Lecture Notes

Hilbert Space Methods

Summer Term 2011

Peter Junghanns

Remark: The present lecture notes contain only a framework of the contents of the lectures. The lectures themselves present detailed expositions, proofs and examples.

Contents

Bibliography

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Chapter 1

Introduction

1.1 The Hilbert space

Let $\mathbf{H}$ be a linear space (in general over the field of complex numbers). A map $\mathbf{H} \times \mathbf{H} \longrightarrow \mathbb{C}$, $(x, y) \mapsto \langle x, y \rangle$ is called **scalar product** or **inner product** on $\mathbf{H}$, if the following conditions are satisfied:

(S1) $\langle x, x \rangle \geq 0 \quad \forall\, x \in \mathbf{H}$ and $\langle x, x \rangle = 0 \iff x = \Theta$,

(S2) $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall\, x, y \in \mathbf{H}$,

(S3) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \quad \forall\, x, y, z \in \mathbf{H}$, $\forall\, \alpha, \beta \in \mathbb{C}$.

One has the **Cauchy-Schwarz inequality**

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle} \quad \forall\, x, y \in \mathbf{H}. \tag{1.1}$$

Using this inequality one can show that

$$\|x\| = \sqrt{\langle x, x \rangle} \tag{1.2}$$

defines a norm on $\mathbf{H}$. A linear space $(\mathbf{H}, \langle ., . \rangle)$ with inner product is called **unitary space**, if $(\mathbf{H}, \|.\|)$ with $\|x\| = \sqrt{\langle x, x \rangle}$ is a Banach space. A unitary space $\mathbf{H}$ is called **Hilbert space**, if for all $n \in \mathbb{N}$ there exists a linearly independent system in $\mathbf{H}$ with exactly n elements.

Proposition 1.1 *Let* $(\mathbf{H}, \langle ., . \rangle)$ *be a Hilbert space.*

1. *The map* $\langle ., . \rangle : \mathbf{H} \times \mathbf{H} \longrightarrow \mathbb{C}$, $(x, y) \mapsto \langle x, y \rangle$ *is continuous.*

2. *If* $\mathbf{L} \subset \mathbf{H}$ *is a linear subspace of* $\mathbf{H}$, *then*

   $$\mathbf{L}^\perp := \{x \in \mathbf{H} : \langle x, y \rangle = 0 \;\forall\, y \in \mathbf{L}\}$$

   is a closed linear subspace of $\mathbf{H}$, *the so called* **orthogonal complement** *of* $\mathbf{L}$, *where* $\mathbf{L} \cap \mathbf{L}^\perp = \{\Theta\}$.

3. *Let* $\mathbf{L} \subset \mathbf{H}$ *a closed linear subspace of* $\mathbf{H}$. *Then, each element* $x \in \mathbf{H}$ *admits the unique representation* $x = y + z$ *with* $y \in \mathbf{L}$ *and* $z \in \mathbf{L}^\perp$. *Moreover,*

   $$\|x - y\| = \inf \{\|x - w\| : w \in \mathbf{L}\}.$$

   The vector y *is called* **orthogonal projection** *of* x *onto* $\mathbf{L}$. *Furthermore, we write* $\mathbf{H} = \mathbf{L} \oplus \mathbf{L}^\perp$.

4. If $\mathbf{L} \subset \mathbf{H}$ *is a linear subspace, then* $\overline{\mathbf{L}} = \mathbf{H}$ *if and only if there exists* **no** $x^* \in \mathbf{H} \setminus \{\Theta\}$ *such that* $\langle x, x^* \rangle = 0$ *for all* $x \in \mathbf{L}$ *.*

A system $B = \{b_0, b_1, \ldots, b_n, \ldots\} = \{b_n\}_{n=0}^{\infty} \subset \mathbf{H}$ is called **linearly independent**, if each finite subsystem is linearly independent. The system B is refered to as an **orthonormal system** (ONS), if $\langle b_j, b_k \rangle = \delta_{jk}$ for all $j, k = 0, 1, 2, \ldots$ Remark, that each ONS is automatically linearly independent.

Corollary 1.2 (Schmidt's orthogonalisation method) *Let* $\{b_n\}_{n=0}^{\infty}$ *be a linearly independent system. Set*

$$a_0 = \frac{1}{\sqrt{\langle b_0, b_0 \rangle}}\, b_0\,.$$

Then $\langle a_0, a_0 \rangle = 1$ *. We determine* $\beta_{10} \in \mathbb{C}$ *such that*

$$\widetilde{a}_1 = b_1 + \beta_{10} a_0$$

is orthogonal w.r.t. a_0 *, i.e.* $\beta_{10} = -\langle b_1, a_0 \rangle$ *. Since* $\widetilde{a}_1 \neq \Theta$ *we can set*

$$a_1 = \frac{1}{\sqrt{\langle \widetilde{a}_1, \widetilde{a}_1 \rangle}}\, \widetilde{a}_1\,.$$

If $a_0, \ldots, a_{m-1} \in \operatorname{span}\{b_0, \ldots, b_{m-1}\}$ *are determined such that*

$$\langle a_j, a_k \rangle = \delta_{jk}\,, \quad j, k = 0, 1, \ldots, m-1\,,$$

then we set

$$\widetilde{a}_m = b_m + \sum_{k=0}^{m-1} \beta_{mk} a_k \quad \textit{mit} \quad \beta_{mk} = -\langle b_m, a_k \rangle$$

and

$$a_m = \frac{1}{\sqrt{\langle \widetilde{a}_m, \widetilde{a}_m \rangle}}\, \widetilde{a}_m\,.$$

In this way we obtain an ONS $\{a_n\}_{n=0}^{\infty}$ *with the property*

$$\operatorname{span}\{a_0, \ldots, a_n\} = \operatorname{span}\{b_0, \ldots, b_n\}\,, \quad n = 0, 1, 2, \ldots$$

Corollary 1.3 *Let* $\{e_n\}_{n=0}^{\infty}$ *be an ONS in* $\mathbf{H}$ *,*

$$\mathbf{L}_m = \operatorname{span}\{e_0, \ldots, e_m\}\,, \quad m = 0, 1, 2, \ldots\,,$$

and

$$\mathbf{L} = \overline{\left\{ \sum_{k=0}^{m} \alpha_k e_k : \alpha_k \in \mathbb{C},\, m = 0, 1, 2, \ldots \right\}}\,.$$

Then

$$\sum_{k=0}^{m} \langle x, e_k \rangle\, e_k$$

is the **best approximation** *to* $x \in \mathbf{H}$ *by elements from* $\mathbf{L}_m$ *. The numbers* $\gamma_k = \langle x, e_k \rangle$ *are called* **Fourier coefficients** *of* x *w.r.t. the ONS* $\{e_n\}_{n=0}^{\infty}$ *.* **Bessel's inequality**

$$\sum_{k=0}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2$$

holds for all $x \in \mathbf{H}$. *If* $\mathbf{L} = \mathbf{H}$, *then*

$$\lim_{m \to \infty} \left\| x - \sum_{k=0}^{m} \langle x, e_k \rangle \, e_k \right\| = 0 \,, \text{ d.h. } x = \sum_{k=0}^{\infty} \langle x, e_k \rangle \, e_k \,, \ \forall \, x \in \mathbf{H} \,.$$

In this case **Parseval's equality**

$$\sum_{k=0}^{\infty} |\langle x, e_k \rangle|^2 = \|x\|^2 \quad \forall \, x \in \mathbf{H} \,,$$

is in force, and we call und man nennt $\{e_n\}_{n=0}^{\infty}$ *a* **complete orthonormal system** *(CONS) in* $\mathbf{H}$ *and the map* $\mathcal{F} : \mathbf{H} \longrightarrow \ell^2$, $x \mapsto (\langle x, e_n \rangle)$ **Fourier transform***, which, due to Parseval's equality, is an* **isometric isomorphism***.*

1.2 Linear and bounded operators

Let $\mathbf{X}$ and $\mathbf{Y}$ be linear spaces over the field $\mathbb{K}$ of real or complex numbers. As it is well known a map $f : \mathbf{X} \longrightarrow \mathbf{Y}$, $x \mapsto f(x)$ is called **linear**, if for arbitrary $x_1, x_2 \in \mathbf{X}$ and $\alpha_1, \alpha_2 \in \mathbb{K}$ the relation

$$f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

holds. Often such a linear map is named **linear operator** , and instead of $f(x)$ there is written Ax . The set of all linear operators between $\mathbf{X}$ and $\mathbf{Y}$ is denoted by $L(\mathbf{X}, \mathbf{Y})$. If we define

$$(\alpha A + \beta B)x = \alpha(Ax) + \beta(Bx) \,, \quad \alpha, \beta \in \mathbb{K} \,, \ A, B \in L(\mathbf{X}, \mathbf{Y}) \,,$$

then $L(\mathbf{X}, \mathbf{Y})$ is a linear space over $\mathbb{K}$.

Proposition 1.4 *Let* $(\mathbf{X}, \|.\|_{\mathbf{X}})$ *and* $(\mathbf{Y}, \|.\|_{\mathbf{Y}})$ *be two normed spaces over the field* $\mathbb{K}$ *as well as* $A \in L(\mathbf{X}, \mathbf{Y})$. *Then the following assertions are equivalent:*

(a) $A : \mathbf{X} \longrightarrow \mathbf{Y}$ *is a continuous map.*

(b) $A : \mathbf{X} \longrightarrow \mathbf{Y}$ *is uniformly continuous.*

(c) $A : \mathbf{X} \longrightarrow \mathbf{Y}$ *is in* $\Theta \in \mathbf{X}$ *continuous.*

(d) *There exists a constant* $M \geq 0$ *such that*

$$\|Ax\|_{\mathbf{Y}} \leq M \, \|x\|_{\mathbf{X}} \quad \forall \, x \in \mathbf{X} \,. \tag{1.3}$$

If there is no risk of misunderstanding, in what follows we will omit the indices of the notation of the norms.

The set of operators $A \in L(\mathbf{X}, \mathbf{Y})$, for which one (and, consequently, each) of the assertions (a)-(d) of the Proposition 1.4 is true, will be denoted by $\mathcal{L}(\mathbf{X}, \mathbf{Y})$. If, for $A \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$, we define

$$\|A\|_{\mathbf{X} \to \mathbf{Y}} := \sup \left\{ \|Ax\| : x \in \mathbf{X}, \ \|x\| \leq 1 \right\} \,. \tag{1.4}$$

then $(\mathcal{L}(\mathbf{X}, \mathbf{Y}), \|.\|_{\mathbf{X} \to \mathbf{Y}})$ becomes a normed space, the space of all **bounded linear operators**.

If $A \in \mathbf{L}(\mathbf{X}, \mathbf{Y})$, then the **nullspace** of A , $N(A) := \{x \in \mathbf{X} : Ax = \Theta\}$, is a closed linear subspace of $\mathbf{X}$.

Proposition 1.5 *If* $\mathbf{Y}$ *is a Banach space, then* $\mathcal{L}(\mathbf{X},\mathbf{Y})$ *is a Banach space, too.*

The elements of $L(\mathbf{X},\mathbb{K})$ are called **linear functionals**. The space $(\mathcal{L}(\mathbf{X},\mathbb{K}),\|.\|_{\mathbf{X}\to\mathbb{K}})$ of all linear and continuous functionals is called the **dual space** of $\mathbf{X}$ and is denoted by $\mathbf{X}^*$.

Theorem 1.6 (Riesz' representation theorem) *Let* $\mathbf{H}$ *be a Hilbert space and* $f\in\mathbf{H}^*$. *Then there exists a unique* $x_f\in\mathbf{H}$, *such that*

$$f(x)=\langle x,x_f\rangle \qquad \forall\, x\in\mathbf{H}\,.$$

Moreover, $\|f\|_{\mathbf{H}^*}=\|x_f\|_{\mathbf{H}}$.

Consequently, $\mathbf{H}^*$ and $\mathbf{H}$ can be identified.

1.3 Spaces of measurable functions

We recall some facts from measure and integration theory. Let (Ω,Σ,P) be a measure space and $f_n:\Omega\longrightarrow\overline{\mathbb{C}}$ be measurable functions.

(A) (**Beppo Levi, Lebesgue**) If $0\le f_1(\omega)\le f_2(\omega)\le\ldots$ a.e. and $f_n(t)\longrightarrow f(t)$ a.e., then $f:\Omega\longrightarrow[0,\infty]$ is meaurable and

$$\int_\Omega f(\omega)\,P(d\omega)=\lim_{n\to\infty}\int_\Omega f_n(\omega)\,P(d\omega)\,.$$

(B) (**Fatou**) For measurable functions $f_n:\Omega\longrightarrow[0,\infty]$

$$\int_\Omega \liminf_{n\to\infty} f_n(\omega)\,P(d\omega)\le\liminf_{n\to\infty}\int_\Omega f_n(\omega)P(d\omega)\,.$$

(C) (**Lebesgue**) Let $g:\Omega\longrightarrow[0,\infty]$ be integrable and $f_n(\omega)\longrightarrow f(\omega)$ a.e. as well as $|f_n(\omega)|\le g(\omega)$ a.e., $n=1,2,\ldots$ Then $f:\Omega\longrightarrow\overline{\mathbb{C}}$ is integrable and

$$\int_\Omega f(\omega)\,P(d\omega)=\lim_{n\to\infty}\int_\Omega f_n(\omega)\,P(d\omega)\,.$$

(D) (**Lusin**) Let $\varepsilon>0$. If $f:[0,1]\longrightarrow\mathbb{C}$ is Lebesgue measurable and bounded, then there is a continuous function $f_0:[0,1]\longrightarrow\mathbb{C}$ with (m - Lebesgue measure)

$$m\,\{t\in[0,1]:f(t)\ne f_0(t)\}<\varepsilon \quad\text{and}\quad \sup\{|f_0(t)|:t\in[0,1]\}\le\sup\{|f(t)|:t\in[0,1]\}\,.$$

(E) (**Fréchet**) Every Lebesgue measurable function $f:[0,1]\longrightarrow\mathbb{C}$ is the limit of an in measure convergent sequence of Polynomials.

(F) (**Riesz**) If f_n converges to f in measure, then there exists a subsequence $(f_{n_k})_{k=1}^\infty$ converging to f a.e.

On the set of all w.r.t. the Lebesgue measure m measurable functions $f:[0,1]\longrightarrow\mathbb{C}$ we consider the equivalence relation

$$f\sim g \qquad\Longleftrightarrow\qquad m\,\{t\in[0,1]:f(t)\ne g(t)\}=0\,,$$

and, in what follows, we identify the measurable function f with its coset $[f]_\sim$. By $\mathbf{S}=\mathbf{S}(0,1)$ we refer to the space of all these cosets equipped with the metric

$$\rho(f,g)=\int_0^1\frac{|f(t)-g(t)|}{1+|f(t)-g(t)|}\,dt\,.$$

Proposition 1.7 *The convergence in* $\mathbf{S}$ *is the convergence w.r.t. the measure, i.e.* $f_n \longrightarrow f$ *in* $\mathbf{S}$ *if and only if for all* $\varepsilon > 0$

$$\lim_{n \longrightarrow \infty} m\left\{t \in [0,1] : |f_n(t) - f(t)| > \varepsilon\right\} = 0\,.$$

Proposition 1.8 *The metric space* $\mathbf{S}$ *is complete and separable.*

By $\mathbf{L}^2 = \mathbf{L}^2(0,1)$ we denote the subset of $\mathbf{S}$ of all functions (more precisely, cosets of functions) f, for which $|f|^2$ is integrable. We equip $\mathbf{L}^2$ with the inner product

$$\langle f, g\rangle_{\mathbf{L}^2} = \int_0^1 f(t)\overline{g(t)}\,dt\,.$$

Proposition 1.9 *The space* $(\mathbf{L}^2, \|.\|_{\mathbf{L}^2})$ *is a separable Hilbert space.*

Chapter 2

Special classes of operators

2.1 Selfadjoint operators

For $A \in \mathcal{L}(\mathbf{H})$, the adjoint operator $A^* \in \mathcal{L}(\mathbf{H})$ is defined by

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \forall\, x, y \in \mathbf{H}\,.$$

We call $A \in \mathcal{L}(\mathbf{H})$ **selfadjoint**, if $A^* = A$. In all what follows we consider complex Hilbert spaces. (i.e., $\mathbf{H}$ is a linear space over the field of complex numbers).

Proposition 2.1 *An operator $A \in \mathcal{L}(\mathbf{H})$ is selfadjoint if and only if $\langle Ax, x \rangle \in \mathbb{R}\ \forall\, x \in \mathbf{H}$.*

Proposition 2.2 *If $A \in \mathcal{L}(\mathbf{H})$ is selfadjoint then*

$$\|A\| = \sup\left\{ |\langle Ax, x \rangle| : x \in \mathbf{H},\ \|x\| \le 1 \right\}\,.$$

Since, for selfadjoint operators $A, B \in \mathcal{L}(\mathbf{H})$, we have $(A\,B)^* = B^* A^* = B\,A$, the product of two selfadjoint operators is selfadjoint if and only iff $B\,A = A\,B$.

We call $A \in \mathcal{L}(\mathbf{H})$ invertible, if $A^{-1} \in \mathcal{L}(\mathbf{H})$ exists. In this case $(A^{-1})^* = (A^*)^{-1}$.

Let $A \in \mathcal{L}(\mathbf{H})$. By $N(A)$ and $R(A)$ we refer to the **nullspace** and the **image space** of the operator A, respectively,

$$N(A) = \{x \in \mathbf{H} : A\,x = \Theta\}\,, \quad R(A) = \{A\,x : x \in \mathbf{H}\}\,.$$

The **spectrum** $\sigma(A)$ of the operator A is the set of all $\lambda \in \mathbb{C}$, for which $A - \lambda I$ is not bounded invertible, i.e., for which $\nexists\, (A - \lambda I)^{-1} \in \mathcal{L}(\mathbf{H})$. A number $\lambda \in \mathbb{C}$ is called **eigenvalue** of A, if $N(A - \lambda I) \neq \{\Theta\}$.

Proposition 2.3 *Let $A \in \mathcal{L}(\mathbf{H})$.*

(a) *We have $N(A^*) = R(A)^\perp$ and $\overline{R(A)} = N(A^*)^\perp$.*

(b) *$A \in \mathcal{L}(\mathbf{H})$ is invertible if and only if there exists an $\varepsilon > 0$ such that*

$$\|A\,x\| \ge \varepsilon\,\|x\| \quad \textit{und} \quad \|A^* x\| \ge \varepsilon\,\|x\| \quad \forall\, x \in \mathbf{H}\,.$$

(c) *Consequently, the selfadjoint operator* $A \in \mathcal{L}(\mathbf{H})$ *is invertible if and only if there exists an* $\varepsilon > 0$ *such that*

$$\|A\,x\| \ge \varepsilon\,\|x\| \quad \forall\, x \in \mathbf{H}\,.$$

(d) *If* $A \in \mathcal{L}(\mathbf{H})$ *is selfadjoint then* $\sigma(A) \subset \mathbb{R}$, *and the eigenvectors w.r.t. different eigenvalues are orthogonal. Moreover,* $\sigma(A) \subset [m_A, M_A]$ *with*

$$m_A = \inf\{\langle Ax, x\rangle : x \in \mathbf{H},\ \|x\| = 1\} \quad \textit{and} \quad M_A = \sup\{\langle Ax, x\rangle : x \in \mathbf{H},\ \|x\| = 1\}\ .$$

Definition 2.4 *We call an operator* $A \in \mathcal{L}(\mathbf{H})$ **positive**, *if* $\langle Ax, x\rangle \ge 0$ *for all* $x \in \mathbf{H}$.

In view of Proposition 2.1 each positive operator is selfadjoint. For two selfadjoint operators $A, B \in \mathcal{L}(\mathbf{H})$ we write $A \le B$ or $B \ge A$, if $B - A$ is positive, i.e. $\langle A\,x, x\rangle \le \langle B\,x, x\rangle\ \forall\, x \in \mathbf{H}$.

Proposition 2.5 *If* $A \in \mathcal{L}(\mathbf{H})$ *is positive then the* **generalized Cauchy-Schwarz inequality**

$$|\langle A\,x, y\rangle|^2 \le \langle A\,x, x\rangle\,\langle A\,y, y\rangle\ , \quad x, y \in \mathbf{H}\,,$$

holds.

Proposition 2.6 *Let* $A, B, C \in \mathcal{L}(\mathbf{H})$ *be selfadjoint and* $T \in \mathcal{L}(\mathbf{H})$. *Then:*

1. $A \le B\,,\ B \le C \Longrightarrow A \le C$
2. $A \le B \Longrightarrow A + C \le B + C$
3. $A \le B\,,\ B \le A \Longrightarrow A = B$
4. $A \ge \Theta \Longrightarrow T^*AT \ge \Theta$
5. $A \ge \Theta \Longrightarrow A^n \ge \Theta\,,\ n \in \mathbb{N}$
6. $\gamma \in \mathbb{R}\,,\ \Theta \le A \le \gamma I \Longrightarrow A^2 \le \gamma\,A$

Since $m_A I \le A \le M_A I$, i.e. $0 \le A - m_A I \le (M_A - m_A)I$, we have

$$(A - m_A I)^2 \le (M_A - m_A)(A - m_A I) = (M_A I - A)(A - m_A I) + (A - m_A I)^2\,,$$

which implies $(M_A I - A)(A - m_A I) \ge \Theta$.

Proposition 2.7 *If* $A \in \mathcal{L}(\mathbf{H})$ *is selfadjoint then* $m_A, M_A \in \sigma(A)$ *(cf. Proposition 2.3).*

A sequence of operators $A_n \in \mathcal{L}(\mathbf{H})$ is called **strongly convergent** if an operator $A \in \mathcal{L}(\mathbf{H})$ exists such that $\lim\limits_{n\to\infty} \|A_n x - Ax\| = 0\ \forall\, x \in \mathbf{H}$. This is also denoted by $A_n \longrightarrow A$.

A sequence of operators $A_n \in \mathcal{L}(\mathbf{H})$ is called **bounded**, if the sequence $(\|A_n\|)_{n=1}^{\infty}$ is bounded. A sequence of selfadjoint operators $A_n \in \mathcal{L}(\mathbf{H})$, $n \in \mathbb{N}$, is said to be **monotone**, if neighter $A_n \le A_{n+1}\ \forall\, n \in \mathbb{N}$ or $A_n \ge A_{n+1}\ \forall\, n \in \mathbb{N}$.

Proposition 2.8 *Every bounded and monotone sequence of selfadjoint operators is strongly convergent.*

We remark, that the strong limit of a sequence of selfadjoint operators is selfadjoint, which follows from $\langle A_n x, y\rangle = \langle x, A_n y\rangle$ by considering $n \longrightarrow \infty$.

- For a continuous function $k : [0,1]^2 \longrightarrow \mathbb{C}$ we consider the operator $K : \mathbf{L}^2(0,1) \longrightarrow \mathbf{L}^2(0,1)$ defined by
$$(Kx)(t) = \int_0^1 k(t,s)x(s)\,ds\,.$$
In case of $\overline{k(t,s)} = k(s,t)$, $(t,s) \in [0,1]^2$ this operator is selfadjoint. Moreover, $\|K\| \le \sup\left\{|k(t,s)| : (t,s) \in [0,1]^2\right\}$. If $k(t,s)$ is of the form
$$k(t,s) = \sum_{r=1}^{N} \overline{g_r(t)}g_r(s)$$
with continuous functions $g_r : [0,1] \longrightarrow \mathbb{C}$, then K is positive.

- The spectrum of an operator $A \in \mathcal{L}(\mathbf{H})$ is closed. This follows from the fact, taht the set $G\mathcal{L}(\mathbf{H})$ of the invertible operators is open in $\mathcal{L}(\mathbf{H})$. Indeed, if $E \in \mathcal{L}(\mathbf{H})$ with $\|E\| < 1$, then $(I-E)^{-1} = \sum_{n=0}^{\infty} E^n$. For any $A, E \in \mathcal{L}(\mathbf{H})$ with $\|E\| < \left\|A^{-1}\right\|^{-1}$, the invertibility of $A+E$ follows from
$$A + E = A(I + A^{-1}E) \quad \text{and} \quad \left\|A^{-1}E\right\| < 1\,.$$
In particular, the operator $A - \lambda I = -\lambda(I - \lambda^{-1}A)$ is invertible, if $|\lambda| > \|A\|$. Hence,
$$\sigma(A) \subset \{\lambda \in \mathbb{C} : |\lambda| \le \|A\|\}\,.$$

- The spectrum of the shift operator
$$V : \ell^2 \longrightarrow \ell^2\,, \quad (\xi_0, \xi_1, \xi_2, \ldots) \mapsto (0, \xi_0, \xi_1, \ldots)$$
is equal to the unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$.

2.2 Orthoprojections

In view of 1.1, for each closed linear subspace $\mathbf{L} \subset \mathbf{H}$ of a Hilbert space $\mathbf{H}$ there exists the orthogonal complement $\mathbf{L}^\perp$, such that $\mathbf{H} = \mathbf{L} \oplus \mathbf{L}^\perp$. Each $x \in \mathbf{H}$ can be uniquely represented in the form $x = x_{\mathbf{L}} + x_{\mathbf{L}^\perp}$ with $x_{\mathbf{L}} \in \mathbf{L}$ and $x_{\mathbf{L}^\perp} \in \mathbf{L}^\perp$.

Definition 2.9 *The operator* $P_{\mathbf{L}} : \mathbf{H} \longrightarrow \mathbf{H}$ *defined by* $P_{\mathbf{L}}x = x_{\mathbf{L}}$ *is called* **orthogonal projection** *or* **orthoprojection** *from* $\mathbf{H}$ *onto* $\mathbf{L}$.

Corollary 2.10 *For* $P = P_{\mathbf{L}}$ *we have* $P^2 = P$ *and* $\|P\|_{\mathcal{L}(\mathbf{H})} = 1$, *if* $P \ne \Theta$ (*i.e.* $P \in \mathcal{L}(\mathbf{H})$).

Corollary 2.11 *A linear operator* $P : \mathbf{H} \longrightarrow \mathbf{H}$ *is an orthoprojection if and only if*
$$\left\langle P^2x, y\right\rangle = \langle Px, y\rangle \quad \textit{and} \quad \langle Px, y\rangle = \langle x, Py\rangle \quad \forall x, y \in \mathbf{H}\,,$$
i.e. if and only if $P^2 = P$ *and* $P^* = P$.

Proposition 2.12 *Let* $P_{\mathbf{L}}, P_{\mathbf{M}} : \mathbf{H} \longrightarrow \mathbf{H}$ *be two orthoprojections.*

(a) *The product* $P_{\mathbf{L}}P_{\mathbf{M}}$ *is a projection iff* $P_{\mathbf{L}}P_{\mathbf{M}} = P_{\mathbf{M}}P_{\mathbf{L}}$. *In this case* $R(P_{\mathbf{L}}P_{\mathbf{M}}) = \mathbf{L} \cap \mathbf{M}$.

(b) *We have* $\mathbf{L}\perp\mathbf{M}$ *if and only if* $P_{\mathbf{L}}P_{\mathbf{M}}=\Theta$.

(c) *The sum* $P_{\mathbf{L}}+P_{\mathbf{M}}$ *is an orthoprojection if and only if* $\mathbf{L}\perp\mathbf{M}$.

(d) *The difference* $P_{\mathbf{L}}-P_{\mathbf{M}}$ *is an orthoprojection if and only if* $\mathbf{M}\subset\mathbf{L}$.

(e) *The inclusion* $\mathbf{M}\subset\mathbf{L}$ *is equivalent to both* $\|P_{\mathbf{M}}x\|\le\|P_{\mathbf{L}}x\|\ \forall\,x\in\mathbf{H}$ *and to* $P_{\mathbf{M}}\le P_{\mathbf{L}}$.

According to Prop. 2.12,(b) we call two orthoprojections P and Q **orthogonal** if $PQ=\Theta$.

Corollary 2.13 *If* $P_0,P_1,\ldots,P_n$ *are orthoprojections then their sum* $P_0+P_1+\ldots+P_n$ *is an orthoprojection if and only if they are pairwise orthogonal. A series* $\sum\limits_{n=0}^{\infty}P_n$ *of pairwise orthogonal orthoprojections converges strongly to an orthoprojection.*

We say that $A_n\in\mathcal{L}(\mathbf{H})$ converges **weakly** to $A\in\mathcal{L}(\mathbf{H})$, if

$$\lim_{n\to\infty}\langle A_nx,y\rangle=\langle Ax,y\rangle\quad\text{for all}\quad x,y\in\mathbf{H}.$$

Proposition 2.14 *Let* $(P_n)_{n=1}^{\infty}$ *be a sequence of orthoprojections* $P_n:\mathbf{H}\longrightarrow\mathbf{H}$.

(a) *If* $(P_n)_{n=1}^{\infty}$ *is monotone, then it converges to an orthoprojection* $P:\mathbf{H}\longrightarrow\mathbf{H}$.

(b) *If the sequence* $(P_n)_{n=1}^{\infty}$ *converges weakly to an orthoprojection, then it converges strongly.*

Definition 2.15 *For two linear subspaces* $\mathbf{M}_1,\mathbf{M}_2\subset\mathbf{H}$, *the number*

$$\mathcal{O}(\mathbf{M}_1,\mathbf{M}_2):=\|P_2-P_1\|_{\mathcal{L}(\mathbf{H})},$$

where $P_j=P_{\overline{\mathbf{M}}_j}$, $j=1,2$, *denotes the so called* **opening** *of these subspaces.*

Corollary 2.16 *For two linear subspaces* $\mathbf{M}_1,\mathbf{M}_2\subset\mathbf{H}$ *we have*

(a) $\mathcal{O}(\mathbf{M}_1,\mathbf{M}_2)=\mathcal{O}(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2)=\mathcal{O}(\mathbf{M}_1^{\perp},\mathbf{M}_2^{\perp})$,

(b) $\mathcal{O}(\mathbf{M}_1,\mathbf{M}_2)=\max\left\{\sup\limits_{x\in\overline{\mathbf{M}}_2,\|x\|=1}\|(I-P_1)x\|\,,\ \sup\limits_{y\in\overline{\mathbf{M}}_1,\|y\|=1}\|(I-P_2)y\|\right\}.$

Proposition 2.17 *If the opening of two linear subspaces is smaller than* 1 *then they have equal dimension.*

Definition 2.18 *A linear subspace* $\mathbf{M}\subset\mathbf{H}$ *is called* **invariant subspace** *of the operator* $A\in\mathcal{L}(\mathbf{H})$, *if* $Ax\in\mathbf{M}\ \forall\,x\in\mathbf{M}$. *A linear subspace* $\mathbf{M}\subset\mathbf{H}$ *is called* **reducing subspace** *of the operator* $A\in\mathcal{L}(\mathbf{H})$ *if* $\mathbf{M}$ *and* $\mathbf{M}^{\perp}$ *are invariant subspaces of* A.

Let $P=P_{\overline{M}}$ and $A\in\mathcal{L}(\mathbf{H})$.

- $\mathbf{M}$ invariant w.r.t. $A\Longrightarrow\overline{M}$ invariant w.r.t. $A\iff PAP=AP$
- $\mathbf{M}^{\perp}$ invariant w.r.t. $A\iff PAP=PA\iff\overline{\mathbf{M}}$ invariant w.r.t. A^*
- $\overline{\mathbf{M}}$ reducing w.r.t. $A\iff\overline{\mathbf{M}}$ invariant w.r.t. A and A^*
- $A=A^*$: $\overline{\mathbf{M}}$ reducing w.r.t. $A\iff\overline{\mathbf{M}}$ invariant w.r.t. $A\iff PA=AP$
- $\mathbf{M}$ reducing w.r.t. $A\Longrightarrow Ax=PAPx+(I-P)A(I-P)x\ \forall\,x\in\mathbf{H}$

Example 2.19 *Let* $\mathbf{H} = \ell^2(\mathbb{Z})$. *For* $\xi \in \ell^2(\mathbb{Z})$, *we define* $f(t) = \sum_{k=-\infty}^{\infty} \xi_k t^k$, $t \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, *so that* $f \in \mathbf{L}^2(\mathbb{T})$ *and* $\xi_k = \widehat{f}_k := \frac{1}{2\pi}\int_0^{2\pi} f(e^{\mathbf{i}s})e^{-\mathbf{i}ks}\,ds$. *For* $a \in \mathbf{C}(\mathbb{T})$, *the operator* $T^o(a) : \mathbf{H} \longrightarrow \mathbf{H}$ *defined by* $\left(T^o(a)\xi\right)_j = \widehat{af}_j$, $j \in \mathbb{Z}$, *is linear and bounded. Moreover,* $T^o(a)\xi = \left(\sum_{k=-\infty}^{\infty} \widehat{a}_{j-k}\xi_k\right)_{j\in\mathbb{Z}}$, *and* $T^o(a)$ *is selfadjoint if and only if* $a(t)$ *is real valued. The subspace* $\ell^2_+(\mathbb{Z}) = \left\{\xi \in \ell^2(\mathbb{Z}) : \xi_k = 0,\ k < 0\right\}$ *is invariant w.r.t.* $T^o(a)$ *if and only if* $\widehat{a}_k = 0\ \forall\, k < 0$. *The operator* $T(a) : \ell^2_+(\mathbb{Z}) \longrightarrow \ell^2_+(\mathbb{Z})$, $\xi \mapsto P_+T^o(a)P_+$, *where* $P_+\xi = (\dots, 0, \dots, 0, \xi_0, \xi_1, \dots)$, *is called* **Toeplitz operator** *with symbol* $a(t)$.

Example 2.20 *For* $[a,b] \subset (0,1)$, *we consider the orthoprojection* $P : \mathbf{L}^2(0,1) \longrightarrow \mathbf{L}^2(0,1)$, $u \mapsto Pu$, *where* $(Pu)(t) = \begin{cases} u(t) & : \quad t \in [a,b], \\ 0 & : \quad t \in (0,1)\setminus[a,b]. \end{cases}$ *Further, let* $k : [0,1]\times[0,1] \longrightarrow \mathbb{C}$ *be a continuous function and* $K : \mathbf{L}^2(0,1) \longrightarrow \mathbf{L}^2(0,1)$ *be defined by*

$$(Ku)(t) = \int_0^1 k(t,s)u(s)\,ds\,.$$

Then PKP *is given by*

$$(PKP)(t) = \begin{cases} \displaystyle\int_a^b k(t,s)u(s)\,ds & : \quad t \in [a,b], \\ 0 & : \quad t \in (0,1)\setminus[a,b]. \end{cases}$$

2.3 Isometric and unitary operators

Let $(\mathbf{H}, \langle .,. \rangle)$ and $(\mathbf{H}_j, \langle .,. \rangle_j)$, $j = 1, 2$, be Hilbert spaces.

Definition 2.21 *A surjective map* $V : \mathbf{H}_1 \longrightarrow \mathbf{H}_2$ *is called* **isometric operator** *if*

$$\langle Vx, Vy\rangle_2 = \langle x, y\rangle_1 \quad \forall\, x, y \in \mathbf{H}_1\,.$$

An isometric operator $U : \mathbf{H} \longrightarrow \mathbf{H}$ *is called* **unitary operator**.

Corollary 2.22 *An isometric operator is invertible and linear.*

Definition 2.23 *An operator* $V \in \mathcal{L}(\mathbf{H})$ *is called a* **partial isometry** *if*

$$\langle Vx, Vx\rangle = \langle x, x\rangle \quad \forall\, x \in N(V)^\perp\,.$$

Corollary 2.24 *If* $V \in L(\mathbf{H}_1, \mathbf{H}_2)$, $R(V) = \mathbf{H}_2$, *and* $\langle Vx, Vx\rangle_2 = \langle x, x\rangle\ \forall\, x \in \mathbf{H}_1$ *then* $V : \mathbf{H}_1 \longrightarrow \mathbf{H}_2$ *is isometric.*

Proposition 2.25 *For* $V \in \mathcal{L}(\mathbf{H})$, *the following assertions are equivalent:*

(a) V *is a partial isometry.*

(b) $V = VV^*V$.

(c) V^*V *is an orthoprojection.*

Example 2.26 *Let* $(e_n)_{n=0}^\infty$ *be a CONS in* $\mathbf{H}$. *Then the operator*

$$V : \mathbf{H} \longrightarrow \mathbf{H}\,, \quad x \mapsto \sum_{n=0}^\infty \langle x, e_{n+1}\rangle\, e_n$$

is a partial isometry.

Example 2.27 *Let* $\varphi(t) = \sqrt{1-t^2}$, $t \in [-1,1]$, *and define the Hilbert space* $\mathbf{L}^2_\varphi$ *as the space of all w.r.t. the weight* $\varphi(t)$ *square integrable (classes of) functions* $f : (-1,1) \longrightarrow \mathbb{C}$ *equipped with the inner product*

$$\langle f, g\rangle_\varphi = \int_{-1}^1 f(t)\overline{g(t)}\varphi(t)\, dt\,.$$

The systems $(\varphi_n)_{n=0}^\infty$ *and* $(U_n)_{n=0}^\infty$ *form CONS in* $\mathbf{L}^2_\varphi$, *where* $\varphi_n(t) = \dfrac{T_n(t)}{\sqrt{1-t^2}}$ *and where*

$$T_0(t) = \frac{1}{\sqrt{\pi}}\,, \ T_n(cos\theta) = \sqrt{\frac{2}{\pi}}\cos(n\theta)\,,\ n>0\,, \quad U_n(\cos\theta) = \sqrt{\frac{2}{\pi}}\frac{\sin((n+1)\theta)}{\sin\theta}\,,\ n \ge 0\,,$$

are the Chebyshev polynomials of first and second kind, respectively. The relations

$$\frac{1}{\pi}\,\text{p.v.}\int_{-1}^1 \frac{T_n(s)\, ds}{(s-t)\sqrt{1-s^2}} = U_{n-1}(t)\,, \quad -1<t<1\,,\ n \in \mathbb{N}_0\,,\ U_{-1}(t) = 0\,,$$

show that the operator defined by

$$(Sf)(t) = \frac{1}{\pi}\,\text{p.v.}\int_{-1}^1 \frac{f(s)\, ds}{s-t}\,, \quad -1<t<1\,,$$

on the linear subspace span$\{\sigma\varphi_n : n \in \mathbb{N}_0\}$ *can be continuously extended to an operator* $S \in \mathcal{L}(\mathbf{L}^2_\varphi)$, *namely by*

$$Sf = \sum_{n=1}^\infty \langle f, \varphi_n\rangle_\varphi\, U_n\,.$$

The operator $S : \mathbf{L}^2_\varphi \longrightarrow \mathbf{L}^2_\varphi$ *defined in this way is a partial isometry.*

Example 2.28 *In* $\mathbf{L}^2(\mathbb{R})$, *the system* $\left\{t^k e^{-\frac{t^2}{2}} : k \in \mathbb{N}_0\right\}$ *is linearly independent. Schmidt's orthogonalization procedure leads to* $\{\varphi_k(t) : k \in \mathbb{N}_0\}$ *with*

$$\varphi_k(t) = H_k(t)e^{-\frac{t^2}{2}}\,, \quad k \in \mathbb{N}_0\,,$$

where $H_k(t)$ *denotes the kth (normalized) Hermite-polynomial,*

$$H_k(t) = \frac{(-1)^k}{\sqrt{\chi_k}}\, e^{t^2}\left(\frac{d}{dt}\right)^k e^{-t^2}\,, \quad \chi_k = \sqrt{\pi}\, 2^k k!\,.$$

The system $(\varphi_n)_{n=0}^\infty$ *is a CONS in* $\mathbf{L}^2(\mathbb{R})$. *The operator* F *defined by*

$$(Fg)(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\mathbf{i}st} g(s)\, ds$$

has the property $F\varphi_k = (-\mathbf{i})^k\varphi_k$, *which allows us to extend this operator in a unique way from* span$\{\varphi_k : k \in \mathbb{N}_0\}$ *to a unitary operator* $F : \mathbf{L}^2(\mathbb{R}) \longrightarrow \mathbf{L}^2(\mathbb{R})$.

Chapter 3

Spectral properties of selfadjoint and compact operators

3.1 Continuous functions of selfadjoint operators

Let $\mathcal{P}(\mathbb{R})$ be the set of all algebraic polynomials with real coefficients. For a selfadjoint operator $A \in \mathcal{L}(\mathbf{H})$, by $\mathbf{C}_A(\mathbb{R})$ we denote the set of all continuous functions $f : [m_A, M_A] \longrightarrow \mathbb{R}$. If $f \in \mathbf{C}_A(\mathbb{R})$ then we put

$$m_A(f) = \min \{f(t) : m_A \le t \le M_A\} \ , \quad M_A(f) = \max \{f(t) : m_A \le t \le M_A\}$$

and

$$\gamma_A(f) = \max \{|f(t)| : m_A \le t \le M_A\} \ .$$

Moreover, by $\mathrm{comm}(A)$ we refer to the set of all operators from $\mathcal{L}(\mathbf{H})$, which commute with $A \in \mathcal{L}(\mathbf{H})$.

Proposition 3.1 *If the polynomial* $p \in \mathcal{P}(\mathbb{R})$ *is nonnegative on the spectral interval* $[m_A, M_A]$ *of the selfadjoint operator* $A \in \mathcal{L}(\mathbf{H})$, *then* $p(A) \ge \Theta$.

Proposition 3.2 *Let* $A \in \mathcal{L}(\mathbf{H})$ *be selfadjoint. The map* $\mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathbf{H})$, $p \mapsto p(A)$ *can be uniquely extended to a linear and multiplicative map*

$$\mathbf{C}_A(\mathbb{R}) \longrightarrow \mathcal{L}(\mathbf{H}) , \quad f \mapsto f(A)$$

satisfying $\|f(A)\| \le \gamma_A(f)$. *For all* $f \in \mathbf{C}_A(\mathbb{R})$ *and* $g \in \mathbf{C}([m_A(f), M_A(f)], \mathbb{R})$ *we have*

(a) $m_A(f)I \le f(A) \le M_A(f)I$,

(b) $\mathrm{comm}(A) \subset \mathrm{comm}(f(A))$,

(c) $g(f(A)) = (g \circ f)(A)$.

Remark 3.3 *Multiplicativity of the map* $f \longrightarrow f(A)$ *(mentioned in Theorem 3.2) means that, for all* $f, g \in \mathbf{C}_A(\mathbb{R})$, *we have* $(fg)(A) = f(A)g(A)$.

Corollary 3.4 *The map* $\mathbf{C}_A(\mathbb{R}) \longrightarrow \mathcal{L}(\mathbf{H})$, $f \mapsto f(A)$ *is positive, i.e.,* $f(t) \ge 0 \ \forall\, t \in [m_A, M_A]$ *implies* $f(A) \ge \Theta$.

Corollary 3.5 *For each positive operator* $A \in \mathcal{L}(\mathbf{H})$ *there exists exactly one positive operator* $B \in \mathcal{L}(\mathbf{H})$ *with* $B^2 = A$, *namely* $B = \sqrt{A}$.

Corollary 3.6 *If the positive operators* $A, B \in \mathcal{L}(\mathbf{H})$ *commute then the operator* AB *is also positive.*

Corollary 3.7 *Let* $A \in \mathcal{L}(\mathbf{H})$ *be selfadjoint. Then the operators*

$$|A| \quad \textit{and} \quad A^{\pm} = \frac{1}{2}(|A| \pm A)$$

are positive, where

$$|A| = \sqrt{A^2} \quad \textit{and} \quad A^+A^- = \Theta .$$

Remark that it is possible to define $|A| = \sqrt{A^*A}$ for all operators $A \in \mathcal{L}(\mathbf{H})$. In case of $A^* = A$ this definition coincides with the above one.

Proposition 3.8 (polar decomposition) *For each operator* $A \in \mathcal{L}(\mathbf{H})$ *there exists exactly one operator* $V \in \mathcal{L}(\mathbf{H})$ *such that* $A = V\sqrt{A^*A}$ *and* $N(A) \subset N(V)$. *Moreover,* V *is a partial isometry,* $\sqrt{A^*A} = V^*A$, *and* $I - V^*V = P_{N(A)}$.

3.2 Spectral decomposition of selfadjoint operators

Let $\lambda \in \mathbb{R}$, $e_\lambda(t) = \begin{cases} 1 & : \ t < \lambda , \\ 0 & : \ \lambda \le t , \end{cases}$ and, for $n \in \mathbb{N}$, $e_{\lambda,n}(t) = \begin{cases} 1 & : \ t < \lambda - \frac{1}{n} , \\ n(\lambda - t) & : \ \lambda - \frac{1}{n} \le t \le \lambda , \\ 0 & : \ \lambda < t . \end{cases}$

Then, for a selfadjoint operator $A \in \mathcal{L}(\mathbf{H})$, $n \in \mathbb{N}$, and $\lambda \le \mu$, we have (see Corollary 3.4)

(Ea) $\Theta \le e_{\lambda,n}(A) \le e_{\lambda,n+1}(A) \le I$,

(Eb) $I - e_{\lambda,2n}(A) \le [I - e_{\lambda,n}(A)]^2 \le I - e_{\lambda,n}(A)$,

(Ec) $e_{\lambda,n}(A) \le e_{\mu,n}(A)$.

By Proposition 2.8 and (Ea) we get the existence of $E_\lambda \in \mathcal{L}(\mathbf{H})$ with

$$e_{\lambda,n}(A) \longrightarrow E_\lambda =: e_\lambda(A) . \tag{3.1}$$

Proposition 3.9 *Assume that* $f \in \mathbf{C}_A(\mathbb{R})$, $a, b \in \mathbb{R}$, *and* $\lambda \le \mu$. *Then*

(a) $a(E_\mu - E_\lambda) \le f(A)(E_\mu - E_\lambda)$ *if* $a \le f(t)$, $t \in [\lambda, \mu]$,

(b) $f(A)(E_\mu - E_\lambda) \le b(E_\mu - E_\lambda)$ *if* $f(t) \le b$, $t \in [\lambda, \mu]$.

Proposition 3.10 *The operators* E_λ, $\lambda \in \mathbb{R}$, *are orthoprojections satisfying the following properties:*

(a) $E_\lambda = \Theta$ *for* $\lambda \le m_A$ *and* $E_\lambda = I$ *for* $M_A < \lambda$,

(b) $(A - \lambda I)E_\lambda \le \Theta \le (A - \lambda I)(I - E_\lambda)$,

(c) $E_\lambda \le E_\mu$ *for* $\lambda \le \mu$,

(d) $\lambda(E_\mu - E_\lambda) \le A(E_\mu - E_\lambda) \le \mu(E_\mu - E_\lambda)$ *for* $\lambda \le \mu$,

(e) $E_\lambda x = \theta$ *if* $A\,x = \lambda x$.

Moreover, for each $\lambda \in \mathbb{R}$ *and each* $B \in \operatorname{comm}(A)$ *the space* $R(E_\lambda)$ *is a reducing subspace of* B .

Proposition 3.11 *For each selfadjoint operator* $A \in \mathcal{L}(\mathbf{H})$ *and for each* $\lambda \in \mathbb{R}$ *there exists exactly one orthoprojection* $E_\lambda \in \mathcal{L}(\mathbf{H})$ *satisfying (b) and (e) of Proposition 3.10 and commuting with* A . *Moreover,*

$$A(I - E_0) = A^+ , \quad A\,E_0 = -A^- , \tag{3.2}$$

and

$$A(I - 2E_0) = |A| , \quad A = |A|(I - 2E_0) . \tag{3.3}$$

Corollary 3.12 *For all selfadjoint operators* $B \in \operatorname{comm}(A)$ *with* $\pm A \le B$, *we have* $|A| \le B$, *i.e.,* $|A|$ *is the smallest selfadjoint operator with this property.*

Corollary 3.13 *In the sense of strong operator convergence, the one-sided limits*

$$E_{\lambda-0} = \lim_{\mu\uparrow\lambda} E_\mu , \quad E_{\lambda+0} = \lim_{\mu\downarrow\lambda} E_\mu , \quad E_{+\infty} = \lim_{\mu\uparrow+\infty} E_\mu , \quad E_{-\infty} = \lim_{\mu\downarrow-\infty} E_\mu$$

exist for each $\lambda \in \mathbb{R}$, *where* $E_{\lambda-0} = E_\lambda \le E_{\lambda+0}$.

Let $\mathcal{O}(\mathbf{H}) \subset \mathcal{L}(\mathbf{H})$ denote the set of orthoprojections in $\mathbf{H}$.

Definition 3.14 *A map* $\mathbb{R} \longrightarrow \mathcal{O}(\mathbf{H})$, $\lambda \longrightarrow E_\lambda$ *is called a* **spectral map** *if* $E_\lambda \le E_\mu$ *for* $\lambda \le \mu$ *and if* $E_{-\infty} = \Theta$, $E_\infty = I$. *It is called left-sided continuous if* $E_{\lambda-0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$.

In what follows let $\lambda \longrightarrow E_\lambda$ be the spectral map defined for the selfadjoint operator $A \in \mathcal{L}(\mathbf{H})$ by (3.1).

Proposition 3.15 *Let* $A \in \mathcal{L}(\mathbf{H})$ *be selfadjoint. Then*

(a) $R(E_{\lambda+0} - E_\lambda) = N(A - \lambda I)\ \forall\, \lambda \in \mathbb{R}$,

(b) $\lambda \in \mathbb{R} \setminus \sigma(A) \iff \exists\, \varepsilon > 0$*:* $E_\lambda = E_\mu\ \forall\, \mu \in (\lambda - \varepsilon, \lambda + \varepsilon)$.

The numbers of $\sigma(A)$ *which are not eigenvalues are called points of the* **continuous spectrum** $\sigma_c(A)$ *of the operator* A , *while the set* $\sigma_p(A) \subset \sigma(A)$ *of eigenvalues is called* **point spectrum** *of* A . *Consequently,* $\lambda \in \sigma_c(A)$ *if and only if* $E_{\lambda+0} = E_\lambda$ *and* $E_{\lambda+\varepsilon} \ne E_{\lambda-\varepsilon}$ *for all* $\varepsilon > 0$.

Example 3.16 *Let* $\mathbf{H} = \mathbf{L}^2(-1,1)$ *and* $A \in \mathcal{L}(\mathbf{H})$ *be defined by* $(A\,x)(t) = t\,x(t)$. *We show that* $\sigma(A) = \sigma_c(A) = [-1,1]$.

Let $f : [a,b) \longrightarrow \mathbb{R}$ be a function, let $Z = \{\lambda_0, \lambda_1, \ldots, \lambda_n\} \in \mathcal{Z}[a,b]$ be a partition of the interval $[a,b]$, and let

$$S(f,Z,\mu) = \sum_{j=0}^{n} f(\mu_j)\left(E_{\lambda_j} - E_{\lambda_{j-1}}\right)$$

be a respective **Riemann-Stieltjes sum** w.r.t. the spectral map E_λ , where μ is a set of numbers $\{\mu_j : j = 1, \ldots, n\}$ with $\lambda_{j-1} \le \mu_j < \lambda_j$. By $d(Z) = \max\{|\lambda_j - \lambda_{j-1}| : j = 1, \ldots, n\}$ we denote the diameter of the partition Z .

Definition 3.17 *A function* $f : [a,b] \longrightarrow \mathbb{R}$ *is said to be* **Riemann-Stieltjes integrable** *or* **uniformly Riemann-Stieltjes integrable** *on* $[a,b]$ *w.r.t.* E_λ *if, for each sequence* $(Z_m)_{m=1}^\infty$ *of partitions* $Z_m \in \mathcal{Z}[a,b]$ *with* $\lim_{m\to\infty} d(Z_m) = 0$, *an arbitrary respective sequence of Riemann-Stieltjes sums* $(S(f, Z_m, \mu^m))_{m=1}^m$ *converges in the strong or in the norm convergence operator topology, respectively. In this case we set*

$$\int_a^{b-0} f(\lambda)\, dE_\lambda := \lim_{m\to\infty} S(Z_m)$$

and

$$\int_a^b f(\lambda)\, dE_\lambda := \int_a^{b-0} f(\lambda)\, dE_\lambda + f(b)\,(E_{b+0} - E_b)\ .$$

Proposition 3.18 *For a left-sided continuous spectral map* E_λ, *any continuous function* $f : [a,b] \longrightarrow \mathbb{R}$ *is uniformly Riemann-Stieltjes integrable.*

Let E_λ be a left-sided continuous spectral map. We set $\mathcal{J}_a^b(f) := \displaystyle\int_a^{b-0} f(\lambda)\, dE_\lambda$ for $f \in \mathbf{C}([a,b],\mathbb{R})$. Then, for $f, g \in \mathbf{C}([a,b],\mathbb{R})$, the following rules hold:

1. $\mathcal{J}_a^b(f) = \mathcal{J}_a^c(f) + \mathcal{J}_c^b(f)$, $a < c < b$.
2. $\mathcal{J}_a^b(f)\,(E_d - E_c) = \mathcal{J}_c^d(f)$, $a \le c < d \le b$.
3. $\mathcal{J}_a^b(f+g) = \mathcal{J}_a^b(f) + \mathcal{J}_a^b(g)$, $\mathcal{J}_a^b(\alpha f) = \alpha \mathcal{J}_a^b(f)$, $\alpha \in \mathbb{R}$.
4. $\mathcal{J}_a^b(fg) = \mathcal{J}_a^b(f)\mathcal{J}_a^b(g)$.
5. $\mathcal{J}_a^b(f) \ge \Theta$ if $f(t) \ge 0\ \forall\, t \in [a,b]$.

To each operator $A \in \mathcal{L}(\mathbf{H})$ one can associate its **real part** $\mathsf{Re}\,A = \frac{1}{2}(A + A^*)$ and its **imaginary part** $\mathsf{Im}\,A = \frac{1}{2\mathbf{i}}(A - A^*)$. Then

$$A = \mathsf{Re}\,A + \mathbf{i}\,\mathsf{Im}\,A \quad \text{and} \quad A^* = \mathsf{Re}\,A - \mathbf{i}\,\mathsf{Im}\,A\,,$$

$\mathsf{Re}\,A$ and $\mathsf{Im}\,A$ are selfadjoint operators. Conversely, if $A = A_1 + \mathbf{i}\,A_2$ with selfadjoint operators $A_j \in \mathcal{L}(\mathbf{H})$ then $A_1 = \mathsf{Re}\,A$ and $A_2 = \mathsf{Im}\,A$.

We call an operator $A \in \mathcal{L}(\mathbf{H})$ **normal** if $A^*A = A\,A^*$. An operator $A \in \mathcal{L}(\mathbf{H})$ is normal if and only if $\mathsf{Re}\,A$ and $\mathsf{Im}\,A$ commute, which is also equivalent to

$$A^*A = A\,A^* = (\mathsf{Re}\,A)^2 + (\mathsf{Im}\,A)^2\,.$$

For a selfadjoint operator $A \in \mathcal{L}(\mathbf{H})$ and a function $f = f_1 + \mathbf{i}\,f_2 \in \mathbf{C}_A(\mathbb{C})$ with $f_j \in \mathbf{C}_A(\mathbb{R})$, the operator

$$f(A) = f_1(A) + \mathbf{i}\,f_2(A)$$

is well defined, where $f_1(A) = \mathsf{Re}\,f(A)$ and $f_2(A) = \mathsf{Im}\,f(A)$. Moreover,

1. $f(A)^* = \overline{f}(A)$,
2. $(\lambda f + \mu g)(A) = \lambda f(A) + \mu g(A)$, $f, g \in \mathbf{C}_A(\mathbb{C})$, $\lambda, \mu \in \mathbb{C}$,
3. $(fg)(A) = f(A)g(A)$, $f, g \in \mathbf{C}_A(\mathbb{C})$.

If, for a selfadjoint operator $A \in \mathcal{L}(\mathbf{H})$, we define $e^{\mathbf{i}A} = \cos(A) + \mathbf{i}\,\sin(A) =: V$, then $V^*V = V\,V^* = I$, i.e., V is a **unitary operator**.

Proposition 3.19 *If $A \in \mathcal{L}(\mathbf{H})$ is selfadjoint, $f \in \mathbf{C}[a,b]$, if $a \le m_A \le M_A < b$, and if E_λ is the spectral map of A, then*

$$f(A) = \int_a^b f(\lambda)\, dE_\lambda\,.$$

Moreover, $\operatorname{comm}(A) = \operatorname{comm}\{E_\lambda : \lambda \in \mathbb{R}\}$.

3.3 Compact operators

It is well known that a linear operator $T : \mathbf{H} \longrightarrow \mathbf{H}$ is **compact** if and only if T maps each bounded sequence $(x_n)_{n=1}^\infty$ with $x_n \in \mathbf{H}$ into a precompact sequence $(T\,x_n)_{n=1}^\infty$. A sequence $(x_n)_{n=1}^\infty$ is called **precompact** if each subsequence possesses a convergent subsequence. The set of linear compact operators is denoted by $\mathcal{K}(\mathbf{H})$, and it is well known that $\mathcal{K}(\mathbf{H}) \subset \mathcal{L}(\mathbf{H})$. Moreover, $\mathcal{K}(\mathbf{H})$ is a closed set (w.r.t. the operator norm topology).

Example 3.20 *Let $h : [0,1]^2 \longrightarrow \mathbb{C}$ be a Lebesgue measurable function with*

$$\iint_{[0,1]^2} |h(t,s)|^2\, ds\, dt < \infty\,.$$

Then the operator $T : \mathbf{L}^2(0,1) \longrightarrow \mathbf{L}^2(0,1)$ defined by

$$(Tu)(t) = \int_0^1 h(t,s)u(s)\, ds$$

is a compact operator. This can be seen, for example, as folows: By $\chi_j^{(n)}(t)$ we denote the characteristic function of the interval $\left[\frac{j-1}{n}, \frac{j}{n}\right]$, $j = 1, \ldots, n \in \mathbb{N}$. For any $\varepsilon > 0$, there exist numbers $n \in \mathbb{N}$ and $\lambda_{jk}^{(n)} \in \mathbb{C}$, such that

$$\iint_{[0,1]^2} \left| h(t,s) - \sum_{j=1}^n \sum_{k=1}^n \lambda_{jk}^{(n)} \chi_j^{(n)}(t) \chi_k^{(n)}(s) \right|^2 ds\, dt < \varepsilon\,.$$

The operator $\widetilde{T} : \mathbf{L}^2(0,1) \longrightarrow \mathbf{L}^2(0,1)$ with

$$(\widetilde{T}u)(t) = \int_0^1 \sum_{j=1}^n \sum_{k=1}^n \lambda_{jk}^{(n)} \chi_j^{(n)}(t) \chi_k^{(n)}(s) u(s)\, ds$$

has finite-dimensional range (the image space is spanned by the functions $\chi_j^{(n)}(t)$), is compact for this reason and fulfils the relation $\left\|\widetilde{T} - T\right\|_{\mathcal{L}(\mathbf{L}^2(0,1))} < \varepsilon$. Consequently, if we choose a sequence (ε_m) of positive numbers tending to zero, then we find operators $\widetilde{T}_m \in \mathcal{K}(\mathbf{L}^2(0,1))$ converging to T in norm. The closedness of $\mathcal{K}(\mathbf{L}^2(0,1))$ w.r.t. the operator norm toplogy implies the compactness of the operator T.

Proposition 3.21 *If $T \in \mathcal{K}(\mathbf{H})$ then $R(I-T)$ is closed.*

Corollary 3.22 *If $T \in \mathcal{K}(\mathbf{H})$ and $\lambda \in \mathbb{C} \setminus \{0\}$ then $R(T - \lambda I)$ is closed.*

Proposition 3.23 *For an operator $T \in \mathcal{K}(\mathbf{H})$ we have:*

(a) *If $\lambda \in \mathbb{C}$ and $\dim N(T - \lambda I) = \infty$ then $\lambda = 0$.*

(b) *The only possible accumulation point of the set of eigenvalues of T is zero.*

(c) *If $\lambda \in \mathbb{C}$ and if there is a sequence $(x_n)_{n=0}^{\infty}$, $x_n \in \mathbf{H}$, such that $x_0 \neq \Theta$, $(T - \lambda I)x_0 = \Theta$, and $(T - \lambda I)x_{n+1} = x_n$, $n = 0, 1, 2, \ldots$, then $\lambda = 0$.*

For an operator $T \in \mathcal{L}(\mathbf{H})$, the following assertions are equivalent:

(A) $T \in \mathcal{K}(\mathbf{H})$,

(B) $T^*T \in \mathcal{K}(\mathbf{H})$,

(C) $T^* \in \mathcal{K}(\mathbf{H})$.

Corollary 3.24 *If $T \in \mathcal{K}(\mathbf{H})$ then $\dim N(T - \lambda I) < \infty$ for all $\lambda \in \mathbb{C} \setminus \{0\}$, $\sigma(T) \setminus \{0\}$ consists only of at most countably many eigenvalues, which can only accumulate in 0. If $\lambda \neq 0$ is an eigenvalue of T then $\overline{\lambda}$ is an eigenvalue of T^*. In any case $0 \in \sigma(T)$.*

Consequently, for a compact operator $T \in \mathcal{L}(\mathbf{H})$ and $A = I - \mu T$, $\mu \in \mathbb{C}$ we have:

1. **Fredholm alternative**: $Ax = y$ is solvable in $\mathbf{H}$ for every $y \in \mathbf{H}$ if and only if this equation is uniquely solvable.
2. **Finite number of solvability conditions**: $Ax = y \in \mathbf{H}$ is solvable in $\mathbf{H}$ if and only if $y \in N(A^*)^\perp$, where $\dim N(A^*) < \infty$.

Proposition 3.25 *To each compact and selfadjoint operator $T \in \mathcal{L}(\mathbf{H})$ there exist a finite sequence or a zero sequence of real numbers $\lambda_0, \lambda_1, \ldots$ with $|\lambda_0| \geq |\lambda_1| \geq |\lambda_2| \geq \ldots$ and pairwise orthonormal elements $e_k \in \mathbf{H}$ such that*

$$T\,x = \sum_k \lambda_k \langle x, e_k \rangle\, e_k \quad \forall\, x \in \mathbf{H}\,.$$

If we define $P_k x = \langle x, e_k \rangle\, e_k$, $x \in \mathbf{H}$, then P_k is an orthoprojection and $P_k P_j = \Theta$ for $k \neq j$ as well as

$$T = \sum_k \lambda_k P_k\,,$$

where in the case of infinite many eigenvalues the series converges in the operator norm.

Proposition 3.26 *A linear operator $T : \mathbf{H} \longrightarrow \mathbf{H}$ is compact if and only if there is a sequence of linear operators with finite dimensional range which converges in the operator norm to T.*

Example 3.27 *The operator $K : \mathbf{L}^2_\sigma(-1,1) \longrightarrow \mathbf{L}^2_\sigma(-1,1)$, where $\sigma(t) = (1-t^2)^{-\frac{1}{2}}$, defined by*

$$(K\,u)(t) = -\frac{1}{\pi} \int_{-1}^{1} \ln|s - t|\, u(s)\, \sigma(s)\, ds$$

is compact. We give its diagonal representation according to Proposition 3.25.

Chapter 4

Spectral integrals

4.1 In general unbounded operators

An operator $A : D(A) \subset \mathbf{H} \longrightarrow \mathbf{H}$ is called **linear**, if its **domain** $D(A)$ is a linear subspace of $\mathbf{H}$ and if for all $x, y \in D(A)$, $\lambda, \mu \in \mathbb{C}$ we have $A(\lambda\, x + \mu\, y) = \lambda\, A\, x + \mu\, A\, y$. The set of all such linear operators is denoted by $L(\mathbf{H})$. An operator $A \in L(\mathbf{H})$ is called a **restriction** of $B \in L(\mathbf{H})$ (or B is an **extension** of A), denoted by $A \subset B$ or by $B \supset A$ of A) if $D(A) \subset D(B)$ and $A\, x = B\, x \;\forall\, x \in D(A)$. For $A, B \in L(\mathbf{H})$ and $\lambda \in \mathbb{C}$ we define

- $D(\lambda\, A) = D(A)$ and $(\lambda\, A)x = \lambda\, A\, x$, $x \in D(A)$,
- $D(A + B) = D(A) \cap D(B)$ and $(A + B)x = A\, x + B\, x$, $x \in D(A + B)$,
- $D(AB) = \{x \in D(B) : B\, x \in D(A)\}$ and $(AB)x = A(B\, x)$, $x \in D(AB)$.

Consequently, $D(A - \lambda\, I) = D(A) \quad \forall\, A \in L(\mathbf{H}), \lambda \in \mathbb{C}$. For $A_n \in L(\mathbf{H})$, $n \in \mathbb{N}$, we say that $A = \lim\limits_{n\to\infty} A_n$ if

- $D(A) = \left\{ x \in \mathbf{H} : \exists\, n_0 = n_0(x) \text{ with } x \in \bigcap\limits_{n=n_0}^{\infty} D(A_n) \right\}$ and

$$A\, x = \lim_{n\to\infty} A_n x \quad \forall\, x \in D(A).$$

Of course, $\lim\limits_{n\to\infty} A_n \in L(\mathbf{H})$. Furthermore, for $A \in L(\mathbf{H})$ we define

- $\operatorname{comm}(A) = \{B \in \mathcal{L}(\mathbf{H}) : B\, x \in D(A),\; AB\, x = BA\, x \;\forall\, x \in D(A)\}$.

Example 4.1 *Let us consider two examples of unbounded operators:*

(a) $\mathbf{H} = \mathbf{L}^2(\mathbb{R})$, $D(A) = \left\{ x \in \mathbf{H} : \int_{\mathbb{R}} t^2 |x(t)|^2 dt < \infty \right\}$, $(A\, x)(t) = t\, x(t)$.

(b) $\mathbf{H} = \mathbf{L}^2(\mathbb{R})$, $D(B) = \mathcal{S}(\mathbb{R}) := \left\{ x \in \mathbf{C}^\infty(\mathbb{R}) : \sup\limits_{t\in\mathbb{R}} |t^n x^{(m)}(t)| < \infty, \forall\, n, m \in \mathbb{N}_0 \right\}$,

$(B\, x)(t) = -x''(t) + t^2 x(t)$.

Definition 4.2 *If* $A \in L(\mathbf{H})$ *and* $\overline{D(A)} = \mathbf{H}$ *then the* **adjoint operator** $A^* \in L(\mathbf{H})$ *can be uniquely defined by*

$$D(A^*) = \Big\{ y \in \mathbf{H} : \sup \{ | \langle Ax, y \rangle | : x \in D(A), \, \|x\| = 1 \} < \infty \Big\}$$

and

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \forall \, x \in D(A) \, , \ \forall \, y \in D(A^*) \, .$$

If $A \subset B$ then $B^* \subset A^*$.

Definition 4.3 *Let* $A \in L(\mathbf{H})$ *and* $\overline{D(A)} = \mathbf{H}$. *A number* $\lambda \in \mathbb{C}$ *is called regular point of* A *if there exists an operator* $R_\lambda \in \mathcal{L}(\mathbf{H})$ *such that*

$$R_\lambda (A - \lambda \, I) x = x \ \forall \, x \in D(A) \quad \text{and} \quad (A - \lambda \, I) R_\lambda x = x \ \forall \, x \in \mathbf{H} \, .$$

The set of all $\lambda \in \mathbb{C}$ *which are not regular points of* A *is called the* **spectrum** *of* A *and denoted by* $\sigma(A)$. *The set* $\rho(A) = \mathbb{C} \setminus \sigma(A)$ *is called the* **resolvent set** *of* A.

Remark: If R_λ fulfils Definition 4.3 then $D(A) = R(R_\lambda)$.

Proposition 4.4 *Let* $A \in L(\mathbf{H})$ *with* $\overline{D(A)} = \mathbf{H}$. *Then,* $\rho(A)$ *is open, i.e.,* $\sigma(A)$ *is closed. Moreover,*

$$\operatorname{comm}(A) = \operatorname{comm}(R_\lambda) \quad \forall \, \lambda \in \rho(A) \, .$$

If $\lambda, \mu \in \rho(A)$ *then*

$$R_\lambda - R_\mu = (\lambda - \mu) R_\mu R_\lambda \quad \text{and} \quad R_\lambda R_\mu = R_\mu R_\lambda \, .$$

If $\lambda \in \rho(A)$ one also writes $R_\lambda = (A - \lambda I)^{-1}$, since in this case $A - \lambda T : D(A) \longrightarrow \mathbf{H}$ is a bijection with a bounded inverse.

Definition 4.5 *An operator* $A \in L(\mathbf{H})$ *is called* **symmetric** *if*

$$\overline{D(A)} = \mathbf{H} \quad \text{and} \quad \langle A \, x, y \rangle = \langle x, A \, y \rangle \ \forall \, x, y \in D(A) \, .$$

We say that $A \in L(\mathbf{H})$ *is* **selfadjoint** *if* $A = A^*$.

Proposition 4.6 *For* $A \in L(\mathbf{H})$ *the following conditions are equivalent:*

(a) A *is symmetric.*

(b) $A \subset A^*$.

(c) $\overline{D(A)} = \mathbf{H}$ *and* $\langle A \, x, x \rangle \in \mathbb{R} \ \forall \, x \in D(A)$.

An operator $A \in L(\mathbf{H})$ *is selfadjoint if and only if*

$$\overline{D(A)} = \mathbf{H} \, , \quad \langle A \, x, x \rangle \in \mathbb{R} \ \forall \, x \in D(A) \, , \quad \text{and} \quad D(A^*) \subset D(A) \, .$$

The **graph** of $A \in L(\mathbf{H})$ is defined as the set

$$\Gamma(A) = \{ (x, A \, x) : x \in D(A) \} \subset \mathbf{H} \times \mathbf{H} \, .$$

For $A, B \in L(\mathbf{H})$, the relation $A \subset B$ is equivalent to $\Gamma(A) \subset \Gamma(B)$. Let us $\mathbf{H} \times \mathbf{H}$ equip with the inner product

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{\mathbf{H} \times \mathbf{H}} := \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle \,.$$

Then, the operator $A \in L(\mathbf{H})$ is called a **closed operator**, if $\Gamma(A)$ is a closed subset of $\mathbf{H} \times \mathbf{H}$. This is equivalent to

$$x_n \in D(A)\,,\ x_n \longrightarrow x\,,\ y_n = A\,x_n \longrightarrow y \implies x \in D(A)\,,\ y = A\,x\,.$$

The **closed graph theorem** says that a closed operator $A \in L(\mathbf{H})$ with $D(A) = \mathbf{H}$ is in $\mathcal{L}(\mathbf{H})$.

Corollary 4.7 *If $A \in L(\mathbf{H})$ is selfadjoint, if $B \in L(\mathbf{H})$ is symmetric, and if $A \subset B$ then $A = B$. If $A \in L(\mathbf{H})$ is symmetric with $D(A) = \mathbf{H}$ then $A \in \mathcal{L}(\mathbf{H})$.*

Definition 4.8 *An operator $A \in L(\mathbf{H})$ is called* **closable**, *if there exists a closed operator $B \in L(\mathbf{H})$ such that $A \subset B$.*

Proposition 4.9 *If A is closable then there exists a smallest closed extension $\overline{A}$, the* **closure** *of A, in the sense that $\overline{A} \subset B$ for every closed extension B of A. If $A \in L(\mathbf{H})$ is closable then $\Gamma(\overline{A}) = \overline{\Gamma(A)}$.*

Example 4.10 *We show that, for each Hilbert space, there exists a linear operator which is not closable.*

Remark 4.11 *If $A \in L(\mathbf{H})$ is closed with $\overline{D(A)} = \mathbf{H}$ then $\lambda \in \rho(A)$ if and only if $A - \lambda I : D(A) \longrightarrow \mathbf{H}$ is a bijection.*

4.2 Selfadjoint operators

Proposition 4.12 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, E_λ be a left-sided continuous spectral map, $J_n(f) = \int_{-n}^{n-0} f(\lambda)\, dE_\lambda$, $D(A) = \left\{ x \in \mathbf{H} : \exists \lim\limits_{n\to\infty} J_n(f)x \right\}$, and $A\,x = \lim\limits_{n\to\infty} J_n(f)x$. Then $A = J(f) \in L(\mathbf{H})$ is a selfadjoint operator, for which we also write*

$$A = J(f) = \int_{-\infty}^{\infty} f(\lambda)\, dE_\lambda := \lim_{n\to\infty} \int_{-n}^{n-0} f(\lambda)\, dE_\lambda\,.$$

If f is bounded then $A \in \mathcal{L}(\mathbf{H})$.

Proposition 4.13 *Let $A \in L(\mathbf{H})$ be a selfadjoint operator. Then*

(a) $\sigma(A) \subset \mathbb{R}$,

(b) $R_\lambda = (A - \lambda I)^{-1}$ *is normal if* $\lambda \in \rho(A)$ *and selfadjoint if* $\lambda \in \rho(A) \cap \mathbb{R}$,

(c) *there exists a unique spectral map* E_λ *with* $A = \displaystyle\int_{-\infty}^{\infty} \lambda\, dE_\lambda$. *Moreover,*

$$\mathrm{comm}(A) = \mathrm{comm}(E_\lambda)\,.$$

The following Lemma is needed for the proof of Proposition 4.13,(c).

Lemma 4.14 *Let* $A \in \mathcal{L}(\mathbf{H})$.

(a) *If* $A = V\sqrt{A^*A}$ *is the polar decomposition of* A (*cf. Proposition 3.8*) *then* $A^* = V^*\sqrt{A\,A^*}$ *is the polar decomposition of* A^* . *Moreover,* $\sqrt{A\,A^*} = V\sqrt{A^*A}V^*$.

(b) *If* A *is normal then there exists a unitary operator* $V \in \mathcal{L}(\mathbf{H})$ *such that* $V\,A = A\,V$ *and* $A = V\sqrt{A^*A}$.

Chapter 5

Differential operators in $\mathbf{L}^2(a,b)$

Recall that by $\Gamma(A)$ we denote the graph of the operator $A \in L(\mathbf{H})$. bezeichnen. Furthermore, we define $V : \mathbf{H} \times \mathbf{H} \longrightarrow \mathbf{H} \times \mathbf{H}$, $(x,y) \mapsto (-y,x)$, where we equip $\mathbf{H} \times \mathbf{H}$ with the inner product $\langle (x,y),(z,w) \rangle := \langle x,z \rangle + \langle y,w \rangle$.

Lemma 5.1 *Let $A \in L(\mathbf{H})$ and $\overline{D(A)} = \mathbf{H}$. The, we have:*

(a) $\Gamma(A^*) = V(\Gamma(A))^{\perp}$.

(b) A^* *is a closed operator.*

(c) A *is closable if and only if* $\overline{D(A^*)} = \mathbf{H}$. *In this case* $\overline{A} = (A^*)^* =: A^{**}$.

(d) *If* A *is closable, then* $(\overline{A})^* = A^*$.

(e) A *is selfadjoint if and only if* $\Gamma(A) \perp V(\Gamma(A))$ *and* $\Gamma(A) + V(\Gamma(A)) = \mathbf{H} \times \mathbf{H}$.

Lemma 5.2 *Let $F : \mathbf{X} \longrightarrow \mathbb{C}$ and $F_j : \mathbf{X} \longrightarrow \mathbb{C}$, $j = 1, \ldots, n$, be linear functionals on the linear space $\mathbf{X}$, where*

$$\bigcap_{j=1}^{n} N(F_j) \subset N(F).$$

Then, there exist $\gamma_1, \ldots, \gamma_n \in \mathbb{C}$, such that $F = \sum_{j=1}^{n} \gamma_j F_j$.

In what follows let $-\infty \le a < b \le \infty$ and $\mathbf{H} = \mathbf{L}^2(a,b)$. A function $f : (a,b) \longrightarrow \mathbb{C}$ is called **absolutely continuous**, if there exists a locally integrable function $g : (a,b) \longrightarrow \mathbb{C}$ (we write $g \in \mathbf{L}^1_{\text{loc}}(a,b)$) such that

$$f(x) = f(c) + \int_c^x g(y)\,dy \quad \forall\, x \in (a,b),\ \forall\, c \in (a,b).$$

In this case we write $g = f'$. By $\mathcal{A}_n(a,b)$ we denote the set of all $n-1$ times continuously differentiable functions $f : (a,b) \longrightarrow \mathbb{C}$, for which $f^{(n-1)} : (a,b) \longrightarrow \mathbb{C}$ is absolutely continuous.

Proposition 5.3 *For all $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists a constant $c_0 = c_0(n,\varepsilon)$ such that*

$$\int_0^1 |f^{(j)}(x)|^2\,dx \le \varepsilon \int_0^1 |f^{(n)}(x)|^2\,dx + c_0 \int_0^1 |f(x)|^2\,dx \quad \forall\, f \in \mathcal{A}_n(0,1),\ \forall\, j = 0,1,\ldots,n-1.$$

Corollary 5.4 *If* $f \in \mathcal{A}_n(a,b) \cap \mathbf{L}^2(a,b)$ *and* $f^{(n)} \in \mathbf{L}^2(a,b)$ *then* $f^{(j)} \in \mathbf{L}^2(a,b)$, $j = 0,1,\ldots,n$.

We set $\mathbf{W}_r^2(a,b) = \left\{ f \in \mathcal{A}_r(a,b) \cap \mathbf{L}^2(a,b) : f^{(r)} \in \mathbf{L}^2(a,b) \right\}$. (Sobolev space of order r)

Corollary 5.5 *If* $f \in \mathbf{W}_r^2(a,b)$ *then* $f^{(j)} \in \mathbf{C}[a,b]$, $j = 0,1,\ldots,r-1$, *where in case of* $b = \infty$ $(a = -\infty)$ *we have* $f^{(j)}(\infty) = 0$ $(f^{(j)}(-\infty) = 0)$, $j = 0,1,\ldots,r-1$.

By $D : \mathcal{A}_1(a,b) \longrightarrow \mathbf{L}^1_{\mathrm{loc}}(a,b)$ we denote the differential operator $Df = f'$. Define

$$D(A_r^0) = \mathbf{C}_0^\infty(a,b), \quad A_r^0 f = (-\mathbf{i}D)^r f$$

and

$$D(A_r) = \mathbf{W}_r^2(a,b), \quad A_r f = (-\mathbf{i}D)^r f.$$

Proposition 5.6 *We have*

(a) A_r^0 *is symmetric,*

(b) $R(A_r^0) = \left\{ g \in \mathbf{C}_0^\infty(a,b) : \int_a^b x^j g(x)\,dx = 0,\ j = 0,1,\ldots,r-1 \right\} =: \mathbf{C}_{0,r}^\infty(a,b)$,

(c) $\left(A_r^0\right)^* = A_r$, $A_r^* = \overline{A_r^0}$,

(d) $D(\overline{A_r^0}) = \left\{ f \in \mathbf{W}_r^2(a,b) : f^{(j)}(a) = f^{(j)}(b) = 0,\ j = 0,1,\ldots,r-1 \right\} =: \mathbf{W}_r^{2,0}(a,b)$.

Corollary 5.7 *In case* $(a,b) = \mathbb{R}$ *the operator* A_r *is selfadjoint. If* $(a,b) \neq \mathbb{R}$ *then* $\overline{A_r^0}$ *and* A_r *are not selfadjoint.*

Proposition 5.8 *In case* $(a,b) = \mathbb{R}$ *we have* $\sigma(A_{2r}) = [0,\infty)$ *and* $\sigma(A_{2r-1}) = \mathbb{R}$.

Chapter 6

The formalism of quantum mechanics

Consider a system of N (in general electrical charged) particles within an electromagnetic field and set $n = 3N$. By $q_1, \ldots, q_n$ we denote the cartesian spatial coordinates (i.e., q_{3j-2}, q_{3j-1} and q_{3j} are the spatial coordinates of the jth particle) and by $p_1, \ldots, p_n$ the momentum coordinates (i.e., $p_{3j-k} = m_j \dot{q}_{3j-k}$, $k = 0, 1, 2$, $j = 1, \ldots, N$). Then, the Hamiltonian differential equations can be written as

$$\dot{q} = \frac{\partial H(q,p)}{\partial p}, \quad \dot{p} = -\frac{\partial H(q,p)}{\partial q},$$

where the Hamiltonian function $H(q,p)$ equals the total energy of the system. In the case of one charged particle in an electrical fiel E with potential V (i.e., $E(q) = -\operatorname{grad} V(q)$) we have

$$H(q,p) = \frac{|p|^2}{2m} + V(q) = \frac{1}{2m} \sum_{j=1}^{3} p_j^2 + V(q_1, q_2, q_3)$$

and, consequently,

$$\dot{q}_j = \frac{p_j}{m}, \quad \dot{p}_j = -\frac{\partial V(q_1, q_2, q_3)}{\partial q_j}, \quad j = 1, 2, 3.$$

These considerations are due to classical mechanics. In quantum mechanics, the **state** of the system at time $t \in \mathbb{R}$ is described by a **state function** $\psi_t : \mathbb{R}^n \longrightarrow \mathbb{C}$, $x \mapsto \psi_t(x)$: If $A \subset \mathbb{R}^n$ is a measurable set, then

$$\int_A |\psi_t(x)|^2 \, dx$$

equals the probability for $q(t) \in A$. Hence, we assume that $\int_{\mathbb{R}^n} |\psi_t(x)|^2 \, dx = 1$, in particular $\psi_t \in \mathbf{L}^2(\mathbb{R}^n)$ $\forall \, t \in \mathbb{R}$. The function $\psi : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{C}$, $(x,t) \mapsto \psi_t(x)$ is called **wave function**. Hence, $\mathbf{L}^2(\mathbb{R}^n)$ is the **state space** of the system. The expectation value of the coordinate q_j at the state $\psi = \psi_t$ is equal to

$$q_j^\psi = \int_{\mathbb{R}^n} x_j |\psi(x)|^2 \, dx = \langle x_j \psi, \psi \rangle ,$$

the respective variance is equal to

$$\operatorname{var}_\psi(q_j) = \int_{\mathbb{R}^n} (x_j - q_j^\psi)^2 |\psi(x)|^2 \, dx = \left\langle (x_j - q_j^\psi)\psi, (x_j - q_j^\psi)\psi \right\rangle ,$$

and the dispersion $\sigma_\psi(q_j) = \sqrt{\mathrm{var}_\psi(q_j)}$. Consequently, we assign the operator of multiplication by x_j to the spatial coordinate q_j . To the momentum coordinate p_j we assign the differentiation operator $D_j = \frac{\hbar}{\mathbf{i}} \frac{\partial}{\partial x_j}$, where $\hbar \approx 10^{-34} Js$ denotes the Planck constant. We obtain

$$\begin{aligned} p_j^\psi &= \frac{\hbar}{\mathbf{i}} \int_{\mathbb{R}^n} \frac{\partial \psi(x)}{\partial x_j} \overline{\psi(x)}\, dx = \langle D_j \psi, \psi \rangle \ , \\ \mathrm{var}_\psi(p_j) &= \int_{\mathbb{R}^n} \left| \left(\frac{\hbar}{\mathbf{i}} \frac{\partial}{\partial x_j} - p_j^\psi \right) \psi(x) \right|^2 dx = \left\langle (D_j - p_j^\psi)\psi, (D_j - pj^\psi)\psi \right\rangle \ , \\ \sigma_\psi(p_j) &= \sqrt{\mathrm{var}_\psi(p_j)} \, . \end{aligned}$$

Remark 6.1 *Let $A, B \in L(\mathbf{H})$ be* **hermitian** *operators (i.e. $\langle Ax, x\rangle = \langle x, Ax\rangle \ \forall\, x \in D(A)$). Define $a_x := \langle Ax, x\rangle$ and $b_x := \langle Bx, x\rangle$ as well as*

$$\sigma_x(A) := \|(A - a_x I)x\| \ , \quad \sigma_x(B) := \|(B - b_x I)x\| \ .$$

Then we have the **uncertainty relation**

$$\sigma_x(A)\sigma_x(B) \geq \frac{1}{2} \left| \langle (AB - BA)x, x \rangle \right| \quad \forall\, x \in D(AB) \cap D(BA) \, .$$

Corollary 6.2 (Heisenberg's uncertainty relation) *For all states $\psi \in \mathbf{C}_0^\infty(\mathbb{R}^n)$,*

$$\sigma_\psi(q_j)\sigma_p si(q_j) \geq \frac{\hbar}{2} \, \|\psi\|^2 \ , \quad j = 1, \ldots, n \, .$$

The wave function satisfies the **Schrödinger equation**

$$\mathbf{i}\hbar \, \frac{\partial \psi(x,t)}{\partial t} = S\psi(x,t) \, , \tag{6.1}$$

where the **Schrödinger operator** S operates on ψ only w.r.t. the spatial coordinates. The operator S is obtained from the Hamiltonian function $H(q,p)$ using the following rules:

1. Represent $H(q,p)$ as a sum of
   (a) pure quadratic terms in p_j ,
   (b) terms $V(q)$ depending only on q ,
   (c) terms of the form $\frac{1}{2}\left[p_j f_j(q) + f_j(q) p_j\right]$.
2. Replace q_j by the multiplication operator by x_j and p_j by the differentiation operator D_j .

The expectation value of the Schrödinger operator at the state ψ equals $\langle S\psi, \psi\rangle$ and should be real for all states $\psi \in D(S)$. Consequently, S has to be a hermitian operator (cf. Proposition 4.6). The evolution operators $U(t)$, $t \in \mathbb{R}$ are defined as follows:

$U(t)\psi$ is equal to the state at time $t \in \mathbb{R}$, if ψ is the state of the system at time 0 .

Consequently, the following conditions have to be satisfied ($\mathbf{H} = \mathbf{L}^2(\mathbb{R}^n)$):

(a) $D(U(t)) = \mathbf{H} \ \forall\, t \in \mathbb{R}$,

(b) $U(0) = I$,

(c) $U(t) \in L(\mathbf{H}) \; \forall\, t \in \mathbb{R}\,,$

(d) $U(t_1 + t_2) = U(t_1)U(t_2) = U(t_2)U(t_1)\,,\; \forall\, t_1, t_2 \in \mathbb{R}\,,$

(e) $\lim\limits_{s \to t} \|U(s)\psi - U(t)\psi\| = 0\,,\; \forall\, t \in \mathbb{R}\,,\; \forall\, \psi \in \mathbf{H}\,,$

(f) $\|U(t)\psi\| = \|\psi\|\;,\; \forall\, t \in \mathbb{R}\,,\; \forall\, \psi \in \mathbf{H}\,,$

(g) $R(U(t)) = \mathbf{H}\,,\; \forall\, t \in \mathbb{R}\,.$

Definition 6.3 *We call a family* $\{U(t) : t \in \mathbb{R}\}$ *of operators with the properties (a)–(g) a (one-parametric)* **strongly continuous group of unitary operators**. *The operator* $A \in L(\mathbf{H})$ *defined by*

$$D(A) = \left\{ x \in \mathbf{H} : \exists \lim_{t \to 0} t^{-1}[U(t) - I]x \in \mathbf{H} \right\}\;, \quad Ax = \lim_{t \to 0} t^{-1}[U(t) - I]x\,,$$

is the **infinitesimal generator** *of this group.*

Proposition 6.4 *Let* $A \in L(\mathbf{H})$ *be the infinitesimal generator of the strongly continuous group* $\{U(t) : t \in \mathbb{R}\}$ *of unitary operators.*

(a) *The operator* $S = \mathbf{i}A$ *is selfadjoint.*

(b) *For each* $x^0 \in D(S)\,,$ *the function* $U(t)x^0\,,\; t \in \mathbb{R}\,,$ *is the unique solution of the initial value problem*

$$\dot{x} = -\mathbf{i}Sx\,, \quad x(0) = x^0\,.$$

In particular, $U(t)x \in D(S)$ *for all* $x \in D(S)$ *and for all* $t \in \mathbb{R}\,.$

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