

Examination emphases in Hilbert space methods

1. Describe the geometric properties of the Hilbert space. (Proposition 1.1) When is an operator $A \in \mathcal{L}(\mathbf{H})$ selfadjoint, when positive? Show that the lower and the upper bound of the spectrum of a selfadjoint operator belong to the spectrum and that each bounded and monotone sequence of selfadjoint operators converges strongly. (Propositions 2.7 and 2.8)
2. Explain the notion of a linear and bounded operator as well as Riesz' representation theorem. (Section 1.2) What is an orthoprojection? Prove the properties of sequences and series of orthoprojections. (Corollary 2.13 and Proposition 2.14)
3. What are isometric resp. unitary operators? Show, that every isometric operator is invertible and linear. Prove necessary and sufficient conditions for the fact that an operator $V \in \mathcal{L}(\mathbf{H})$ is a partial isometry. (Corollary 2.22 and Proposition 2.25)
4. Define $f(A)$ for a continuous function f and a selfadjoint operator $A \in \mathcal{L}(\mathbf{H})$. (Proposition 3.1 and Proposition 3.2) Prove one of the properties (a), (b) or (c) of $f(A)$ from Proposition 3.2.
5. How can one define \sqrt{A} and $|A|$? (Corollaries 3.5 and 3.7) Under which conditions is the product of two positive operators positive? Justify your statement. (Corollary 3.6) Show, that for every operator $A \in \mathcal{L}(\mathbf{H})$ there exists exactly one polar decomposition $A = V\sqrt{A^*A}$ with a partial isometry $V \in \mathcal{L}(\mathbf{H})$ and $N(A) \subset N(V)$. (Proposition 3.8)
6. Describe the construction of the spectral map E_λ , $\lambda \in \mathbb{R}$, w.r.t a selfadjoint operator $A \in \mathcal{L}(\mathbf{H})$ and prove its fundamental properties. (Proposition 3.10 and Proposition 3.15)
7. Define the Riemann-Stieltjes integral w.r.t. a spectral map and show, that every continuous function is uniformly Riemann-Stieltjes integrable. (Proposition 3.18) Prove the fundamental properties of the Riemann-Stieltjes integral and the formula

$$f(A) = \int_a^{b-0} f(\lambda) dE_\lambda.$$

(1.-5. of Section 3.2 and Proposition 3.19)

8. Show the fundamental properties of the spectrum of a compact operator. (Proposition 3.23 and Corollary 3.24) How does the spectral representation of a compact operator follow? (Proposition 3.25)
9. How can one define the concept of an in general unbounded linear operator? What is a restriction resp. extension of such an operator? Define the adjoint operator of an operator $A \in L(\mathbf{H})$, its resolvent set and its spectrum. Show, that the resolvent set is an open set. (Proposition 4.4) Prove equivalent conditions for symmetry and selfadjointness of operators $A \in L(\mathbf{H})$. (Proposition 4.6)
10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and E_λ be a left-sided continuous spectral map. Define the integral $J(f) = \int_{-\infty}^{\infty} f(\lambda) dE_\lambda$. Show that $J(f)$ is selfadjoint and that $J(f) \in \mathcal{L}(\mathbf{H})$ if f is bounded. (Proposition 4.12)
11. Define the differential operators A_r^0 and A_r . Prove $(A_r^0)^* = A_r$ and $A_r^* = \overline{A_r^0}$. What is the domain of the operator $\overline{A_r^0}$? Prove your assertion. (Proposition 5.6,(c),(d))