Integer Polynomial Programming

Robert Weismantel
Otto-von-Guericke Universität Magdeburg

based on joint work with

- Dennis Michaels;
- Raymond Hemmecke, Matthias Köppe, Jesus De Loera;

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Computational Complexity

$$\max \ f(x) \text{ st } x \in P \cap \mathbf{Z}^d.$$

Assumption: The number of variables is fixed.
f is a polynomial with integral coefficients and P is a rational polytope.

Lemma Minimizing a degree four polynomial over the lattice points of a convex polygon is NP-hard.

Sketch of proof Reduction from NP-complete problem: Given $a, b, c \in \mathbf{Z}_+ \setminus \{0\}$, does there exist $x < c$, $x \in \mathbf{Z}_+$ with $x^2 \equiv a \bmod b$?

$$\text{minimize } (x^2 - a - by)^2$$

over the integers of

$$\{(x,y) \in \mathbf{Z}^2 : \quad 1 \leq x \leq c-1, \quad \tfrac{1-a}{b} \leq y \leq \tfrac{(c-1)^2-a}{b} \}.$$

If the value is zero, we obtain a solution.

Optimizing polynomials over polytopes

Given a polynomial $f \in \mathbf{Z}[x_1, x_2, \ldots, x_d]$ of maximum total degree D. Let P be a polytope,

$$P = \{x \mid Ax \leq b\}$$

where A is an $m \times d$ integral matrix and b is an integral vector.

Main theorem Let d be fixed. Let $f(x)$ be an integer polynomial of fixed degree D and P be a full-dimensional d-polytope defined by linear inequalities. We obtain a sequence of lower bounds $\{L_k\}$ and upper bounds $\{U_k\}$ which after polynomially many steps reaches

$$f^* = \text{max} \;\; f(x) \text{ st } \; x \in P \cap \mathbf{Z}^d.$$

The bounds satisfy $U_k - L_k \leq f^* \cdot (\sqrt[k]{|P \cap \mathbf{Z}^d|} - 1)$. When k is constant, L_k, U_k can be computed in polynomial time.

Some background material

Idea: (A. Barvinok) Lattice points encoded as exponent vectors of monomials.

$z_1^2 z_2^{-11}$ encodes the lattice point $(2, -11)$.

The set of lattice points is represented by a polynomial: $g_P(z) = \sum_{\alpha \in P \cap \mathbf{Z}^d} z^\alpha$.

$$g_P(z) = \sum_{i \in I} E_i \frac{z^{u_i}}{\prod_{j=1}^{d} (1 - z^{v_{ij}})},$$

where I is a polynomial-size indexing set, and where $E_i \in \{1, -1\}$ and $u_i, v_{ij} \in \mathbf{Z}^d$ for all i and j.

Example: Consider $P = \mathbf{R}_+$. Then,

$$g_P(z) = \sum_{\alpha \in P \cap \mathbf{Z}} z^\alpha = \sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}.$$

Operations on rational functions

Lemma

(a) Given the rational function

$$g_P(z) = \sum_{i \in I} E_i \frac{z^{u_i}}{\prod_{j=1}^{d} (1 - z^{v_{ij}})},$$

for fixed number of variables, there is a polynomial time algorithm which computes a rational function for

$$z_k \frac{\partial}{\partial z_k} \cdot g_P(z).$$

(b) Given a polynomial f of fixed total degree D. There exists a polynomial time algorithm which computes a rational function for

$$\sum_{\alpha \in P \cap \mathbf{Z}^d} f(\alpha) z^\alpha.$$

Definition[Barvinok,Woods 2003]
Given

$$g_1(z) = \sum_{m \in \mathbf{Z}^d} \beta_{1m} z^m,\; g_2(z) = \sum_{m \in \mathbf{Z}^d} \beta_{2m} z^m.$$

The Hadamard product $g = g_1 \star g_2$ is the power series $g(z) = \sum_{m \in \mathbf{Z}^d} \beta_{1m}\beta_{2m} z^m$.

Lemma [Barvinok,Woods 2003]
For k fix, there is a polynomial time algorithm, which, given

$$g_1(z) = \frac{z^{p_1}}{\prod_{j=1}^{k}(1 - z^{a_{1j}})},\; g_2(z) = \frac{z^{p_1}}{\prod_{j=1}^{k}(1 - z^{a_{2j}})}.$$

computes a function

$$h(z) = \sum_{i \in I} \beta_i \frac{z^{q_i}}{\prod_{j=1}^{s}(1 - z^{b_{ij}})}$$

with integer vectors q_i, b_{ij}, rational numbers β_i, and with $s \leq 2k$ such that $h(z)$ is the Hadamard product

$$g_1(z) * g_2(z).$$

Algorithm

Input: polytope P, a polynomial f.

Output: A sequence of bounds L_k, U_k reaching the maximal value of f over $P \cap \mathbf{Z}^d$.

1. Transform f such that $0 \leq f(x)\ \forall x \in P$.

2. Compute $g_P(z) = \sum_{\alpha \in P \cap \mathbf{Z}^d} z^\alpha$.
Determine $|P \cap \mathbf{Z}^d| = g_P(1)$.

3. Compute $g_{P,f}(z) = \sum\limits_{\alpha \in P \cap \mathbf{Z}^d} f(\alpha) z^\alpha$.

4. Let $h_1(z) := g_{P,f}(z)$, and

$$h_{2^s}(z) := h_{2^{s-1}}(z) \star h_{2^{s-1}}(z) = \sum_{\alpha \in P \cap \mathbf{Z}^d} f(\alpha)^{2^s} z^\alpha,$$

While $U_{2^{s-1}} - L_{2^{s-1}} \geq 1$, compute

$$L_{2^s} := \sqrt[2^s]{\frac{h_{2^s}(1)}{|P \cap \mathbf{Z}^d|}} \text{ and } U_{2^s} := \sqrt[2^s]{h_{2^s}(1)},$$

using repeated Hadamard products.

Example: nvs04 from MINLPLIB.

$$\begin{aligned} \min \quad & 100\left(\frac{1}{2}+i_2-\left(\frac{3}{5}+i_1\right)^2\right)^2+\left(\frac{2}{5}-i_1\right)^2 \\ \text{s.t.} \quad & i_1, i_2 \in [0,200]\cap \mathbf{Z}. \end{aligned}$$

The short generating function for the feasible region is

$$g(z_1,z_2)=\left(\frac{1}{1-z_1}-\frac{z_1^{201}}{1-z_1}\right)\left(\frac{1}{1-z_2}-\frac{z_2^{201}}{1-z_2}\right).$$

$$|P\cap \mathbf{Z}^2| = \lim_{\mathbf{z}\to(1,1)} g(z_1,z_2) = 40401.$$

Iteration	Lower bound	Upper bound
0	132544300737	5354922294111089
1	139463892042	28032242300500
2	146482339474	2076745586576
3	152495816311	574191743622
4	157024255611	304695647474
5	160129592292	223059822091
6	162121980035	191344849230
7	163341185909	177452917013
8	164062155566	171002371183
9	164477815613	167920681907
10	164712846940	166427811960
11	164843709354	165699650910
12	164915648176	165343251366
13	164954762445	165168476306
14	164975823059	165082659040
15	164987068761	165040481747
16	164993043030	165019748329
17	164996224761	165009577128
18	164997940416	165004616533
19	164998877141	165002215185
20	164999390043	165001059062
21	164999670092	165000504601
22	164999822326	165000239580
23	164999904682	165000113309
24	164999949041	165000053354
25	164999972844	165000025001
26	164999985569	165000011647
27	164999992342	165000005382
28	164999995937	165000002457
29	164999997839	165000001099
30	164999998845	165000000475

Determining an Optimal Process Design

Example:

2,3-Dimethylbutene-2 $\rightleftharpoons$ 2,3-Dimethylbutene-1.

two different types of processes are available:

$\longrightarrow$ mathematical descriptions lead to nonlinear models.

$\longrightarrow$ local minimal solution w. r. t. a cost function is known for the first process.

Aim: Prove that the second process type cannot work with lower costs!

Approach: Construct a MIP such that

– the set of feasible solutions for the second process is contained in the region defined by the MIP,

– the optimal value of its objective function is
  - a lower bound for the total cost of the process,
  - greater than the cost for the known local optimal solution of the first process.

Theoretical Background (1)

Consider

$$\max\{c^T x \mid p_i(x) \le 0,\ i \in M;\ x \in [l,u] \cap \mathbf{Z}^n\},$$

Let $p = (p_1, \ldots, p_{|M|})$. Introduce a new variable π_i for each p_i.

Determine for all $j \in M$:

$$P^j_{[p]} := \text{conv}\left(\left\{(x,\pi) \in \mathbf{R}^{n+|M|} \mid x \in [l,u] \cap \mathbf{Z}^n,\ \pi_j = p_j(x),\ \pi_i \in \mathbf{R},\ i \neq j\right\}\right).$$

Theorem.

$$\{x \in [l,u] \cap \mathbf{Z}^n \mid p(x) \le 0\} \subseteq \left(\bigcap_{j \in M} P^j_{[p]} \cap \{(x,\pi) \in \mathbf{R}^{n+|M|} \mid \pi \le 0\}\right)_x \cap \mathbf{Z}^n,$$

where equality holds, if and only if every p_i satisfies

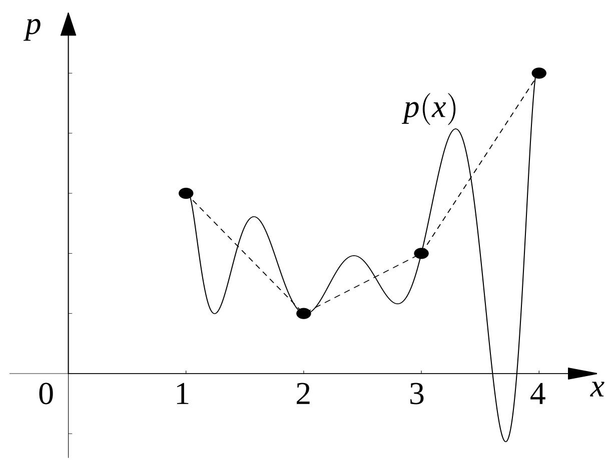
$\forall\, \lambda_k \ge 0,\ k \in [l,u] \cap \mathbf{Z}^n,$ with $\sum_k \lambda_k = 1$ and $\sum_k \lambda_k k \in \mathbf{Z}^n$:

$$p_i\left(\sum_k \lambda_k k\right) - \sum_k \lambda_k p_i(k) \ < \ 1.$$

Theoretical Background (2)

A $p \in \mathbf{Z}[x]$ is called **integer-convex** on $[l,u] \in \mathbf{R}^n$, if for all $\lambda_k \geq 0$, $k \in [l,u] \cap \mathbf{Z}^n$, $\sum_k \lambda_k = 1$, $\sum_k \lambda_k k \in \mathbf{Z}^n$:

$$p\Big(\sum_k \lambda_k k\Big) - \sum_k \lambda_k p(k) \;\leq\; 0.$$



Lemma. Let $p = (p_1, \ldots, p_m) \in \mathbf{Z}[x]^m$ be a vector of integer-convex polynomials on $[l,u]$,

$$\{x \in [l,u] \cap \mathbf{Z}^n \mid \sum_{j=1}^{m} p_i(x) \leq 0\}$$
$$\subseteq \Big(\bigcap_{i=1}^{m} P^i_{[p]} \cap \{(x,\pi) \in \mathbf{R}^{n+m} \mid \sum_{i=1}^{m} \pi_i \leq 0\}\Big)_x \cap \mathbf{Z}^n.$$