lambda_{n-1}\Delta t + p_{m,n+1}(t)\eta_{n+1}\Delta t \\ &\qquad + p_{mn}(t)[1 - (\lambda_n + \eta_n)\Delta t] + o(\Delta t). \end{aligned}$$

(Quatities with negative indizees are asumed to be zero.) It follows

$$\frac{p_{mn}(t+\Delta t) - p_{mn}(t)}{\Delta t} = \lambda_{n-1}p_{m,n-1}(t) + \eta_{n+1}p_{m,n+1}(t)$$

$$-(\lambda_n + \eta_n)p_{mn}(t) + \frac{o(\Delta t)}{\Delta t},$$

and, for $\Delta t \longrightarrow +0$,

$$p'_{mn}(t) = \lambda_{n-1}p_{m,n-1}(t) + \eta_{n+1}p_{m,n+1}(t) - (\lambda_n + \eta_n)p_{mn}(t)\,.$$

Analogously, using the relation $P(t+\Delta t) = P(\Delta t)P(t)$ leads to

$$p'_{mn}(t) = \lambda_m p_{m+1,n}(t) + \eta_m p_{m-1,n}(t) - (\lambda_m + \eta_m)p_{mn}(t).$$

[2]By $o(t)$ we denote quatities satisfying $\lim_{t\to+0} o(t)/t = 0$

the last two equations ar called **backward** and **forward Chapman-Kolmogorov equations**, respectively. Taking the separarion ansatz $p_{mn}(t) = f(t)Q_m F_n$, from the backward Chapman-Kolmogorov equations we get

$$\frac{f'(t)}{f(t)} = \frac{\lambda_{n-1}F_{n-1} + \eta_{n+1}F_{n+1} - (\lambda_n + \eta_n)F_n}{F_n} =: -x.$$

hence, up to a multiplicative constant $f(t) = e^{-x\,t}$. Obviously, the F_n's depend on x and we write $F_n(x)$. With the agreement $F_{-1}(x) \equiv 0$ we get

$$\eta_{n+1}F_{n+1}(x) = (\lambda_n + \eta_n - x)F_n(x) - \lambda_{n-1}F_{n-1}(x), \tag{1.23}$$

$n \in \mathbb{N}_0$. The function $F_0(x)$ can be chosen arbitrarily. Hence, the functions $F_n(x)$ are uniquely determined by, for example, the initial conditions $F_{-1}(x) \equiv 0$, $F_0(x) \equiv 1$. Analogously, the functions $Q_n(x)$ are determined uniquely by the initial consditions $Q_{-1}(x) \equiv 0$, $Q_0(x) \equiv 1$ and by the recursive relation

$$\lambda_n Q_{n+1}(x) = (\lambda_n + \eta_n - x)Q_n(x) - \eta_n Q_{n-1}(x),$$

$n \in \mathbb{N}_0$. The polynomial system $(\varepsilon_n Q_n)_{n=0}^{\infty}$ with

$$\varepsilon_0 = 1 \quad \text{und} \quad \varepsilon_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\eta_1 \cdots \eta_n}, \; n \in \mathbb{N},$$

satisfies the same initial conditions and recurrence relation as the polynomial system $(F_n)_{n=0}^{\infty}$. Thus, our separation ansatz leads to

$$p_{mn}(t) = \frac{1}{\varepsilon_m}\, e^{-x\,t} F_m(x) F_n(x),$$

where $x \geq 0$ in order to guarantee that $p_{mn}(t)$ remains bounded for $t \longrightarrow +\infty$. Remark that we again arrive at a system of polynomials which fulfills a recurrence relation analogous to (1.11) (cf. (1.23)). We will see that such a recursion formula implies the existence of a moment sequence $(\mu_n)_{n=0}^{\infty}$, w.r.t. which the system $(F_n)_{n=0}^{\infty}$ is an OPS, and that a respective distribution function $\Omega(x)$ with the property

$$\mu_n = \int_0^{\infty} x^n \, d\Omega(x), \quad n \in \mathbb{N}_0$$

exists.

1.4 Exercises

1. Define[3]

$$\widetilde{T}_n(x) = \frac{(-1)^n\sqrt{1-x^2}}{(2n-1)!!}\,\mathbf{D}^n(1-x^2)^{n-\frac{1}{2}}, \; n \in \mathbb{N}\,, \; \mathbf{D} = \frac{d}{dx}\,, \; \widetilde{T}_0(x)01\,.$$

   (a) Prove that

$$T_n(x) = \widetilde{T}_n(x)\,, \; n \in \mathbb{N}_0\,,$$

   holds true.

   (b) Use this representation of $T_n(x)$ to prove the respective orthogonality relations (1.10).

[3] $(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$

2. The **Chebyshev polynomials** $U_n(x)$ **of second kind** can be defined by

$$U_n(x) = \frac{\sin[(n+1)\vartheta]}{\sin\vartheta}\,,\ x = \cos\vartheta, \quad n \in \mathbb{N}_0\,.$$

   (a) Show that $U_n(x)$ is a polynomial of degree n and that

$$\int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}\,dx = \frac{\pi}{2}\delta_{mn}\,. \tag{1.24}$$

   (b) Prove the formula

$$U_n(x) = \frac{(-1)^n(n+1)}{(2n+1)!!\,\sqrt{1-x^2}}\,\mathbf{D}^n(1-x^2)^{n+\frac{1}{2}}\,, \quad n \in \mathbb{N}_0\,,$$

   and use this formula to prove the orthogonality relations (1.24).

3. For $x = \cos\vartheta$, define

$$R_n(x) = \frac{\cos[(n+\frac{1}{2})\vartheta]}{\cos\frac{\vartheta}{2}}\,, \quad n \in \mathbb{N}_0\,.$$

   Prove

   (a) $R_n(x) = \dfrac{T_n(x) + T_{n+1}(x)}{1+x},\ n \in \mathbb{N}_0\,,$

   (b) $R_0(x) = 1\,,\ R_1(x) = 2x - 1\,,$
   $R_{n+1}(x) = 2xR_n(x) - R_{n-1}(x)\,,\ n \in \mathbb{N}\,,$

   (c) $\displaystyle\int\limits_{-1}^{1} R_m(x)R_n(x)\sqrt{\frac{1+x}{1-x}}\,dx = \pi\delta_{mn}\,.$

4. Let $F(x,w) = e^{-(x-w)^2}$. Show that

   (a) $H_n(x) = e^{x^2}\dfrac{\partial^n}{\partial w^n}F(x,0) = e^{x^2}(-1)^n\mathbf{D}^n e^{-x^2}$
   is a polynomial of degree n,

   (b) $G(x,w) := e^{2xw-w^2} = \displaystyle\sum_{n=0}^{\infty} H_n(x)\frac{w^n}{n!}\,,$

   (c) $\displaystyle\int_{-\infty}^{+\infty} G(x,v)G(x,w)e^{-x^2}\,dx = \sqrt{\pi}\,e^{2vw}\,,\ \Big(\text{Hint: } \int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}\,,\Big)$

   (d) $\displaystyle\int_{-\infty}^{+\infty} H_m(x)H_n(x)e^{-x^2}\,dx = \sqrt{\pi}\,2^n n!\,\delta_{mn}$ .right)

   $H_n(x)$ is the so-called **Hermite polynomial** of degree n.

Chapter 2

Basic Theory of Orthogonal Polynomials

2.1 Moment functionals. Existence of orthogonal systems of polynomials

Definition 2.1 *For a given sequence* $(\mu_n)_{n=0}^{\infty} \subset \mathbb{C}$ *of numbers we define the respective* **moment functional** $\mathcal{L}$ *on the linear space* $\mathbb{C}[x]$ *of all (algebraic) polynomials in* x *by*

$$\mathcal{L}[M_n] = \mu_n\,, \quad n \in \mathbb{N}_0\,,\, M_n(x) = x^n\,,$$

and

$$\mathcal{L}[\alpha_1\pi_1 + \alpha_2\pi_2] = \alpha_1\mathcal{L}[\pi_1] + \alpha_2\mathcal{L}[\pi_2]\,, \quad \alpha_j \in \mathbb{C}\,,\; \pi_j \in \mathbb{C}[x]\,.$$

The number μ_n *is called* **moment** *of* n*th*-**order**.

The coefficients of the polynomials are in general complex numbers, but the independent variable will be considered here as real.

Definition 2.2 *A sequence* $(P_n)_{n=0}^{\infty} \subset \mathbb{C}[x]$ *is called* **orthogonal polynomial system** *(OPS) w.r.t. the moment functional* $\mathcal{L}$, *if* $\forall\, m, n \in \mathbb{N}_0$ *the following conditions are satisfied:*

1. $\deg P_n = n\,,$
2. $\mathcal{L}[P_m P_n] = k_n \delta_{mn}\,, \quad k_n \neq 0\,.$

In case $k_n = 1$, $n \in \mathbb{N}_0$, *the system* $(P_n)_{n=0}^{\infty}$ *is a* **orthonormal** *polynomial system (ONPS).*

It is easy to see that an OPS does not exist for all sequences $(\mu_n)_{n=0}^{\infty} \subset \mathbb{C}$, for example if $\mu_0 = 0$. But, for example, also in case $\mu_0 = \mu_1 = \mu_2 = 1$ there does not exist an OPS, since otherwise for $P_0(x) = a$ and $P_1(x) = bx + c$ we have

$$0 = \mathcal{L}[P_0 P_1] = a(b + c)\,,\; \text{d.h. } c = -b\,,$$

which implies

$$\mathcal{L}[P_1^2] = \mathcal{L}[b^2(x-1)^2] = b^2(\mu_2 - 2\mu_1 + \mu_0) = 0\,.$$

Proposition 2.3 *Let $\mathcal{L}$ be a moment functional and $(P_n)_{n=0}^{\infty} \subset \mathbb{C}[x]$ be a sequence of polynomials with* $\deg P_n = n$ *. Then the following statements are equivalent:*

(a) $(P_n)_{n=0}^{\infty}$ *is an OPS w.r.t.* $\mathcal{L}$ *.*

(b) For all $n \in \mathbb{N}_0$ *,*

$$\mathcal{L}[\pi P_n] = 0 \quad \forall\, \pi \in \mathbb{C}_n[x]$$

and

$$\mathcal{L}[\pi P_n] \neq 0 \quad \forall \pi \in \mathbb{C}[x] \text{ with } \deg \pi = n\,.$$

(c) For all $n \in \mathbb{N}_0$ *,*

$$\mathcal{L}[M_m P_n] = \widetilde{k}_n \delta_{mn}\,, \quad m = 0, 1, \ldots, n\,,\ \widetilde{k}_n \neq 0\,.$$

Remark 2.4 *Let* $(P_n)_{n=0}^{\infty}$ *be an OPS w.r.t. the moment functional* $\mathcal{L}$ *.*

(a) If $\pi = \sum_{k=0}^{n} \gamma_k P_k \in \mathbb{C}[x]$ *then* $\gamma_k = \dfrac{\mathcal{L}[\pi P_k]}{\mathcal{L}[P_k^2]}$, $k = 0, 1, \ldots, n$.

(b) If $(Q_n)_{n=0}^{\infty}$ *is a further OPS w.r.t.* $\mathcal{L}$ *, then, due to (a) and Prop. 2.3,(b),*

$$Q_n = \delta_n P_n\,,\ \delta_n \neq 0\,, \quad n \in \mathbb{N}_0\,.$$

(c) If all $P_n(x)$ *are monic, i.e.* $P_n(x) = x^n + \cdots$ *, then* $(P_n)_{n=0}^{\infty}$ *is called* **monic** *OPS.*

(d) The polynomial system $(p_n)_{n=0}^{\infty}$ *with*

$$p_n(x) = \left(\mathcal{L}[P_n^2]\right)^{-\frac{1}{2}} P_n(x)\,, \quad n \in \mathbb{N}_0\,,$$

is an ONPS w.r.t. $\mathcal{L}$ *, where by* $\left(\mathcal{L}[P_n^2]\right)^{\frac{1}{2}}$ *we refer to one solution of* $z^2 = \mathcal{L}[P_n^2]$ *.*

For a given moment sequence $(\mu_n)_{n=0}^{\infty}$, we define

$$\Delta_n = \det\left[\ \mu_{j+k}\ \right]_{j,k=0}^{n} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}.$$

Proposition 2.5 *Let* $\mathcal{L}$ *be a moment functional with the moment sequence* $(\mu_n)_{n=0}^{\infty}$ *. Then, there exists an OPS w.r.t.* $\mathcal{L}$ *if and only if*

$$\Delta_n \neq 0 \quad \forall\, n \in \mathbb{N}_0\,.$$

Proof. Let $(P_n)_{n=0}^{\infty}$ be an OPS w.r.t. $\mathcal{L}$, $P_n(x) = \sum_{k=0}^{n} \gamma_{nk} x^k$. Prop. 2.3,(c) leads to

$$\mathcal{L}[M_m P_n] = \sum_{k=0}^{n} \gamma_{nk} \mu_{k+m} = \widetilde{k}_n \delta_{mn}\,,\ m \leq n\,,\ \widetilde{k}_n \neq 0\,,$$

i.e.

$$\begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix} \begin{bmatrix} \gamma_{n0} \\ \gamma_{n1} \\ \vdots \\ \gamma_{n,n-1} \\ \gamma_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \widetilde{k}_n \end{bmatrix} . \tag{2.1}$$

Remark 2.4,(a),(b) implies, that $P_n(x)$ is uniquely determined if $\widetilde{k}_n$ is given, which yields the unique solvability of the linear system (2.1). If, vice versa, $\Delta_n \neq 0$, then (2.1) is uniquely solvable, which means that $P_n(x)$ exists, where

$$\gamma_{nn} = \frac{\widetilde{k}_n \Delta_{n-1}}{\Delta_n} \neq 0 , \quad n \in \mathbb{N}_0 , \tag{2.2}$$

$(\Delta_{-1} := 1)$. □

Let $(P_n)_{n=0}^{\infty}$ be an OPS w.r.t. $\mathcal{L}$ and $\pi_n \in \mathbb{C}[x]$ be a polynomial of degree n with the leading coefficient α_n, i.e. $\pi_n(x) = \alpha_n x^n + \cdots$. Then, due to (2.2),

$$\mathcal{L}[\pi_n P_n] = \alpha_n \widetilde{k}_n = \frac{\alpha_n \gamma_n \Delta_n}{\Delta_{n-1}} , \tag{2.3}$$

where γ_n denotes the leading coefficient of $P_n(x)$.

Definition 2.6 *A moment functional $\mathcal{L}$ is called* **positive definite**, *if $\mathcal{L}[\pi] > 0$ for all poynomials $\pi \in \mathbb{C}[x]$ with $\pi(x) \geq 0$, $x \in \mathbb{R}$, and $\pi(x) \not\equiv 0$.*

If $\mathcal{L}$ is positive definite, then $\mu_{2n} = \mathcal{L}[M_{2n}] > 0$, $n \in \mathbb{N}_0$, and, since

$$0 < \mathcal{L}[(x+1)^{2n}] = \sum_{k=0}^{2n} \binom{2n}{k} \mu_k ,$$

also $\mu_{2n-1} \in \mathbb{R}$, $n \in \mathbb{N}$. Moreover, by $\langle p, q \rangle := \mathcal{L}[p\overline{q}]$ there is defined a (positive definite) inner product on $\mathbb{C}[x]$. Schmidt's orthogonalization procedure applied to the system $\{M_0, M_1, M_2, \ldots\}$ yields an ONPS $(p_n)_{n=0}^{\infty}$ with $p_n \in \mathbb{R}[x]$.

Lemma 2.7 *Let $\pi \in \mathbb{C}[x]$. Then $\pi(x) \geq 0$, $x \in \mathbb{R}$, if and only if there exist polynomials $p, q \in \mathbb{R}[x]$ such that $\pi(x) = [p(x)]^2 + [q(x)]^2$.*

Proposition 2.8 *A moment functional $\mathcal{L}$ is positive definite, if and only if all moments are real and $\Delta_n > 0$ for all $n \in \mathbb{N}_0$.*

Corollary 2.9 *The proof of Prop. 2.8 shows, that a moment functional $\mathcal{L}$ is positive definite if an OPS $(P_n)_{n=0}^{\infty} \subset \mathbb{R}[x]$ exists with $\mathcal{L}[P_n^2] > 0$, $n \in \mathbb{N}_0$.*

Definition 2.10 *A moment functional $\mathcal{L}$ is called* **quasi-definite**, *if $\Delta_n \neq 0$, $n \in \mathbb{N}_0$.*

2.2 Exercises

1. Assume that $\mathcal{L}[M_n] = a^n$, $n \geq 0$, where $a \in \mathbb{C} \setminus \{0\}$. Show that there exists **no** OPS w.r.t. $\mathcal{L}$.

2. Show that there is **no** moment functional w.r.t. which $(M_n)_{n=0}^{\infty}$ is an OPS.

3. Let $\mathcal{L}$ be a moment functional, for which an OPS exists and let $\{c_n\}$ be a sequence of numbers different from 0 . Show that each of the following conditions defines uniquely an OPS $(P_n(x))_{n=0}^{\infty}$ w.r.t. $\mathcal{L}$:

   (a) $\mathcal{L}[M_n P_n] = c_n$, $n \in \mathbb{N}_0$,

   (b) $\lim\limits_{x \to 0} x^n P_n(1/x) = c_n$, $n \in \mathbb{N}_0$.

4. Use known OPS's to determine polynomial systems $(P_n(x))_{n=0}^{\infty}$ satisfying the following conditions:

   (a) $\displaystyle\int_0^1 P_m(x)\, P_n(x)\, (1-x)^{-\frac{1}{2}} x^{-\frac{1}{2}} dx = k_n \delta_{mn}$, $k_0 = \pi$, $k_n = \dfrac{\pi}{2}$, $n \in \mathbb{N}_0$,

   (b) $\displaystyle\int_{-\infty}^{+\infty} P_m(x)\, P_n(x)\, e^{-x^2/2} dx = \sqrt{2\pi}\, n!\, \delta_{mn}$,

   (c) $\displaystyle\int_0^1 P_m(x)\, P_n(x)\, dx = \frac{1}{2n+1}\, \delta_{mn}$.

5. Let $\mathcal{L}$ be a quasi-definite moment functional with the moment sequence $\{\mu_n\}$. Prove that $\{(\Delta_{n-1})^{-1} D_n(x)\}$ with

$$D_n(x) = \det \begin{bmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-1} \\ 1 & x & \dots & x^n \end{bmatrix}$$

   is the monic OPS w.r.t. $\mathcal{L}$. Determine an ONPS w.r.t. $\mathcal{L}$.

6. Let $\mathcal{L}$ be a quasi-definite moment functional. Prove: If $\deg \pi_n = n$, $n \in \mathbb{N}_0$, and $\mathcal{L}[\pi_m \pi_n] = 0$ for $m \neq n$, then $(\pi_n)_{n=0}^{\infty}$ is an OPS w.r.t. $\mathcal{L}$.

7. Let $\mathcal{L}$ be a quasi-definite moment functional. Show that $\pi \in \mathbb{C}[x]$ and $\mathcal{L}[\pi\, M_n] = 0$, $n \in \mathbb{N}_0$, implies $\pi(x) \equiv 0$.

8. Let $\mathcal{L}$ be positive definite and $(P_n)_{n=0}^{\infty}$ the monic OPS w.r.t. $\mathcal{L}$. Prove that the inequality $\mathcal{L}[P_n^2] < \mathcal{L}[|\pi|^2]$ holds for all monic polynomials $\pi(x) \not\equiv P_n(x)$ with $\deg \pi(x) = n$.

2.3 Recursion formulas and the formula of Christoffel-Darboux

Proposition 2.11 *If the moment functional $\mathcal{L}$ is quasi-definite and $(P_n)_{n=0}^{\infty}$ is the respective monic OPS, then there exists numbers $\alpha_n, \beta_n \in \mathbb{C}$ with $\beta_n \neq 0$ and*

$$P_{n+1}(x) = (x - \alpha_n) P_n(x) - \beta_n P_{n-1}(x) , \quad n \in \mathbb{N}_0 , \tag{2.4}$$

where $P_{-1}(x) \equiv 0$, $P_0(x) \equiv 1$ and β_0 is arbitrary. In case of a positive definite moment functional $\mathcal{L}$ we have $\alpha_n \in \mathbb{R}$ and $\beta_n > 0$.

Corollary 2.12 *From (2.4) and from the proof of Prop. 2.11 we get the following relations:*

1. *It holds*
$$\beta_n = \frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2} = \frac{\mathcal{L}[P_n^2]}{\mathcal{L}[P_{n-1}^2]}\,, \quad n \in \mathbb{N}\,.$$

2. *If we agree that* $\beta_0 = \mu_0 = \Delta_0$ *, then*
$$\mathcal{L}[P_n^2] = \beta_0\beta_1\cdots\beta_n\,, \quad n \in \mathbb{N}_0\,.$$

3. *We have*
$$\alpha_n = \frac{\mathcal{L}[M_1 P_n^2]}{\mathcal{L}[P_n^2]}\,, \quad n \in \mathbb{N}_0\,.$$

4. *The* n*th monic orthogonal polynomial admits the representation*
$$P_n(x) = x^n - (\alpha_0 + \alpha_1 + \ldots + \alpha_{n-1})x^{n-1} + \ldots\,,$$
since, for $P_{n+1}(x) = x^{n+1} + d_n x^n + \ldots$ *, we get* $d_n = d_{n-1} - \alpha_n$ *from (2.4).*

Example 2.13 *The monic OPS* $\left(\widehat{T}_n(x)\right)_{n=0}^{\infty}$ *w.r.t. the Chebyshev weight of first kind is given by*
$$\widehat{T}_0(x) = T_0(x) \quad \textit{und} \quad \widehat{T}_n(x) = 2^{1-n}T_n(x)\,,\ n \in \mathbb{N}\,.$$
Using (1.9) we get
$$\widehat{T}_1(x) = x\,\widehat{T}_0(x)\,,\ \widehat{T}_2(x) = x\,\widehat{T}_1(x) - \frac{1}{2}\,\widehat{T}_0(x)$$
and
$$\widehat{T}_{n+1}(x) = x\,\widehat{T}_n(x) - \frac{1}{4}\,\widehat{T}_{n-1}(x)\,, \quad n = 2, 3, \ldots\,,$$
so that in this case $\alpha_n = 0$ *for* $n \geq 0$ *and* $\beta_1 = \frac{1}{2}$ *as well as* $\beta_n = \frac{1}{4}$ *for* $n \geq 2$ *.*

Proposition 2.14 (Favard/Shohat/Natanson) *For arbitrary sequences* $(\alpha_n)_{n=0}^{\infty}$ *,* $(\beta_n)_{n=0}^{\infty}$ *of complex numbers and a polynomial system* $\{P_n\}_{n=-1}^{\infty}$ *with* $P_{-1}(x) \equiv 0$ *and* $P_0(x) \equiv 1$ *, satisfying (2.4), there exists exactly one moment functional* $\mathcal{L}$ *with the properties*
$$\mathcal{L}[P_0] = \beta_0 \quad \textit{und} \quad \mathcal{L}[P_m P_n] = 0\,,\ m \neq n\,.$$
This moment functional $\mathcal{L}$ *is quasi-definite if and only if* $\beta_n \neq 0$ *for all* $n \in \mathbb{N}_0$ *and positive definite if and only if* $(\alpha_n)_{n=0}^{\infty} \subset \mathbb{R}$ *and* $\beta_n > 0$ *for all* $n \in \mathbb{N}_0$ *.*

Proposition 2.15 (Christoffel/Darboux) *If the polynomial system* $\{P_n\}_{n=-1}^{\infty}$ *satisfies the recursion formula (2.4) with* $\beta_n \neq 0$ *,* $n \in \mathbb{N}_0$ *, then*
$$\sum_{k=0}^{n} \frac{P_k(x)P_k(t)}{\beta_0\cdots\beta_k} = \frac{1}{\beta_0\cdots\beta_n}\,\frac{P_{n+1}(x)P_n(t) - P_n(x)P_{n+1}(t)}{x-t}\,. \tag{2.5}$$

From a monic OPS $(P_n(x))_{n=0}^{\infty}$ w.r.t. a positive definite moment functional we get a respective ONPS $(\widetilde{p}_n(x))_{n=0}^{\infty}$ via the formula

$$\widetilde{p}_n(x) = k_n P_n(x)\,, \quad k_n = (\beta_0 \cdots \beta_n)^{-\frac{1}{2}}\,.$$

From (2.4) it follows

$$\sqrt{\beta_{n+1}}\widetilde{p}_{n+1}(x) = (x-\alpha_n)\widetilde{p}_n(x) - \sqrt{\beta_n}\widetilde{p}_{n-1}(x)\,, \quad n \in \mathbb{N}_0 \tag{2.6}$$

and by (2.5)

$$\sum_{k=0}^{n} \widetilde{p}_k(x)\widetilde{p}_k(t) = \frac{k_n}{k_{n+1}} \frac{\widetilde{p}_{n+1}(x)\widetilde{p}_n(t) - \widetilde{p}_n(x)\widetilde{p}_{n+1}(t)}{x-t}\,. \tag{2.7}$$

Moreover, from (2.5) we get for $t \longrightarrow x$

$$\sum_{k=0}^{n} \frac{[P_k(x)]^2}{\beta_0 \cdots \beta_k} = \frac{P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x)}{\beta_0 \cdots \beta_n}\,. \tag{2.8}$$

Consequently, in case of a positive definite moment functional,

$$P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x) > 0\,. \tag{2.9}$$

2.4 On the zeros of orthogonal polynomials. Gaussian quadrature rule

Definition 2.16 *We say that $E \subset \mathbb{R}$ is a* **supporting set** *of $\mathcal{L}$, if $P(x) \geq 0$ on E and $P(x) \not\equiv 0$ on E implies $\mathcal{L}[P] > 0$. In this case $\mathcal{L}$ is called* **positive definite on** E.

In what follows $\mathcal{L}$ is assumed to be positive definite and $(P_n)_{n=0}^{\infty}$ to be the respective monic OPS.

Proposition 2.17 *Let (a,b) be a supporting set of $\mathcal{L}$ and $n \in \mathbb{N}$. Then all zeros of $P_n(x)$ are real, simple and in (a,b).*

Provided that the assumptions of Prop. 2.17 are fulfilled, we denote the zeros of $P_n(x)$ by x_{nk}, where $x_{nn} < x_{n,n-1} < \cdots < x_{n1}$. It follows $\operatorname{sgn} P_n(x) = 1$ for $x > x_{n1}$ and $\operatorname{sgn} P_n(x) = (-1)^n$ for $x < x_{nn}$. The polynomial $P'_n(x)$ possesses exactly one zero in the interval $(x_{nk}, x_{n,k-1})$, $k = 2, \ldots, n$, which implies

$$\operatorname{sgn} P'_n(x_{nk}) = (-1)^{k-1}\,. \tag{2.10}$$

Proposition 2.18 *We have $x_{n+1,k+1} < x_{nk} < x_{n+1,k}$, $k = 1, \ldots, n$.*

Corollary 2.19 *For every $k \geq 1$, the sequence $\{x_{nk}\}_{n=k}^{\infty}$ is monotone increasing and the sequence $\{x_{n,n-k+1}\}_{n=k}^{\infty}$ is monotone decreasing. Consequently, the limits*

$$\xi_k := \lim_{n\to\infty} x_{nk} \quad und \quad \eta_k := \lim_{n\to\infty} x_{n,n-k+1}$$

exist ($\xi_k = +\infty$ or/and $\eta_k = -\infty$ is possible).

Definition 2.20 *The interval* (η_1, ξ_1) *is named* **support** *of the moment functional* $\mathcal{L}$.

By $\ell_{nk}(x)$ we denote the kth **Lagrange fundamental polynomial**

$$\ell_{nk}(x) = \prod_{j=1, j\neq k}^{n} \frac{x - x_{nj}}{x_{nk} - x_{nj}} = \frac{P_n(x)}{(x - x_{nk})P_n'(x_{nk})}, \quad k = 1, \ldots, n.$$

Obviously, $\ell_{nk}(x_{nj}) = \delta_{jk}$. For a continuous function $f : (\eta_1, \xi_1) \longrightarrow \mathbb{C}$, i.e. $f \in \mathbf{C}(\eta_1, \xi_1)$, we define the Lagrange **interpolation polynomial**

$$(L_n f)(x) = \sum_{k=1}^{n} f(x_{nk})\ell_{nk}(x)$$

and the functional, the so-called **Gaussian quadrature rule** ,

$$\mathcal{L}_n[f] = \mathcal{L}[L_n f] = \sum_{k=1}^{n} A_{nk} f(x_{nk}) \quad \text{with} \quad A_{nk} = \mathcal{L}[\ell_{nk}].$$

Proposition 2.21 *We have*

$$\mathcal{L}_n[P] = \mathcal{L}[P] \quad \forall\, P \in \mathbb{C}_{2n}[x].$$

Corollary 2.22 *It holds*

$$\sum_{k=1}^{n} A_{nk} = \mu_0 \quad \textit{und} \quad A_{nk} > 0, \; k = 1, \ldots, n.$$

Corollary 2.23 *By Corollary 2.22 and* $L_n p = p$ *for* $p \in \mathbb{C}[x]$ *and for all* $n \geq n_0(p)$ *we get that* (η_1, ξ_1) *is a supporting set of* $\mathcal{L}$.

Now, in case $-\infty < \eta_1 < \xi_1 < \infty$ we are able to define $\mathcal{L}[f]$ for each $f \in \mathbf{C}[\eta_1, \xi_1]$ via

$$\mathcal{L}[f] = \lim_{n\to\infty} \mathcal{L}[p_n],$$

where $p_n \in \mathbb{C}[x]$ and $\lim_{n\to\infty} \|f - p_n\|_\infty = 0$, $\|f\|_\infty = \|f\|_{\infty,[\eta_1,\xi_1]} = \max\{|f(x)| : \eta_1 \leq x \leq \xi_1\}$. This definition is correct. Moreover, $\mathcal{L}[f_1] \leq \mathcal{L}[f_2]$ if $f_1(x) \leq f_2(x)$, $x \in [\eta_1, \xi_1]$.

Proposition 2.24 *If* $-\infty < \eta_1 < \xi_1 < +\infty$, *then* $\lim_{n\to\infty} \mathcal{L}_n[f] = \mathcal{L}[f] \quad \forall\, f \in \mathbf{C}[\eta_1, \xi_1]$.

Proposition 2.25 *For* $-\infty < \eta_1 < \xi_1 < \infty$ *and* $f \in \mathbf{C}[\eta_1, \xi_1]$, *we have* $\lim_{n\to\infty} \mathcal{L}[|f - L_n f|^2] = 0$.

For $z \in \mathbb{C}$, we define

$$\mathcal{L}_z^*[M_n] = \mathcal{L}[(x - z)x^n] = \mu_{n+1} - z\,\mu_n$$

and

$$P_n^z(x) = \frac{1}{x - z}\left[P_{n+1}(x) - \frac{P_{n+1}(z)}{P_n(z)}\, P_n(x)\right],$$

where we suppose that $P_n(z) \neq 0$ for all $n \geq 1$. Taking into account (2.5) and Corollary 2.12, 2. we conclude

$$P_n^z(x) = \frac{\beta_0\beta_1 \cdots \beta_n}{P_n(z)} \sum_{k=0}^{n} \widetilde{p}_k(x)\widetilde{p}_k(z). \tag{2.11}$$

Proposition 2.26 *The moment functional $\mathcal{L}_z^*$ is quasi-definite and $(P_n^z)_{n=0}^\infty$ is the respective monic OPS. The moment functional $\mathcal{L}_z^*$ is positive definite if and only if $z \le \eta_1$.*

In view of (2.11) it holds

$$K_n(z,x) := \frac{1}{\beta_0\beta_1\cdots\beta_n} P_n(z)P_n^z(x) = \sum_{k=0}^n \widetilde{p}_k(x)\widetilde{p}_k(z)\,, \quad x,z \in \mathbb{R}\,.$$

For $w, z \in \mathbb{C}$, we define

$$K_n(z,w) = \sum_{k=0}^n \widetilde{p}_k(w)\overline{\widetilde{p}_k(z)}\,.$$

Proposition 2.27 *For all $z_0 \in \mathbb{C}$ and $n \in \mathbb{N}_0$,*

$$\frac{1}{K_n(z_0,z_0)} = \min\left\{\mathcal{L}[|\pi|^2] : \pi \in \mathbb{C}_{n+1}[x],\ \pi(z_0) = 1\right\}$$

and

$$A_{nk} = \frac{1}{K_{n-1}(x_{nk},x_{nk})} = \frac{1}{K_n(x_{nk},x_{nk})}\,.$$

2.5 The Jacobi polynomials

Let $\alpha > -1$, $\beta > -1$. For $n \in \mathbb{N}_0$, the **Jacobi polynomials** can be defined by **Rodrigues' formula** (cf. also Section 1.4, Exercise 1)

$$P_n^{\alpha,\beta}(x) = \frac{(1-x)^{-\alpha}(1+x)^{-\beta}}{(-2)^n\, n!}\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]\,. \tag{2.12}$$

Lemma 2.28 *It holds*

$$\sum_{k=0}^n \binom{n+\alpha}{n-k}\binom{n+\beta}{k} = \binom{2n+\alpha+\beta}{n}, \quad n \in \mathbb{N}_0$$

Proof. Using

$$(1+z)^\alpha = \sum_{j=0}^\infty \binom{\alpha}{j} z^j$$

we get

$$\sum_{j=0}^\infty \binom{2n+\alpha+\beta}{j} z^j = (1+z)^{n+\alpha}(1+z)^{n+\beta} = \sum_{j=0}^\infty\sum_{k=0}^j \binom{n+\alpha}{j-k}\binom{n+\beta}{k} z^j\,.$$

Comparing the coefficients at z^n proves the lemma. □

From (2.12) and

$$\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left[\frac{d^{n-k}}{dx^{n-k}} (1-x)^{n+\alpha} \right] \left[\frac{d^k}{dx^k} (1+x)^{n+\beta} \right]$$

$$= (1-x)^{\alpha}(1+x)^{\beta} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (-1)^{n-k} (n+\alpha)\cdots(k+\alpha+1)(1-x)^k \cdot$$

$$\cdot (n+\beta)\cdots(n+\beta-k+1)(1+x)^{n-k}$$

$$= (-1)^n (1-x)^{\alpha}(1+x)^{\beta} n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (x-1)^k (x+1)^{n-k}$$

it follows

$$P_n^{\alpha,\beta}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k}. \tag{2.13}$$

In view of Lemma 2.28, $P_n^{\alpha,\beta}(x)$ has the leading coefficient

$$k_n^{\alpha,\beta} = 2^{-n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} = 2^{-n} \binom{2n+\alpha+\beta}{n}. \tag{2.14}$$

The monic Jacobi polynomials we denote by $\widehat{P}_n^{\alpha,\beta}(x)$, i.e.

$$\widehat{P}_n^{\alpha,\beta}(x) = \frac{1}{k_n^{\alpha,\beta}} P_n^{\alpha,\beta}(x).$$

With the help of partial integration one can prove the following Lemma.

Lemma 2.29 *For $n \in \mathbb{N}_0$,*

$$\int_{-1}^{1} x^k P_n^{\alpha,\beta}(x)(1-x)^{\alpha}(1+x)^{\beta}\, dx \begin{cases} = 0 & , \quad k = 0, \ldots, n-1, \\ > 0 & , \quad k = n. \end{cases}$$

Hence, $\left(P_n^{\alpha,\beta}(x)\right)_{n=0}^{\infty}$ is an OPS. Let us look for formulas of the coefficients α_n and β_n in the recursion formula

$$\widehat{P}_{n+1}^{\alpha,\beta}(x) = (x-\alpha_n)\widehat{P}_n^{\alpha,\beta}(x) - \beta_n \widehat{P}_{n-1}^{\alpha,\beta}(x), \quad n = 0, 1, \ldots \tag{2.15}$$

of the monic Jacobi polynomials. Define

$$(\alpha)_n := \begin{cases} 1 & , \quad n = 0, \\ \prod_{k=1}^{n} (k+\alpha) & , \quad n \in \mathbb{N} \end{cases}$$

Relation (2.13) implies

$$P_n^{\alpha,\beta}(1) = \binom{n+\alpha}{n} \quad \text{und} \quad P_n^{\alpha,\beta}(-1) = (-1)^n \binom{n+\beta}{n},$$

so that

$$\widehat{P}_n^{\alpha,\beta}(1) = \frac{2^n \binom{n+\alpha}{n}}{\binom{2n+\alpha+\beta}{n}} = \frac{2^n (\alpha)_n (\alpha+\beta)_n}{(\alpha+\beta)_{2n}}$$

and

$$\widehat{P}_n^{\alpha,\beta}(-1) = \frac{(-2)^n \binom{n+\beta}{n}}{\binom{2n+\alpha+\beta}{n}} = \frac{(-2)^n (\beta)_n (\alpha+\beta)_n}{(\alpha+\beta)_{2n}}.$$

Consequently,

$$1-\alpha_0 = \widehat{P}_1(1) = \frac{2(1+\alpha)}{2+\alpha+\beta}, \quad \text{d.h.} \quad \alpha_0 = \frac{\beta-\alpha}{2+\alpha+\beta}.$$

For $n \in \mathbb{N}$, we solve the system

$$\begin{aligned} \widehat{P}_{n+1}^{\alpha,\beta}(1) &= (1-\alpha_n)\widehat{P}_n^{\alpha,\beta}(1) - \beta_n \widehat{P}_{n-1}^{\alpha,\beta}(1), \\ \widehat{P}_{n+1}^{\alpha,\beta}(-1) &= -(1+\alpha_n)\widehat{P}_n^{\alpha,\beta}(-1) - \beta_n \widehat{P}_{n-1}^{\alpha,\beta}(-1), \end{aligned}$$

which is a consequence of (2.15) for $x = \pm 1$ and which can be written in the form

$$\frac{4(\alpha)_{n+1}(\alpha+\beta)_{n+1}}{(\alpha+\beta)_{2n+2}} = (1-\alpha_n)\frac{2(\alpha)_n(\alpha+\beta)_n}{(\alpha+\beta)_{2n}} - \beta_n \frac{(\alpha)_{n-1}(\alpha+\beta)_{n-1}}{(\alpha+\beta)_{2n-2}}, \tag{2.16}$$

$$\frac{4(\beta)_{n+1}(\alpha+\beta)_{n+1}}{(\alpha+\beta)_{2n+2}} = (1+\alpha_n)\frac{2(\beta)_n(\alpha+\beta)_n}{(\alpha+\beta)_{2n}} - \beta_n \frac{(\beta)_{n-1}(\alpha+\beta)_{n-1}}{(\alpha+\beta)_{2n-2}}. \tag{2.17}$$

Multiply equation (2.16) by $(\beta)_{n-1}$, equation (2.17) by $(\alpha)_{n-1}$, and substract the first from the second one to obtain

$$\frac{2(\alpha)_{n-1}(\beta)_{n-1}(\alpha+\beta)_{n+1}(2n+\alpha+\beta+1)(\beta-\alpha)}{(\alpha+\beta)_{2n+2}}$$

$$= \frac{(\alpha)_{n-1}(\beta)_{n-1}(\alpha+\beta)_n[\beta-\alpha+\alpha_n(2n+\alpha+\beta)]}{(\alpha+\beta)_{2n}},$$

where we took into account

$$(n+\beta)(n+1+\beta) - (n+\alpha)(n+1+\alpha) = (2n+\alpha+\beta+1)(\beta-\alpha).$$

Thus,

$$\alpha_n = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \quad n \in \mathbb{N}$$

Now, we multiply (2.16) by $(\beta)_n$, (2.17) by $(\alpha)_n$ and consider the sum of both equations. It follows

$$\frac{4(\alpha)_n(\beta)_n(\alpha+\beta)_{n+1}}{(\alpha+\beta)_{2n+1}} = \frac{4(\alpha)_n(\beta)_n(\alpha+\beta)_n}{(\alpha+\beta)_{2n}} - \beta_n \frac{(\alpha)_{n-1}(\beta)_{n-1}(\alpha+\beta)_{n-1}(2n+\alpha+\beta)}{(\alpha+\beta)_{2n-2}},$$

so that

$$\beta_n = \begin{cases} \dfrac{4(1+\alpha)(1+\beta)}{(2+\alpha+\beta)^2(3+\alpha+\beta)}, & n=1 \\[2ex] \dfrac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, & n=2,3,\ldots \end{cases}$$

2.6 Exercises

In what follows by $(P_n)_{n=0}^{\infty}$ we denote the monic OPS w.r.t. the quasi-definite moment functional $\mathcal{L}$ with the moment sequence $\{\mu_n\}$ and the respective recursion formula (2.4).

1. Prove that the following assertions are equivalent:

   (a) $\mathcal{L}$ is symmetric, i.e. $\mu_{2n+1} = 0 \;\forall\, n \in \mathbb{N}_0$.

   (b) $P_n(-x) = (-1)^n P_n(x) \;\forall\, x \in \mathbb{R}\,, \forall\, n \in \mathbb{N}_0$.

   (c) In the recursion formula (2.4) there holds $\alpha_n = 0 \;\forall\, n \in \mathbb{N}_0$.

2. Define $Q_n(x) = a^{-n} P_n(ax+b)$ $(a \neq 0)$ and show that

   (a) $Q_{n+1}(x) = \left(x - \dfrac{\alpha_n - b}{a}\right) Q_n(x) - \dfrac{\beta_n}{a^2}\, Q_{n-1}(x)\,,$

   (b) $(Q_n)_{n=0}^{\infty}$ is OPS w.r.t. the moment sequence $\{\eta_n\}$ with

   $$\eta_n = a^{-n} \sum_{k=0}^{n} \binom{n}{k} (-b)^{n-k} \mu_k\,.$$

3. Show that the polynomials $P_n(x)$ in formula (1.15) satisfy the recursion formula

   $$Q_{n+1}(x) = (x-n-a)\, Q_n(x) - a\, n\, Q_{n-1}(x),\;\; n = 0, 1, \ldots,\;\; Q_n(x) := n!\, P_n(x)\,.$$

4. Let $\alpha_n = 0$ and $\beta_n < 0$, $n \in \mathbb{N}_0$. Then $(P_n)_{n=0}^{\infty}$ is OPS w.r.t. a quasi-definit moment functional $\mathcal{L}$. Define $\mathcal{L}^*[M_n] := \mathbf{i}^{-n}\mathcal{L}[M_n]$ and prove that $\mathcal{L}^*$ is positive definite. Determine the respective monic OPS.

5. Let $\alpha_n = 0$, $\beta_n < 0$, $n \in \mathbb{N}$, and $\alpha_0 \in \mathbb{R} \setminus \{0\}$. Define $R_n(x) = \mathsf{Re}\,[\mathbf{i}^{-n} P_n(\mathbf{i}x)]$ and $I_n(x) = \mathsf{Im}\,[\mathbf{i}^{-n} P_n(\mathbf{i}x)]$. Show that $(R_n)_{n=0}^{\infty}$ and $\left(\alpha_0^{-1} I_{n+1}\right)_{n=0}^{\infty}$ are monic OPS w.r.t. some positive definite moment functionals.

6. Prove that

   (a) $\dfrac{1 - x\,w}{1 - 2\,x\,w + w^2} = \displaystyle\sum_{n=0}^{\infty} T_n(x)\, w^n\,,$

   (b) $\dfrac{1}{1 - 2\,x\,w + w^2} = \displaystyle\sum_{n=0}^{\infty} U_n(x)\, w^n\,.$

7. Show that a monic OPS $(P_n)_{n=0}^{\infty}$ fulfils a condition of the form

   $$P_{n-1}(x)\, P_n(-x) + P_{n-1}(-x)\, P_n(x) = a_n \neq 0\,, \quad n \in \mathbb{N}\,,$$

   if and only if $\beta_n \neq 0$, $n > 0$, and $\alpha_n = 0$, $n \geq 1$, $\alpha_0 \neq 0$ in the recursion formula (2.4). Moreover, show that the respective moment functional is positive definite if and only if $(-1)^n a_1 a_n < 0$, $n \geq 1$, and $\beta_0 > 0$.

8. Let $(P_n)_{n=0}^{\infty}$ be an OPS and $\mathcal{M}$ a moment functional with $\mathcal{M}[P_0] \neq 0$ and $\mathcal{M}[P_n] = 0$, $n \in \mathbb{N}$. Prove that $\{P_n\}$ is an OPS w.r.t. $\mathcal{M}$.

9. Show that the weights in the Gaussian rule satisfy $A_{n,n-k+1} = A_{nk}$ if the moment functional is symmetric.

10. Let $\mathcal{L}$ be positive definit and $K_n(z, x)$ be defined as in Section 2.4. Show that $\pi(t) = \mathcal{L}[\pi\, K_n(., t)]$ for each polynomial $\pi \in \mathbb{C}[x]$ and $n \geq \deg \pi(x)$.

11. Show, that the normalized Jacobi polynomials $p_n^{\alpha,\beta}(x)$ are given by

$$p_n^{\alpha,\beta}(x) = \left[h_n^{\alpha,\beta}\right]^{-1} P_n^{\alpha,\beta}(x) \tag{2.18}$$

with

$$h_n^{\alpha,\beta} = \sqrt{\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+2)}}$$

(cf. [4, Equation (3.1)]).

12. Prove, that, for $n \in \mathbb{N}$, the formulas

$$\frac{d\, P_n^{\alpha,\beta}(x)}{dx} = \frac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{\alpha+1,\beta+1}(x)\,, \tag{2.19}$$

$$\frac{d\, p_n^{\alpha,\beta}(x)}{dx} = \gamma_n^{\alpha,\beta} p_{n-1}^{\alpha+1,\beta+1}(x) \tag{2.20}$$

with $\gamma_n^{\alpha,\beta} = \sqrt{n(n+\alpha+\beta+1)}$ and

$$(1-x)^\alpha(1+x)^\beta p_n^{\alpha,\beta}(x) = -\frac{1}{\gamma_n^{\alpha,\beta}} \frac{d}{dx}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1} p_{n-1}^{\alpha+1,\beta+1}(x)\right] \tag{2.21}$$

(cf. [4, Equation (3.4) and Exercise 3.1]) are valid.

Chapter 3

Singular Integral Operators

3.1 Cauchy singular integral operators

Define

$$(\mathcal{S}u)(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{u(y)\,dy}{y-x} := \lim_{\varepsilon\to+0}\left(\int_{-1}^{x-\varepsilon} + \int_{x+\varepsilon}^{1}\right)\frac{u(y)\,dy}{y-x}\,, \quad -1<x<1\,, \tag{3.1}$$

where we assume that this Cauchy principal value integral exists.

Lemma 3.1 *For* $0<\nu<1$ *and* $0<t<\infty$,

$$\int_0^\infty \frac{s^{\nu-1}ds}{s-t} = -\pi t^{\nu-1}\cot(\pi\nu)\,, \tag{3.2}$$

where the integral has to be understood in the sense of a Cauchy principal value integral.

Corollary 3.2 *For* $\alpha,\beta>-1$, $\alpha+\beta=-1$, *and* $-1<x<1$,

$$\frac{1}{\pi}\int_{-1}^{1}\frac{v^{\alpha,\beta}(y)\,dy}{y-x} = -\cot(\pi\beta)v^{\alpha,\beta}(x)\,, \tag{3.3}$$

where $v^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$.

For $\alpha,\beta>-1$, define

$$\left(\mathcal{S}_{\alpha,\beta}u\right)(x) = \cos(\pi\beta)v^{\alpha,\beta}(x)u(x) + \frac{\sin(\pi\beta)}{\pi}\int_{-1}^{1}\frac{v^{\alpha,\beta}(y)u(y)}{y-x}\,dy\,, \quad -1<x<1\,.$$

Proposition 3.3 *For* $\alpha,\beta>-1$, $\alpha+\beta=-1$, *and* $-1<x<1$,

$$\left(\mathcal{S}_{\alpha,\beta}\widehat{P}_n^{\alpha,\beta}\right)(x) = -\widehat{P}_{n-1}^{-\alpha,-\beta}(x)\,, \quad n\in\mathbb{N}_0\,. \tag{3.4}$$

Remark 3.4 *Proposition 3.3 can be generalized in the following sense. Let* $a, b \in \mathbb{R}$, $a - \mathbf{i}b = e^{\mathbf{i}\pi\beta_0}$, $\beta_0 \in (0,1)$, $\lambda, \nu \in \mathbb{Z}$ *with* $\alpha := \lambda + \beta_0 \in (-1,1)$ *and* $\beta := \nu - \beta_0 \in (-1,1)$. *Then, for*

$$(\mathcal{A}u)(x) = a v^{\alpha,\beta}(x) u(x) + \frac{b}{\pi} \int_{-1}^{1} \frac{v^{\alpha,\beta}(y) u(y)}{y - x} \, dy \,, \tag{3.5}$$

we have the relations

$$\left(\mathcal{A}\widehat{P}_n^{\alpha,\beta}\right)(x) = (-1)^\lambda \widehat{P}_{n-\kappa}^{-\alpha,-\beta}(x) \,, \quad n \in \mathbb{N}_0 \,, \ -1 < x < 1 \,,$$

and

$$\left(\mathcal{A}p_n^{\alpha,\beta}\right)(x) = (-1)^\lambda p_{n-\kappa}^{-\alpha,-\beta}(x) \,, \quad n \in \mathbb{N}_0 \,, \ -1 < x < 1 \,, \tag{3.6}$$

where $\kappa = -(\lambda + \nu) = -(\alpha + \beta)$ *(cf. [4, Corollary 3.7, Exercises 3.8, 3.9]).*

Corollary 3.5 *Let* $\mathbf{L}^2_{\alpha,\beta}$ *denote the Hilbert space of w.r.t. the Jacobi weight* $v^{\alpha,\beta}(x)$ *square integrable functions* $u : (-1,1) \longrightarrow \mathbb{C}$ *equipped with the inner product*

$$\langle u, v \rangle_{\alpha,\beta} = \int_{-1}^{1} u(x) \overline{v(x)} v^{\alpha,\beta}(x) \, dx \,.$$

Then the operator $\mathcal{A}$ *defined by* (3.5) *on all polynomials can be uniquely extended to a bounded linear operator* $\mathcal{A} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{-\alpha,-\beta}$. *This operator is invertible if* $\kappa = 0$, *left-sided invertible if* $\kappa = -1$, *and right-sided invertible if* $\kappa = 1$.

For $s \geq 0$, define the weighted Sobolev spaces $\mathbf{L}^{2,s}_{\alpha,\beta}$ as

$$\mathbf{L}^{2,s}_{\alpha,\beta} = \left\{ u \in \mathbf{L}^{2,s}_{\alpha,\beta} : \sum_{n=0}^{\infty} (1+n)^{2s} \left| \langle u, p_n^{\alpha,\beta} \rangle_{\alpha,\beta} \right|^2 < \infty \right\}$$

equipped with the norm

$$\|u\|_{\alpha,\beta,s} = \sqrt{\sum_{n=0}^{\infty} (1+n)^{2s} \left| \langle u, p_n^{\alpha,\beta} \rangle_{\alpha,\beta} \right|^2} \,.$$

Corollary 3.6 *For* $s > 0$, *the operator* $\mathcal{A} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{-\alpha,-\beta}$ *defined in Remark 3.4 is also bounded from* $\mathbf{L}^{2,s}_{\alpha,\beta}$ *into* $\mathbf{L}^{2,s}_{-\alpha,-\beta}$.

Remark 3.7 *For* $s \geq 0$ *and* $\delta > 0$, *the space* $\mathbf{L}^{2,s+\delta}\alpha, \beta$ *is compactly embedded into the space* $\mathbf{L}^{2,s}_{\alpha,\beta}$.

3.2 Singular integro-differential operators

Remark 3.8 ([1], Section 2) *If* $r \in \mathbb{N}_0$ *then* $u \in \mathbf{L}^{2,r}_{\alpha,\beta}$ *if and only if* $u^{(k)}\varphi^k \in \mathbf{L}^2_{\alpha,\beta}$ *for all* $k = 0, \ldots, r$, *where* $\varphi(x) = \sqrt{1 - x^2}$. *Moreover, the norms* $\|u\|_{\alpha,\beta,s}$ *and* $\sum_{k=0}^{r} \left\| u^{(k)} \varphi^k \right\|_{\alpha,\beta}$ *are equivalent.*

Lemma 3.9 *For $s \ge 0$ and $\alpha, \beta > -1$, the operator $\mathcal{D}$ of generalized differentiation is a continuous isomorphism from $\mathbf{L}^{2,s+1,0}_{\alpha,\beta}$ onto $\mathbf{L}^{2,s}_{\alpha+1,\beta+1}$, where $\mathbf{L}^{2,s,0}_{\alpha,\beta} = \left\{ u \in \mathbf{L}^{2,s}_{\alpha,\beta} : \langle u, 1 \rangle_{\alpha,\beta} = 0 \right\}$.*

With the help of the operator $\mathcal{A}$ given in Remark 3.4 we define $\mathcal{B} = \mathcal{D}\mathcal{A}$.

Proposition 3.10 *We have*

$$\mathcal{B} p_n^{\alpha,\beta} = (-1)^\lambda \sqrt{(n-\kappa)(n+1)}\, p_{n-\kappa-1}^{1-\alpha,1-\beta}\,, \quad n \in \mathbb{N}_0\,.$$

Hence, in case $\alpha = \beta = \frac{1}{2}$, i.e. $a = 0$, $b = -1$, $\lambda = 0$, and $\nu = 1$, this leads to

$$-\frac{d}{dx}\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-y^2}\, U_n(y)}{y-x}\, dy = (n+1) U_n(x)\,, \quad n \in \mathbb{N}_0\,,\ -1 < x < 1\,.$$

Consequently, the operator $\mathcal{D}\mathcal{S} : \mathbf{L}^{2,s+1}_{\frac{1}{2},\frac{1}{2}} \longrightarrow \mathbf{L}^{2,s}_{\frac{1}{2},\frac{1}{2}}$ is an isometrical isomorphism.

3.3 Weakly singular integral operators

We use the same notations as in Remark 3.4 and define

$$(\mathcal{W}u)(x) = a \int_{-1}^{x} v^{\alpha,\beta}(y) u(y)\, dy - \frac{b}{\pi} \int_{-1}^{1} v^{\alpha,\beta}(y) \ln|y-x|\, u(y)\, dy\,, \quad -1 < x < 1\,.$$

In case $\alpha + \beta = -1$ we have, due to (2.21),

$$\int_{-1}^{x} v^{\alpha,\beta}(y) p_n^{\alpha,\beta}(y)\, dy = -\frac{1}{n}\, v^{\alpha+1,\beta+1}(x) p_{n-1}^{\alpha+1,\beta+1}(x)\,, \quad n \in \mathbb{N}\,,$$

and, by partial integration,

$$\int_{-1}^{1} v^{\alpha,\beta}(y) \ln|y-x|\, p_n^{\alpha,\beta}(y)\, dy = \frac{1}{n} \int_{-1}^{1} \frac{v^{\alpha+1,\beta+1}(y) p_{n-1}^{\alpha+1,\beta+1}(y)}{y-x}\, dy\,, \quad -1 < x < 1\,,\ n \in \mathbb{N}\,.$$

Hence,

$$\mathcal{W} p_n^{\alpha,\beta} = -\frac{1}{n}\, p_n^{-\alpha-1,-\beta-1}\,, \quad n \in \mathbb{N}\,.$$

Remark 3.11 (cf. [5], Example 3.27) *In case $\alpha = \beta = -\frac{1}{2}$ it yields, for $-1 < x < 1$,*

$$-\frac{1}{\pi}\int_{-1}^{1} \ln|y-x|\, p_n^{-\frac{1}{2},-\frac{1}{2}}(y)\, \frac{dy}{\sqrt{1-y^2}} = \begin{cases} \ln 2\, p_0^{-\frac{1}{2},-\frac{1}{2}}(x) & : \ n = 0\,, \\ \dfrac{1}{n}\, p_n^{-\frac{1}{2},-\frac{1}{2}}(x) & : \ n \in \mathbb{N}\,. \end{cases}$$

Further considerations in this direction one find in [4, Section 3.4].

Chapter 4

Continued Fractions and Orthogonal Polynomials

4.1 Basics

By an (infinite) **continued fraction** we refer to a tripel $\left((a_n)_{n=1}^{\infty}, (b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty} \right)$ of number sequences, where

$$\begin{aligned} c_0 &= b_0 \\ c_1 &= b_0 + \frac{a_1}{b_1} \\ c_2 &= b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2}} \\ &\vdots \\ c_n &= b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n}}} \end{aligned}$$

The number c_n is called nth **approximant** of the infinite continued fraction

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + \ddots}}} \tag{4.1}$$

In what follows, c_n is written shortly as

$$c_n = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n}$$

and (4.1) as

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n} + \cdots$$

In case of $a_k = -d_k$, we write $-\frac{d_k|}{|b_k}$ instead of $+\frac{-d_k|}{|b_k}$.

Definition 4.1 *We say that the continued fraction (4.1)* **converges** *to K, if at most finitely many approximants c_n are not defined and if*

$$\lim_{n\to\infty} c_n = K\,.$$

In this case, we also write

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n} + \cdots = K\,.$$

We write c_n in the form

$$c_n = \frac{A_n}{B_n}, \quad n = 0, 1, 2, \ldots\,,$$

where, for example,

$$A_0 = b_0\,, \quad B_0 = 1\,,$$

$$A_1 = b_0 b_1 + a_1 \quad B_1 = b_1\,,$$

$$A_2 = b_0 b_1 b_2 + b_0 a_2 + a_1 b_2\,, \quad B_2 = b_1 b_2 + a_2\,.$$

In general, it is possible to define the sequences $(A_n)_{n=0}^{\infty}$ and $(B_n)_{n=0}^{\infty}$ can be defined in such a way that

$$A_n = b_n A_{n-1} + a_n A_{n-2}\,,\ n = 1, 2, \ldots\,, \quad A_{-1} = 1\,,\ A_0 = b_0\,, \tag{4.2}$$

and

$$B_n = b_n B_{n-1} + a_n B_{n-2}\,,\ n = 1, 2, \ldots\,, \quad B_{-1} = 0\,,\ B_0 = 1\,. \tag{4.3}$$

A_n und B_n are called the nth **partial numerator** and **denominator** of the continued fraction (4.1), respectively. We have

$$A_n B_{n-1} - B_n A_{n-1} = (-1)^{n+1} a_1 a_2 \cdots a_n\,, \quad n = 1, 2, \ldots\,, \tag{4.4}$$

and

$$\frac{A_n}{B_n} = b_0 + \sum_{k=1}^{n} \frac{(-1)^{k+1} a_1 a_2 \cdots a_k}{B_{k-1} B_k} \tag{4.5}$$

provided that $B_k \neq 0$, $k = 1, \ldots, n$

Lemma 4.2 *If $m_0 = 0$, then the nth partial denominator of the continued fraction*

$$1 - \frac{1|}{|1} - \frac{(1-m_0)m_1|}{|1} - \frac{(1-m_1)m_2|}{|1} - \cdots$$

equals

$$B_n = (1-m_1)\cdots(1-m_{n-1})\,, \quad n = 1, 2, \ldots \tag{4.6}$$

Lemma 4.3 *Let* $a_n = (1-m_{n-1})m_n$, $m_0 = 0$ *and* $0 < m_n < 1$, $n = 1, 2, \ldots$ *Then*

$$1 - \frac{a_1|}{|1} - \frac{a_2|}{|1} - \frac{a_3|}{|1} - \cdots = \frac{1}{1+L},$$

where

$$L = \sum_{n=1}^{\infty} \frac{m_1 m_2 \cdots m_n}{(1-m_1)(1-m_2)\cdots(1-m_n)}.$$

Proposition 4.4 *Let* $b_n = (1-g_{n-1})g_n$, $0 \le g_0 < 1$ *and* $0 < g_n < 1$, $n = 1, 2, \ldots$ *Then*

$$1 - \frac{b_1|}{|1} - \frac{b_2|}{|1} - \frac{b_3|}{|1} - \cdots = g_0 + \frac{1-g_0}{1+G}, \tag{4.7}$$

wobei

$$G = \sum_{n=1}^{\infty} \frac{g_1 g_2 \cdots g_n}{(1-g_1)(1-g_2)\cdots(1-g_n)}.$$

Example 4.5 *If*

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \frac{a_3|}{|b_3} + \cdots = K \neq 0$$

then

$$b_{-1} + \frac{a_0|}{|b_0} + \frac{a_2|}{|b_2} + \frac{a_3|}{|b_3} + \cdots = b_{-1} + \frac{a_0}{K}.$$

Example 4.6 *Assuming that the continued fraction*

$$1 + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|1} + \cdots$$

converges we show that its value is equal to $\dfrac{1+\sqrt{5}}{2}$.

4.2 Jacobi fractions and orthogonal polynomials

Let α_n and β_n be given numbers with $\beta_n \neq 0$. Let us denote the nth partial denominator of the so-called **Jacobi fraction**

$$\frac{\beta_0|}{|x-\alpha_0} - \frac{\beta_1|}{|x-\alpha_1} - \frac{\beta_2|}{|x-\alpha_2} - \cdots \tag{4.8}$$

by $P_n(x)$. Formula (4.3) gives

$$P_{n+1}(x) = (x-\alpha_n)P_n(x) - \beta_n P_{n-1}(x), \quad n = 0, 1, 2, \ldots, \quad P_0(x) = 1, \; P_{-1}(x) = 0. \tag{4.9}$$

The nth partial numerator $A_n(x)$ satisfies the recursion formula

$$A_{n+1}(x) = (x-\alpha_n)A_n(x) - \beta_n A_{n-1}(x) \quad n = 1, 2, \ldots, \quad A_1(x) = \beta_0, \; A_0(x) = 0,$$

where $\beta_0^{-1}A_n(x)$ is a monic polynomial of degree $n-1$, which is independent of β_0. For this reason we write $Q_n(x) = \beta_0^{-1}A_{n+1}(x)$, $n = -1, 0, 1, \ldots$ Then

$$Q_{n+1}(x) = (x - \alpha_{n+1})Q_n(x) - \beta_{n+1}Q_{n-1}\,, \quad n = 0, 1, 2, \ldots\,, \tag{4.10}$$

$Q_0(x) = 1$, $Q_{-1}(x) = 0$. The polynomials $Q_n(x)$ are called **monic numerator polynomials** w.r.t. the polynomial system $(P_n(x))_{n=0}^{\infty}$. From (4.4) we deduce

$$P_{n+1}(x)Q_{n-1}(x) - P_n(x)Q_n(x) = -\beta_1\beta_2\cdots\beta_n\,, \quad n = 1, 2, \ldots \tag{4.11}$$

Proposition 4.7 *If $\alpha_n \in \mathbb{R}$ and $\beta_n > 0$, then the zeros x_{nk} and y_{nk} of the polynomials $P_n(x)$ and $Q_n(x)$, respectively, fulfil the relations*

$$x_{n+1,k+1} < y_{nk} < x_{n+1,k}\,.$$

Corollary 4.8 *Let (η_1, ξ_1) and (η_1^1, ξ_1^1) be the supports of the OPS*

$$(P_n(x))_{n=0}^{\infty} \quad und \quad (Q_n(x))_{n=0}^{\infty}\,,$$

respectively. Then $(\eta_1^1, \xi_1^1) \subset (\eta_1, \xi_1)$. Moreover, if, for example, $\xi_1^1 < \xi_1$, then, for all sufficiently large n, there lies exactly one zero of $P_n(x)$ in the interval (ξ_1^1, ξ_1).

Proposition 4.9 *If the numbers α_n are real and the numbers β_n are positive, then*

$$\frac{\beta_0 Q_{n-1}(x)}{P_n(x)} = \sum_{k=1}^{n} \frac{A_{nk}}{x - x_{nk}}\,.$$

where the A_{nk}-s denote the weights in the Gaussian quadrature rule w.r.t. the OPS $(P_n(x))_{n=0}^{\infty}$.

Example 4.10 *Die monic Chebyshev polynomials $2^{-n}U_n(x)$, $n \in \mathbb{N}_0$ are the numerator polynomials for $2^{1-n}T_n(x)$, $n \in \mathbb{N}$ and $T_0(x)$.*

4.3 Chain sequences

Definition 4.11 *A sequence $(a_n)_{n=1}^{\infty}$ of the form*

$$a_n = (1 - g_{n-1})g_n \quad mit \quad 0 \le g_0 < 1 \quad and \quad 0 < g_n < 1\,,\; n = 1, 2, \ldots\,,$$

is called **chain sequence**. *The sequence $(g_n)_{n=0}^{\infty}$ is named a* **parameter sequence** *and g_0 a* **initial parameter** *of the chain sequence $(a_n)_{n=1}^{\infty}$.*

Example 4.12 *The constant sequence $\left(\frac{1}{4}\right)_{n=1}^{\infty}$ is a chain sequence, where $\left(\frac{n}{2(n+1)}\right)_{n=0}^{\infty}$ as well as the constant sequence $\left(\frac{1}{2}\right)_{n=0}^{\infty}$ are parameter sequences. The equations*

$$a = \left(1 - \frac{1-\sqrt{1-4a}}{2}\right)\frac{1-\sqrt{1-4a}}{2} = \left(1 - \frac{1+\sqrt{1-4a}}{2}\right)\frac{1+\sqrt{1-4a}}{2}$$

show that each constant sequence $(a)_{n=1}^{\infty}$ with $0 < a \le \dfrac{1}{4}$ is a chain sequence.

Lemma 4.13 *Let* $(g_n)_{n=0}^{\infty}$ *and* $(h_n)_{n=0}^{\infty}$ *be parameter sequences of the chain sequence* $(a_n)_{n=1}^{\infty}$. *Then* $g_n < h_n$, $n = 1, 2, \dots$, *if and only if* $g_0 < h_0$.

Lemma 4.14 *If a chain sequence* $(a_n)_{n=1}^{\infty}$ *possesses the parameter sequence* $(g_n)_{n=0}^{\infty}$ *with* $g_0 > 0$, *then, for each* $h_0 \in [0, g_0]$, *there exists a parameter sequence* $(h_n)_{n=0}^{\infty}$ *of* $(a_n)_{n=1}^{\infty}$.

Corollary 4.15 *Each chain sequence possesses a parameter sequence* $(m_n)_{n=0}^{\infty}$ *with* $m_0 = 0$. *It holds* $m_n < g_n$, $n = 0, 1, 2, \dots$, *for each other parameter sequence* $(g_n)_{n=0}^{\infty}$ *of this chain sequence. The sequence* $(m_n)_{n=1}^{\infty}$ *is called* **minimal parameter sequence** *of the respective chain sequence. A parameter sequence* $(M_n)_{n=0}^{\infty}$, *for which* $M_n \geq g_n$, $n = 0, 1, 2, \dots$, *holds for any parameter sequence* $(g_n)_{n=0}^{\infty}$, *is named* **maximal parameter sequence** *of the respective chain sequence.*

Lemma 4.16 *To each chain sequence there exists a maximal parameter sequence.*

In what follows $(m_n)_{n=0}^{\infty}$ denotes the minimal parameter sequence and $(M_n)_{n=0}^{\infty}$ the maximal parameter sequence of the chain sequence $(a_n)_{n=1}^{\infty}$.

Proposition 4.17 *If* $(b_n)_{n=1}^{\infty}$ *is a chain sequence with the parameter sequence* $(h_n)_{n=0}^{\infty}$ *and if* $a_n \leq b_n$, $n = 1, 2, \dots$, *then*

$$m_n \leq h_n \leq M_n , \quad n = 0, 1, 2, \dots$$

Lemma 4.18 *Let the chain sequence* $(a_n)_{n=1}^{\infty}$ *be monotonously non-decreasing. Then the minimal parameter sequence* $(m_n)_{n=0}^{\infty}$ *is monotonously increasing and the maximal parameter sequence* $(M_n)_{n=0}^{\infty}$ *monotonously non-increasing.*

Corollary 4.19 *If* $(a_n)_{n=1}^{\infty} = \left(\frac{1}{4}\right)_{n=1}^{\infty}$ *then* $(M_n)_{n=0}^{\infty} = \left(\frac{1}{2}\right)_{n=0}^{\infty}$.

Corollary 4.20 *Let* $(a_n)_{n=1}^{\infty}$ *be a chain sequence. If there exists an index* $N \in \mathbb{N}$ *with* $a_n \geq \frac{1}{4}$, $n = N, N+1, \dots$, *then* $\lim_{n\to\infty} a_n = \frac{1}{4}$. *Consequently, if* $b_n \geq b > \frac{1}{4}$, $n = N, N+1, \dots$, *then* $(b_n)_{n=1}^{\infty}$ *is* **not** *a chain sequence.*