# Nonconvex Quadratic Programming

Dieter Vandenbussche

Department of Mechanical and Industrial Engineering
University of Illinois Urbana-Champaign

Workshop on Integer Programming and Continuous Optimization, 2004

# Outline

1. General QP
   - Introduction
   - Reformulation to LP with Complementarity Constraints
   - Bounding LP Relaxation
2. Special Case
   - Formulation
   - Valid Inequalities
   - Example
3. Fixed Cost Variables
   - Formulation
   - Lifting
   - Facets not from lifting

# Collaborators

- George Nemhauser (Georgia Institute of Technology)
- Sam Burer (University of Iowa)
- Tin-Chi Lin (University of Illinois U-C)

# Outline

1. General QP
   - Introduction
   - Reformulation to LP with Complementarity Constraints
   - Bounding LP Relaxation
2. Special Case
3. Fixed Cost Variables

# QP

Quadratic Program (QP):

$$\max \ \frac{1}{2}x^T Qx + c^T x$$
$$Ax \leq b$$

- If $Q$ is negative semidefinite, then QP is solvable in polynomial time.
- If $Q$ is indefinite, QP is $\mathcal{NP}$-hard.

# Reformulating QP

KKT conditions:

$$A^T y - Qx = c$$
$$Ax \le b \quad y \ge 0$$
$$y^T(b - Ax) = 0$$

For any KKT point,

$$\frac{1}{2}x^T Q x + c^T x = \frac{1}{2}\left(c^T x + y^T b\right)$$

**Reformulation**

$$\max \frac{1}{2}\left(c^T x + y^T b\right)$$
$$A^T y - Qx = c$$
$$Ax \le b \quad y \ge 0$$
$$y^T(b - Ax) = 0$$

# Reformulating QP

KKT conditions:

$$A^T y - Qx = c$$
$$Ax \le b \quad y \ge 0$$
$$y^T(b - Ax) = 0$$

For any KKT point,

$$\frac{1}{2}x^T Q x + c^T x = \frac{1}{2}\left(c^T x + y^T b\right)$$

## Reformulation

$$\max \frac{1}{2}\left(c^T x + y^T b\right)$$
$$A^T y - Qx = c$$
$$Ax \le b \quad y \ge 0$$
$$y^T(b - Ax) = 0$$

# Just make it an IP

Slack variables $s := b - Ax$
Can replace each complementarity $y_i s_i = 0$ with

$$\begin{aligned} y_i &\leq M\delta_i^y \\ s_i &\leq M\delta_i^s \\ \delta_i^y + \delta_i^s &\leq 1 \\ \delta_i^y, \delta_i^s &\in \{0, 1\} \end{aligned}$$

for a sufficiently large M.

Problems:

- Introduces many new variables
- big $M$ yields poor LP relaxations
- Not as much fun

# Just make it an IP

Slack variables $s := b - Ax$
Can replace each complementarity $y_i s_i = 0$ with

$$\begin{aligned} y_i &\leq M\delta_i^y \\ s_i &\leq M\delta_i^s \\ \delta_i^y + \delta_i^s &\leq 1 \\ \delta_i^y, \delta_i^s &\in \{0, 1\} \end{aligned}$$

for a sufficiently large M.

Problems:

- Introduces many new variables
- big $M$ yields poor LP relaxations
- Not as much fun

# Just make it an IP

Slack variables $s := b - Ax$
Can replace each complementarity $y_i s_i = 0$ with

$$\begin{aligned} y_i &\le M\delta_i^y \\ s_i &\le M\delta_i^s \\ \delta_i^y + \delta_i^s &\le 1 \\ \delta_i^y, \delta_i^s &\in \{0, 1\} \end{aligned}$$

for a sufficiently large M.

Problems:

- Introduces many new variables
- big $M$ yields poor LP relaxations
- Not as much fun

# Just make it an IP

Slack variables $s := b - Ax$
Can replace each complementarity $y_i s_i = 0$ with

$$\begin{aligned} y_i &\le M\delta_i^y \\ s_i &\le M\delta_i^s \\ \delta_i^y + \delta_i^s &\le 1 \\ \delta_i^y, \delta_i^s &\in \{0, 1\} \end{aligned}$$

for a sufficiently large M.

Problems:

- Introduces many new variables
- big $M$ yields poor LP relaxations
- Not as much fun

# Direct Approach

Classic idea:

- solve LP relaxation
- branch on complementarity
- use branch-and-bound

$$\max \frac{1}{2}\left(c^T x + y^T b\right)$$
$$A^T y - Qx = c$$
$$Ax \le b \quad y \ge 0$$

Other contexts

- Beale and Tomlin: SOS sets.
- Disjunctive Programs: Balas, Beaumont, etc...
- DeFarias and Nemhauser: complementarity, cardinality, SOS.

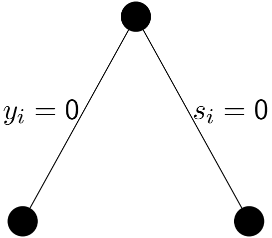
Challenges:

1. Branching Strategies.
2. Development of cutting planes.
3. Bounding the dual variables.

# Direct Approach

Classic idea:

- solve LP relaxation
- branch on complementarity
- use branch-and-bound




Other contexts

- Beale and Tomlin: SOS sets.
- Disjunctive Programs: Balas, Beaumont, etc...
- DeFarias and Nemhauser: complementarity, cardinality, SOS.

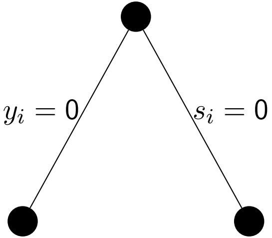
Challenges:

1. Branching Strategies.
2. Development of cutting planes.
3. Bounding the dual variables.

# Direct Approach

Classic idea:

- solve LP relaxation
- branch on complementarity
- use branch-and-bound



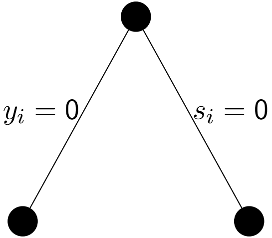

Other contexts

- Beale and Tomlin: SOS sets.
- Disjunctive Programs: Balas, Beaumont, etc...
- DeFarias and Nemhauser: complementarity, cardinality, SOS.

Challenges:

1. Branching Strategies.
2. Development of cutting planes.
3. Bounding the dual variables.

# Direct Approach




Classic idea:

- solve LP relaxation
- branch on complementarity
- use branch-and-bound

Other contexts

- Beale and Tomlin: SOS sets.
- Disjunctive Programs: Balas, Beaumont, etc...
- DeFarias and Nemhauser: complementarity, cardinality, SOS.

Challenges:

1. Branching Strategies.
2. Development of cutting planes.
3. Bounding the dual variables.

# Bounding the LP relaxation

LP relaxation of complementarity reformulation:

$$\max \frac{1}{2}\left(c^T x + y^T b\right)$$
$$A^T y - Qx = c$$
$$Ax \le b \quad y \ge 0$$

Assume $\{x \in \mathbb{R}^n : Ax \le b\}$ nonempty and bounded, i.e. $\{r \in \mathbb{R}^n : Ar \le 0\} = \{0\}$.

Recession cone of the LP relaxation:

$$\mathcal{C} = \{(\gamma, r) \in \mathbb{R}^{m+n} : Ar \le 0, A^T\gamma - Qr = 0, \gamma \ge 0\} =$$
$$\{(\gamma, 0) \in \mathbb{R}^{m+n} : A^T\gamma = 0, \gamma \ge 0\}$$

# Bounding the LP relaxation

LP relaxation of complementarity reformulation:

$$\max \frac{1}{2}\left(c^T x + y^T b\right)$$
$$A^T y - Qx = c$$
$$Ax \leq b \quad y \geq 0$$

Assume $\{x \in \mathbb{R}^n : Ax \leq b\}$ nonempty and bounded, i.e. $\{r \in \mathbb{R}^n : Ar \leq 0\} = \{0\}$.

Recession cone of the LP relaxation:

$$\mathcal{C} = \left\{(\gamma, r) \in \mathbb{R}^{m+n} : Ar \leq 0, A^T\gamma - Qr = 0, \gamma \geq 0\right\} = \left\{(\gamma, 0) \in \mathbb{R}^{m+n} : A^T\gamma = 0, \gamma \geq 0\right\}$$

# Extreme rays of LP relaxation

$\exists$ interior point, i.e. $\bar{x}$ such that $A\bar{x} < b \quad \Rightarrow$

$$b^T\gamma > 0 \ \forall(\gamma, 0) \in \mathcal{C} \setminus \{0\}$$

$\Rightarrow$ LP relaxation is unbounded.

If $(y, x)$ is a feasible, complementary solution and $(\gamma, 0) \in \mathcal{C} \setminus \{0\}$, then

$$(y + t\gamma)^T(b - Ax) = t\gamma^T b > 0 \text{ for any } t > 0$$

- Convex hull of complementary solutions is closed and bounded
- Every vertex of this convex hull is a vertex of LP relaxation
- Optimize over just the vertices of the LP relaxation?
- Will not necessarily yield complementary solutions

Unfortunately, this is not easy in general

# Extreme rays of LP relaxation

$\exists$ interior point, i.e. $\bar{x}$ such that $A\bar{x} < b \quad \Rightarrow$

$$b^T\gamma > 0 \ \forall (\gamma, 0) \in \mathcal{C} \setminus \{0\}$$

$\Rightarrow$ LP relaxation is unbounded.

If $(y, x)$ is a feasible, complementary solution and $(\gamma, 0) \in \mathcal{C} \setminus \{0\}$, then

$$(y + t\gamma)^T(b - Ax) = t\gamma^T b > 0 \text{ for any } t > 0$$

- Convex hull of complementary solutions is closed and bounded
- Every vertex of this convex hull is a vertex of LP relaxation
- Optimize over just the vertices of the LP relaxation?
- Will not necessarily yield complementary solutions

Unfortunately, this is not easy in general

# Extreme rays of LP relaxation

$\exists$ interior point, i.e. $\bar{x}$ such that $A\bar{x} < b \quad \Rightarrow$

$$b^T\gamma > 0 \ \forall(\gamma, 0) \in \mathcal{C} \setminus \{0\}$$

$\Rightarrow$ LP relaxation is unbounded.

If $(y, x)$ is a feasible, complementary solution and $(\gamma, 0) \in \mathcal{C} \setminus \{0\}$, then

$$(y + t\gamma)^T(b - Ax) = t\gamma^T b > 0 \text{ for any } t > 0$$

- Convex hull of complementary solutions is closed and bounded
- Every vertex of this convex hull is a vertex of LP relaxation
- Optimize over just the vertices of the LP relaxation?
- Will not necessarily yield complementary solutions

Unfortunately, this is not easy in general

# Extreme rays of LP relaxation

$\exists$ interior point, i.e. $\bar{x}$ such that $A\bar{x} < b \quad \Rightarrow$

$$b^T\gamma > 0 \; \forall (\gamma, 0) \in \mathcal{C} \setminus \{0\}$$

$\Rightarrow$ LP relaxation is unbounded.

If $(y, x)$ is a feasible, complementary solution and $(\gamma, 0) \in \mathcal{C} \setminus \{0\}$, then

$$(y + t\gamma)^T(b - Ax) = t\gamma^T b > 0 \text{ for any } t > 0$$

- Convex hull of complementary solutions is closed and bounded
- Every vertex of this convex hull is a vertex of LP relaxation
- Optimize over just the vertices of the LP relaxation?
- Will not necessarily yield complementary solutions

Unfortunately, this is not easy in general

# Extreme rays of LP relaxation

$\exists$ interior point, i.e. $\bar{x}$ such that $A\bar{x} < b \quad \Rightarrow$

$$b^T\gamma > 0 \ \forall(\gamma, 0) \in \mathcal{C} \setminus \{0\}$$

$\Rightarrow$ LP relaxation is unbounded.

If $(y, x)$ is a feasible, complementary solution and $(\gamma, 0) \in \mathcal{C} \setminus \{0\}$, then

$$(y + t\gamma)^T(b - Ax) = t\gamma^T b > 0 \text{ for any } t > 0$$

- Convex hull of complementary solutions is closed and bounded
- Every vertex of this convex hull is a vertex of LP relaxation
- Optimize over just the vertices of the LP relaxation?
- Will not necessarily yield complementary solutions

Unfortunately, this is not easy in general

# Extreme rays of LP relaxation

$\exists$ interior point, i.e. $\bar{x}$ such that $A\bar{x} < b \quad \Rightarrow$

$$b^T\gamma > 0 \; \forall (\gamma, 0) \in \mathcal{C} \setminus \{0\}$$

$\Rightarrow$ LP relaxation is unbounded.

If $(y, x)$ is a feasible, complementary solution and $(\gamma, 0) \in \mathcal{C} \setminus \{0\}$, then

$$(y + t\gamma)^T(b - Ax) = t\gamma^T b > 0 \text{ for any } t > 0$$

- Convex hull of complementary solutions is closed and bounded
- Every vertex of this convex hull is a vertex of LP relaxation
- Optimize over just the vertices of the LP relaxation?
- Will not necessarily yield complementary solutions

Unfortunately, this is not easy in general

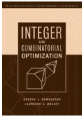
# Complexity

**Observation**

*Given an LP in the form of a poly-time separation oracle, optimizing an arbitrary linear objective function over the extreme points of this LP is $\mathcal{NP}$-hard.*

Dominant of convex hull of incidence vectors of $s-t$ paths:

$$\left\{x \in \mathbb{R}_{+}^{|E|} : a_H^T x \geq 1 \quad \forall H \in \mathcal{H}\right\}$$




- $\mathcal{H}$: set of $s-t$ cuts
- $a_H$: incidence vector of a cut $H$.

Finding vertex that maximizes arbitrary linear objective $\Leftrightarrow$ Longest Path

# Bounding LP relaxation

Need a way to "truncate" the polyhedron

$$\begin{aligned} A^T y - Qx &= c \\ Ax &\leq b \\ y &\geq 0 \end{aligned}$$

without eliminating solutions that satisfy $y^T(b - Ax) = 0$.

- Could use information about sizes of vertices: vertex size $\leq 4n^2 \times$ inequality size
- Add valid inequalities
- Examine special cases

# Outline

1. General QP
2. Special Case
   - Formulation
   - Valid Inequalities
   - Example
3. Fixed Cost Variables

# QP with simple bounds

Quadratic program with simple bounds (QPB):

$$\max \ \frac{1}{2}x^T Qx + c^T x$$
$$0 \le x \le e$$

- Can show this generalizes 0-1 QP
- $\mathcal{NP}$-complete

# KKT reformulation

Equivalent complementarity problem:

$$\max \frac{1}{2}c^T x + \frac{1}{2}y^T e$$

$$y_i - \sum_{j\in N} q_{ij}x_j - z_i = c_i \quad \forall i \in N$$

$$y_i(1 - x_i) = 0 \quad \forall i \in N$$

$$z_i x_i = 0 \quad \forall i \in N$$

$$x \leq e \quad x, y, z \geq 0$$

$y_i(1 - x_i) = 0$ and $z_i x_i = 0 \Rightarrow y_i z_i = 0$

# KKT reformulation

Equivalent complementarity problem:

$$\max \frac{1}{2}c^T x + \frac{1}{2}y^T e$$

$$y_i - \sum_{j \in N} q_{ij} x_j - z_i = c_i \quad \forall i \in N$$

$$y_i(1 - x_i) = 0 \quad \forall i \in N$$

$$z_i x_i = 0 \quad \forall i \in N$$

$$x \le e \quad x, y, z \ge 0$$

$y_i(1 - x_i) = 0$ and $z_i x_i = 0 \Rightarrow y_i z_i = 0$

# KKT reformulation

Equivalent complementarity problem:

$$\max \frac{1}{2}c^T x + \frac{1}{2}y^T e$$

$$y_i - \sum_{j \in N} q_{ij} x_j - z_i = c_i \quad \forall i \in N$$

$$y_i(1 - x_i) = 0 \quad \forall i \in N$$

$$z_i x_i = 0 \quad \forall i \in N$$

$$x \le e \quad x, y, z \ge 0$$

$y_i(1 - x_i) = 0$ and $z_i x_i = 0 \Rightarrow y_i z_i = 0$

# KKT reformulation

Equivalent complementarity problem:

$$
\begin{aligned}
\max\ & \frac{1}{2}c^T x + \frac{1}{2}y^T e \\
& y_i - \sum_{j \in N} q_{ij} x_j - z_i = c_i \quad \forall i \in N \\
& y_i(1 - x_i) = 0 \quad \forall i \in N \\
& z_i x_i = 0 \quad \forall i \in N \\
& x \le e \quad x, y, z \ge 0
\end{aligned}
$$

$y_i(1 - x_i) = 0$ and $z_i x_i = 0 \Rightarrow y_i z_i = 0$

# KKT reformulation

Equivalent complementarity problem:

$$\max \frac{1}{2}c^T x + \frac{1}{2}y^T e$$

$$y_i - \sum_{j \in N} q_{ij}x_j - z_i = c_i \quad \forall i \in N$$

$$y_i(1 - x_i) = 0 \quad \forall i \in N$$

$$z_i x_i = 0 \quad \forall i \in N$$

$$x \leq e \quad x, y, z \geq 0$$

$y_i(1 - x_i) = 0$ and $z_i x_i = 0 \Rightarrow y_i z_i = 0$

# KKT reformulation

Equivalent complementarity problem:

$$\max \frac{1}{2}c^T x + \frac{1}{2}y^T e$$
$$y_i - \sum_{j \in N} q_{ij} x_j - z_i = c_i \quad \forall i \in N$$
$$y_i(1 - x_i) = 0 \quad \forall i \in N$$
$$z_i x_i = 0 \quad \forall i \in N$$
$$x \le e \quad x, y, z \ge 0$$

$y_i(1 - x_i) = 0$ and $z_i x_i = 0 \Rightarrow y_i z_i = 0$

# Bounding the LP relaxation

$$\text{Row } i: \quad y_i - z_i - \sum_{j \in N} q_{ij} x_j = c_i$$

Define

- $\bar{y}_i \equiv c_i + q_{ii} + \sum_{j \in N \setminus i} q_{ij}^+$ where $a^+ = \max\{0, a\}$
- $\bar{z}_i \equiv -c_i - \sum_{j \in N \setminus i} q_{ij}^-$ where $a^- = \min\{0, a\}$

- If $y_i > 0$, then $y_i \leq \bar{y}_i$
- If $z_i > 0$, then $z_i \leq \bar{z}_i$

Assume $\bar{y}_i > 0$ and $\bar{z}_i > 0$

Yields two sets of valid inequalities:

1. $y_i \leq \bar{y}_i x_i$
2. $z_i + \bar{z}_i x_i \leq \bar{z}_i$

Make LP relaxation bounded

# Bounding the LP relaxation

$$\text{Row } i: \quad y_i - z_i - \sum_{j \in N} q_{ij} x_j = c_i$$

Define

- $\bar{y}_i \equiv c_i + q_{ii} + \sum_{j \in N \setminus i} q_{ij}^+$ where $a^+ = \max\{0, a\}$
- $\bar{z}_i \equiv -c_i - \sum_{j \in N \setminus i} q_{ij}^-$ where $a^- = \min\{0, a\}$

- If $y_i > 0$, then $y_i \leq \bar{y}_i$
- If $z_i > 0$, then $z_i \leq \bar{z}_i$

Assume $\bar{y}_i > 0$ and $\bar{z}_i > 0$

Yields two sets of valid inequalities:

1. $y_i \leq \bar{y}_i x_i$
2. $z_i + \bar{z}_i x_i \leq \bar{z}_i$

Make LP relaxation bounded

# Bounding the LP relaxation

$$\text{Row } i: \quad y_i - z_i - \sum_{j \in N} q_{ij} x_j = c_i$$

Define

- $\bar{y}_i \equiv c_i + q_{ii} + \sum_{j \in N \setminus i} q_{ij}^+$ where $a^+ = \max\{0, a\}$
- $\bar{z}_i \equiv -c_i - \sum_{j \in N \setminus i} q_{ij}^-$ where $a^- = \min\{0, a\}$

- If $y_i > 0$, then $y_i \leq \bar{y}_i$
- If $z_i > 0$, then $z_i \leq \bar{z}_i$

Assume $\bar{y}_i > 0$ and $\bar{z}_i > 0$

Yields two sets of valid inequalities:

1. $y_i \leq \bar{y}_i x_i$
2. $z_i + \bar{z}_i x_i \leq \bar{z}_i$

Make LP relaxation bounded

# Bounding the LP relaxation

$$\text{Row } i: \quad y_i - z_i - \sum_{j \in N} q_{ij} x_j = c_i$$

Define

- $\bar{y}_i \equiv c_i + q_{ii} + \sum_{j \in N \setminus i} q_{ij}^+$ where $a^+ = \max\{0, a\}$
- $\bar{z}_i \equiv -c_i - \sum_{j \in N \setminus i} q_{ij}^-$ where $a^- = \min\{0, a\}$

- If $y_i > 0$, then $y_i \leq \bar{y}_i$
- If $z_i > 0$, then $z_i \leq \bar{z}_i$

Assume $\bar{y}_i > 0$ and $\bar{z}_i > 0$

Yields two sets of valid inequalities:

1. $y_i \leq \bar{y}_i x_i$
2. $z_i + \bar{z}_i x_i \leq \bar{z}_i$

Make LP relaxation bounded

# Bounding the LP relaxation

$$\text{Row } i: \quad y_i - z_i - \sum_{j \in N} q_{ij} x_j = c_i$$

Define

- $\bar{y}_i \equiv c_i + q_{ii} + \sum_{j \in N \setminus i} q_{ij}^+$ — where $a^+ = \max\{0, a\}$
- $\bar{z}_i \equiv -c_i - \sum_{j \in N \setminus i} q_{ij}^-$ — where $a^- = \min\{0, a\}$

- If $y_i > 0$, then $y_i \leq \bar{y}_i$
- If $z_i > 0$, then $z_i \leq \bar{z}_i$

Assume $\bar{y}_i > 0$ and $\bar{z}_i > 0$

Yields two sets of valid inequalities:

1. $y_i \leq \bar{y}_i x_i$
2. $z_i + \bar{z}_i x_i \leq \bar{z}_i$

Make LP relaxation bounded

# One-row relaxations

Find relaxations of the problem for which generating cuts is easy.

Consider the set

$$S_i \equiv \Big\{ (y_i, z_i, x) \in \mathbb{R}^{n+2} : y_i - \sum_{j \in N} q_{ij} x_j - z_i = c_i \\ y_i(1 - x_i) = 0, z_i x_i = 0 \\ x_j \leq 1 \; \forall j \in N, \; y_i, z_i, x \geq 0 \Big\}$$

Want valid inequalities for conv$(S_i)$.

- Much like knapsack relaxation
- Only 2 non-convexities (unlike knapsack)

# Nontrivial facets

Facets not induced by bounds belong to one of two classes

- $z_i + \sum_j \alpha_j x_j \leq \sum_j \alpha_j^+$

$$\left.\begin{array}{rl} \sum_j |\alpha_j| = \bar{z}_i & \\ \alpha_i \geq 0 & \\ 0 \leq \alpha_j \leq q_{ij} & \forall j \in N^+ \\ q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^- \end{array}\right\} SEP^z$$

- $y_i + \sum_j \alpha_j x_j \leq \sum_j \alpha_j^+$

$$\left.\begin{array}{rl} \sum_j |\alpha_j| = \bar{y}_i & \\ \alpha_i \leq 0 & \\ -q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^+ \\ 0 \leq \alpha_j \leq -q_{ij} & \forall j \in N^- \end{array}\right\} SEP^y$$

# Example

$$Q = \begin{bmatrix} -3 & 4 & -5 & -3 \\ 4 & -5 & -1 & 5 \\ -5 & -1 & 6 & 2 \\ -3 & 5 & 2 & -2 \end{bmatrix} \qquad c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{llllllllllllll}
\text{Max} & \frac{1}{2}y_1 & + & \frac{1}{2}y_2 & + & \frac{1}{2}y_3 & + & \frac{1}{2}y_4 \\
\text{s.t.} & y_1 & - & z_1 & + & 3x_1 & - & 4x_2 & + & 5x_3 & + & 3x_4 & = 0 \\
& y_2 & - & z_2 & - & 4x_1 & + & 5x_2 & + & x_3 & - & 5x_4 & = 0 \\
& y_3 & - & z_3 & + & 5x_1 & + & x_2 & - & 6x_3 & - & 2x_4 & = 0 \\
& y_4 & - & z_4 & + & 3x_1 & - & 5x_2 & - & 2x_3 & + & 2x_4 & = 0 \\
& y_1 & \le & 1x_1 & & & & z_1 & + & 8x_1 & \le 8 \\
& y_2 & \le & 4x_2 & & & & z_2 & + & x_2 & \le 1 \\
& y_3 & \le & 8x_3 & & & & z_3 & + & 6x_3 & \le 6 \\
& y_4 & \le & 5x_4 & & & & z_4 & + & 3x_4 & \le 3 \\
& y, & z, & x & \ge 0 & & & x & \le 1
\end{array}$$

# Example

$$Q = \begin{bmatrix} -3 & 4 & -5 & -3 \\ 4 & -5 & -1 & 5 \\ -5 & -1 & 6 & 2 \\ -3 & 5 & 2 & -2 \end{bmatrix} \qquad c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{llllllllllll}
\text{Max} & \frac{1}{2}y_1 & + & \frac{1}{2}y_2 & + & \frac{1}{2}y_3 & + & \frac{1}{2}y_4 \\
\text{s.t.} & y_1 & - & z_1 & + & 3x_1 & - & 4x_2 & + & 5x_3 & + & 3x_4 = 0 \\
& y_2 & - & z_2 & - & 4x_1 & + & 5x_2 & + & x_3 & - & 5x_4 = 0 \\
& y_3 & - & z_3 & + & 5x_1 & + & x_2 & - & 6x_3 & - & 2x_4 = 0 \\
& y_4 & - & z_4 & + & 3x_1 & - & 5x_2 & - & 2x_3 & + & 2x_4 = 0 \\
& y_1 & \leq & 1x_1 & & & & z_1 & + & 8x_1 & \leq 8 \\
& y_2 & \leq & 4x_2 & & & & z_2 & + & x_2 & \leq 1 \\
& y_3 & \leq & 8x_3 & & & & z_3 & + & 6x_3 & \leq 6 \\
& y_4 & \leq & 5x_4 & & & & z_4 & + & 3x_4 & \leq 3 \\
& y, & z, & x & \geq 0 & & & x & \leq 1
\end{array}$$

# Example (cont.)

Optimal solution to LP:

$$y^* = \begin{pmatrix} 0 \\ 2 \\ 7.5 \\ 2.5 \end{pmatrix} \quad z^* = \begin{pmatrix} 6 \\ 0.5 \\ 0 \\ 0 \end{pmatrix} \quad x^* = \begin{pmatrix} 0 \\ 0.5 \\ 1 \\ 1 \end{pmatrix}$$

Adding the following inequalities:

$$\begin{aligned} z_2 + x_4 &\leq 1 \\ y_2 - 3x_1 + x_3 &\leq 1 \end{aligned}$$

Cuts off the optimal LP solution and produces the optimal complementary solution.

# Example (cont.)

Optimal solution to LP:

$$y^* = \begin{pmatrix} 0 \\ 2 \\ 7.5 \\ 2.5 \end{pmatrix} \quad z^* = \begin{pmatrix} 6 \\ 0.5 \\ 0 \\ 0 \end{pmatrix} \quad x^* = \begin{pmatrix} 0 \\ 0.5 \\ 1 \\ 1 \end{pmatrix}$$

Adding the following inequalities:

$$\begin{aligned} z_2 + x_4 &\le 1 \\ y_2 - 3x_1 + x_3 &\le 1 \end{aligned}$$

Cuts off the optimal LP solution and produces the optimal complementary solution.

# Comments

- Computational Issues
  - Cuts can be separated in $\mathcal{O}(n \log n)$
  - To be effective, cuts must be added and deleted aggressively
  - Use of cuts significantly expedites branch-and-bound
  - Direct method much better than conversion to IP
  - Strong branching very effective for "hard" instances
  - Tables of computational results available for those interested (and can't find anything better to do)

---

- Theoretical Issues
  - Only vertices of $SEP^z$ and $SEP^y$ can yield facets
  - Possible to identify which vertices yield facets
  - Possible to characterize convex hull when $\bar{y}_i \leq 0$ or $\bar{z}_i \leq 0$

# Comments

- Computational Issues
  - Cuts can be separated in $\mathcal{O}(n \log n)$
  - To be effective, cuts must be added and deleted aggressively
  - Use of cuts significantly expedites branch-and-bound
  - Direct method much better than conversion to IP
  - Strong branching very effective for "hard" instances
  - Tables of computational results available for those interested (and can't find anything better to do)

---

- Theoretical Issues
  - Only vertices of $SEP^z$ and $SEP^y$ can yield facets
  - Possible to identify which vertices yield facets
  - Possible to characterize convex hull when $\bar{y}_i \leq 0$ or $\bar{z}_i \leq 0$

# Comments

- Computational Issues
  - Cuts can be separated in $\mathcal{O}(n \log n)$
  - To be effective, cuts must be added and deleted aggressively
  - Use of cuts significantly expedites branch-and-bound
  - Direct method much better than conversion to IP
  - Strong branching very effective for "hard" instances
  - Tables of computational results available for those interested (and can't find anything better to do)

---

- Theoretical Issues
  - Only vertices of $SEP^z$ and $SEP^y$ can yield facets
  - Possible to identify which vertices yield facets
  - Possible to characterize convex hull when $\bar{y}_i \leq 0$ or $\bar{z}_i \leq 0$

# Comments

- Computational Issues
  - Cuts can be separated in $\mathcal{O}(n \log n)$
  - To be effective, cuts must be added and deleted aggressively
  - Use of cuts significantly expedites branch-and-bound
  - Direct method much better than conversion to IP
  - Strong branching very effective for "hard" instances
  - Tables of computational results available for those interested (and can't find anything better to do)

---

- Theoretical Issues
  - Only vertices of $SEP^z$ and $SEP^y$ can yield facets
  - Possible to identify which vertices yield facets
  - Possible to characterize convex hull when $\bar{y}_i \leq 0$ or $\bar{z}_i \leq 0$

# Comments

- Computational Issues
  - Cuts can be separated in $\mathcal{O}(n \log n)$
  - To be effective, cuts must be added and deleted aggressively
  - Use of cuts significantly expedites branch-and-bound
  - Direct method much better than conversion to IP
  - Strong branching very effective for "hard" instances
  - Tables of computational results available for those interested (and can't find anything better to do)

---

- Theoretical Issues
  - Only vertices of $SEP^z$ and $SEP^y$ can yield facets
  - Possible to identify which vertices yield facets
  - Possible to characterize convex hull when $\bar{y}_i \leq 0$ or $\bar{z}_i \leq 0$

# Comments

- Computational Issues
  - Cuts can be separated in $\mathcal{O}(n \log n)$
  - To be effective, cuts must be added and deleted aggressively
  - Use of cuts significantly expedites branch-and-bound
  - Direct method much better than conversion to IP
  - Strong branching very effective for "hard" instances
  - Tables of computational results available for those interested (and can't find anything better to do)

---

- Theoretical Issues
  - Only vertices of $SEP^z$ and $SEP^y$ can yield facets
  - Possible to identify which vertices yield facets
  - Possible to characterize convex hull when $\bar{y}_i \leq 0$ or $\bar{z}_i \leq 0$

# Comments

- Computational Issues
  - Cuts can be separated in $\mathcal{O}(n \log n)$
  - To be effective, cuts must be added and deleted aggressively
  - Use of cuts significantly expedites branch-and-bound
  - Direct method much better than conversion to IP
  - Strong branching very effective for "hard" instances
  - Tables of computational results available for those interested (and can't find anything better to do)

---

- Theoretical Issues
  - Only vertices of $SEP^z$ and $SEP^y$ can yield facets
  - Possible to identify which vertices yield facets
  - Possible to characterize convex hull when $\bar{y}_i \leq 0$ or $\bar{z}_i \leq 0$

# Comments

- Computational Issues
  - Cuts can be separated in $\mathcal{O}(n \log n)$
  - To be effective, cuts must be added and deleted aggressively
  - Use of cuts significantly expedites branch-and-bound
  - Direct method much better than conversion to IP
  - Strong branching very effective for "hard" instances
  - Tables of computational results available for those interested (and can't find anything better to do)

---

- Theoretical Issues
  - Only vertices of $SEP^z$ and $SEP^y$ can yield facets
  - Possible to identify which vertices yield facets
  - Possible to characterize convex hull when $\bar{y}_i \leq 0$ or $\bar{z}_i \leq 0$

# Comments

- Computational Issues
  - Cuts can be separated in $\mathcal{O}(n \log n)$
  - To be effective, cuts must be added and deleted aggressively
  - Use of cuts significantly expedites branch-and-bound
  - Direct method much better than conversion to IP
  - Strong branching very effective for "hard" instances
  - Tables of computational results available for those interested (and can't find anything better to do)

---

- Theoretical Issues
  - Only vertices of $SEP^z$ and $SEP^y$ can yield facets
  - Possible to identify which vertices yield facets
  - Possible to characterize convex hull when $\bar{y}_i \leq 0$ or $\bar{z}_i \leq 0$

# Comments

- Computational Issues
  - Cuts can be separated in $\mathcal{O}(n \log n)$
  - To be effective, cuts must be added and deleted aggressively
  - Use of cuts significantly expedites branch-and-bound
  - Direct method much better than conversion to IP
  - Strong branching very effective for "hard" instances
  - Tables of computational results available for those interested (and can't find anything better to do)

---

- Theoretical Issues
  - Only vertices of $SEP^z$ and $SEP^y$ can yield facets
  - Possible to identify which vertices yield facets
  - Possible to characterize convex hull when $\bar{y}_i \leq 0$ or $\bar{z}_i \leq 0$

# Comments

- Computational Issues
  - Cuts can be separated in $\mathcal{O}(n \log n)$
  - To be effective, cuts must be added and deleted aggressively
  - Use of cuts significantly expedites branch-and-bound
  - Direct method much better than conversion to IP
  - Strong branching very effective for "hard" instances
  - Tables of computational results available for those interested (and can't find anything better to do)

---

- Theoretical Issues
  - Only vertices of $SEP^z$ and $SEP^y$ can yield facets
  - Possible to identify which vertices yield facets
  - Possible to characterize convex hull when $\bar{y}_i \leq 0$ or $\bar{z}_i \leq 0$

# Outline

# Fixed cost variables

Consider the problem

$$\max \frac{1}{2}x^TQx + c^Tx - f^T\delta$$
$$0 \le x_j \le \delta_j \ \forall j \in N$$
$$\delta_j \in \{0,1\} \ \forall j \in N$$

Can reformulate as

$$\max \frac{1}{2}\left(c^Tx + e^Ty^1\right) - f^T\delta$$
$$y^1 + y^0 - Qx - z = c$$
$$y_i^1(1 - x_i) = 0 \quad y_i^0 x_i = 0 \quad z_i x_i = 0 \quad \forall i \in N$$
$$0 \le x_j \le \delta_j \ \forall j \in N \quad \delta_j \in \{0,1\} \ \forall j \in N$$
$$y^1, y^0, z \ge 0$$

WLOG, can also require that $y_i^0 z_i = 0$.

# Fixed cost variables

Consider the problem

$$\max \ \frac{1}{2}x^T Q x + c^T x - f^T \delta$$
$$0 \le x_j \le \delta_j \ \forall j \in N$$
$$\delta_j \in \{0,1\} \ \forall j \in N$$

Can reformulate as

$$\max \ \frac{1}{2}\left(c^T x + e^T y^1\right) - f^T \delta$$
$$y^1 + y^0 - Qx - z = c$$
$$y_i^1(1 - x_i) = 0 \quad y_i^0 x_i = 0 \quad z_i x_i = 0 \quad \forall i \in N$$
$$0 \le x_j \le \delta_j \ \forall j \in N \quad \delta_j \in \{0,1\} \ \forall j \in N$$
$$y^1, y^0, z \ge 0$$

WLOG, can also require that $y_i^0 z_i = 0$.

# Fixed cost variables

Consider the problem

$$\begin{aligned} \max \ & \frac{1}{2} x^T Q x + c^T x - f^T \delta \\ & 0 \leq x_j \leq \delta_j \ \forall j \in N \\ & \delta_j \in \{0, 1\} \ \forall j \in N \end{aligned}$$

Can reformulate as

$$\begin{aligned} \max \ & \frac{1}{2} \left( c^T x + e^T y^1 \right) - f^T \delta \\ & y^1 + y^0 - Qx - z = c \\ & y_i^1 (1 - x_i) = 0 \quad y_i^0 x_i = 0 \quad z_i x_i = 0 \quad \forall i \in N \\ & 0 \leq x_j \leq \delta_j \ \forall j \in N \quad \delta_j \in \{0, 1\} \ \forall j \in N \\ & y^1, y^0, z \geq 0 \end{aligned}$$

WLOG, can also require that $y_i^0 z_i = 0$.

# Fixed cost variables

Consider the problem

$$\max \frac{1}{2}x^TQx + c^Tx - f^T\delta$$
$$0 \le x_j \le \delta_j \ \forall j \in N$$
$$\delta_j \in \{0,1\} \ \forall j \in N$$

Can reformulate as

$$\max \frac{1}{2}\left(c^Tx + e^Ty^1\right) - f^T\delta$$
$$y^1 + y^0 - Qx - z = c$$
$$y_i^1(1 - x_i) = 0 \quad y_i^0 x_i = 0 \quad z_i x_i = 0 \quad \forall i \in N$$
$$0 \le x_j \le \delta_j \ \forall j \in N \quad \delta_j \in \{0,1\} \ \forall j \in N$$
$$y^1, y^0, z \ge 0$$

WLOG, can also require that $y_i^0 z_i = 0$.

# Fixed cost variables

Consider the problem

$$\max \frac{1}{2}x^TQx + c^Tx - f^T\delta$$
$$0 \leq x_j \leq \delta_j \;\forall j \in N$$
$$\delta_j \in \{0,1\} \;\forall j \in N$$

Can reformulate as

$$\max \frac{1}{2}\left(c^Tx + e^Ty^1\right) - f^T\delta$$
$$y^1 + y^0 - Qx - z = c$$
$$y_i^1(1 - x_i) = 0 \quad y_i^0 x_i = 0 \quad z_i x_i = 0 \quad \forall i \in N$$
$$0 \leq x_j \leq \delta_j \;\forall j \in N \quad \delta_j \in \{0,1\} \;\forall j \in N$$
$$y^1, y^0, z \geq 0$$

WLOG, can also require that $y_i^0 z_i = 0$.

# Fixed cost variables

Consider the problem

$$\begin{aligned} \max\ & \frac{1}{2}x^TQx + c^Tx - f^T\delta \\ & 0 \le x_j \le \delta_j \ \forall j \in N \\ & \delta_j \in \{0,1\} \ \forall j \in N \end{aligned}$$

Can reformulate as

$$\begin{aligned} \max\ & \frac{1}{2}\left(c^Tx + e^Ty^1\right) - f^T\delta \\ & y^1 + y^0 - Qx - z = c \\ & y_i^1(1 - x_i) = 0 \quad y_i^0 x_i = 0 \quad z_i x_i = 0 \quad \forall i \in N \\ & 0 \le x_j \le \delta_j \ \forall j \in N \quad \delta_j \in \{0,1\} \ \forall j \in N \\ & y^1, y^0, z \ge 0 \end{aligned}$$

WLOG, can also require that $y_i^0 z_i = 0$.

# One-row relaxations

$$S_i \equiv \Big\{ (y_i^1, y_i^0, z_i, x) \in \mathbb{R}^{n+2} : y_i^1 + y_i^0 - \sum_{j \in N} q_{ij} x_j - z_i = c_i$$
$$y_i^1 (1 - x_i) = 0, y_i^0 x_i = 0, z_i x_i = 0$$
$$x_j \le 1 \; \forall j \in N, \; y_i^1, y_i^0, z_i, x \ge 0 \quad \Big\}$$

## Projecting

$S_i|_{y_i^0 = 0}$ gives the one-row relaxation from before

## Definition

$$\bar{y}_i^0 := c_i + \sum_{j \in N^+} q_{ij}$$

# One-row relaxations

$$S_i \equiv \Big\{ (y_i^1, y_i^0, z_i, x) \in \mathbb{R}^{n+2} : y_i^1 + y_i^0 - \sum_{j \in N} q_{ij} x_j - z_i = c_i$$

$$y_i^1 (1 - x_i) = 0, y_i^0 x_i = 0, z_i x_i = 0$$

$$x_j \leq 1 \; \forall j \in N, \; y_i^1, y_i^0, z_i, x \geq 0 \Big\}$$

## Projecting

$S_i|_{y_i^0 = 0}$ gives the one-row relaxation from before

## Definition

$$\bar{y}_i^0 := c_i + \sum_{j \in N^+} q_{ij}$$

# One-row relaxations

$$S_i \equiv \Big\{ (y_i^1, y_i^0, z_i, x) \in \mathbb{R}^{n+2} : y_i^1 + y_i^0 - \sum_{j \in N} q_{ij} x_j - z_i = c_i$$
$$y_i^1(1 - x_i) = 0, y_i^0 x_i = 0, z_i x_i = 0$$
$$x_j \leq 1 \; \forall j \in N, \; y_i^1, y_i^0, z_i, x \geq 0 \quad \Big\}$$

## Projecting

$S_i|_{y_i^0 = 0}$ gives the one-row relaxation from before

## Definition

$$\bar{y}_i^0 := c_i + \sum_{j \in N^+} q_{ij}$$

# Lifting

Facet of $S_i|_{y_i^0=0}$: $z_i + \sum_j \alpha_j x_j \leq \sum_j \alpha_j^+$

$$\left.\begin{aligned} \sum_j |\alpha_j| &= \bar{z}_i \\ \alpha_i &\geq 0 \\ 0 \leq \alpha_j &\leq q_{ij} \quad \forall j \in N^+ \\ q_{ij} \leq \alpha_j &\leq 0 \quad \forall j \in N^- \end{aligned}\right\} SEP^z$$

# Lifting

Facet of $S_i|_{y_i^0=0}$: $z_i + \sum_j \alpha_j x_j \leq \sum_j \alpha_j^+$

$$\left.\begin{array}{rl} \sum_j |\alpha_j| = \bar{z}_i & \\ \alpha_i \geq 0 & \\ 0 \leq \alpha_j \leq q_{ij} & \forall j \in N^+ \\ q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^- \end{array}\right\} SEP^z$$

## Lifting problem

$$\theta = \min \frac{\sum_j (\alpha_j^+ - \alpha_j x_j) - z_i}{y_i^0}$$

$$(y_i^1, y_i^0, z_i, x) \in S_i, y_i^0 > 0$$

# Lifting

Facet of $S_i|_{y_i^0=0}$: $z_i + \sum_j \alpha_j x_j \leq \sum_j \alpha_j^+$

$$\left.\begin{array}{rl} \sum_j |\alpha_j| = \bar{z}_i & \\ \alpha_i \geq 0 & \\ 0 \leq \alpha_j \leq q_{ij} & \forall j \in N^+ \\ q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^- \end{array}\right\} SEP^z$$

## Lifting problem simplified

$$\theta = \min \frac{\alpha_i + \sum_{j \in N^+} \alpha_j (1 - x_j) - \sum_{j \in N^-} \alpha_j x_j}{y_i^0}$$

$$(0, y_i^0, 0, x) \in S_i, y_i^0 > 0$$

# Lifting

Facet of $S_i|_{y_i^0=0}$: $z_i + \sum_j \alpha_j x_j \leq \sum_j \alpha_j^+$

$$\left.\begin{aligned} \sum_j |\alpha_j| &= \bar{z}_i \\ \alpha_i &\geq 0 \\ 0 \leq \alpha_j &\leq q_{ij} \quad \forall j \in N^+ \\ q_{ij} \leq \alpha_j &\leq 0 \quad \forall j \in N^- \end{aligned}\right\} SEP^z$$

## Lifting problem solution

$$\theta = \frac{\alpha_i}{\bar{y}_i^0}$$

$$z_i + \frac{\alpha_i}{\bar{y}_i^0} y_i^0 + \sum_j \alpha_j x_j \leq \sum_j \alpha_j^+$$

# Lifting (cont.)

Facet of $S_i|_{y_i^0=0}$: $y_i^1 + \sum_j \alpha_j x_j \leq \sum_j \alpha_j^+$

$$\left.\begin{aligned} \sum_j |\alpha_j| &= \bar{y}_i \\ \alpha_i &\leq 0 \\ -q_{ij} \leq \alpha_j &\leq 0 \qquad \forall j \in N^+ \\ 0 \leq \alpha_j &\leq -q_{ij} \quad \forall j \in N^- \end{aligned}\right\} SEP^y$$

# Lifting (cont.)

Facet of $S_i|_{y_i^0=0}$: $y_i^1 + \sum_j \alpha_j x_j \leq \sum_j \alpha_j^+$

$$\left.\begin{array}{rl} \sum_j |\alpha_j| = \bar{y}_i & \\ \alpha_i \leq 0 & \\ -q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^+ \\ 0 \leq \alpha_j \leq -q_{ij} & \forall j \in N^- \end{array}\right\} SEP^y$$

## Lifting Problem

$$\theta = \min \frac{\sum_j(\alpha_j^+ - \alpha_j x_j) - y_i^1}{y_i^0}$$

$$(y_i^1, y_i^0, z_i, x) \in S_i, y_i^0 > 0$$

# Lifting (cont.)

Facet of $S_i|_{y_i^0=0}$: $y_i^1 + \sum_j \alpha_j x_j \leq \sum_j \alpha_j^+$

$$\left.\begin{aligned} \sum_j |\alpha_j| &= \bar{y}_i \\ \alpha_i &\leq 0 \\ -q_{ij} \leq \alpha_j &\leq 0 \qquad \forall j \in N^+ \\ 0 \leq \alpha_j &\leq -q_{ij} \quad \forall j \in N^- \end{aligned}\right\} SEP^y$$

## Lifting Problem simplified

$$\theta = \min \frac{\sum_{j \in N^-} \alpha_j (1 - x_j) - \sum_{j \in N^+} \alpha_j x_j}{c_i + \sum_j q_{ij} x_j}$$

$$(0, y_i^0, 0, x) \in S_i, y_i^0 > 0$$

# Lifting (cont.)

## Lifting Problem simplified

$$\theta = \min \frac{\sum_{j \in N^-} \alpha_j (1 - x_j) - \sum_{j \in N^+} \alpha_j x_j}{c_i + \sum_j q_{ij} x_j}$$

$$(0, y_i^0, 0, x) \in S_i, y_i^0 > 0$$

Set $x_j = 1$ if $\alpha_j > 0$ or if $j \in N^+$ and $\alpha_j = 0$

Feasible point if $c_i + \sum_{j \in N^- : \alpha_j > 0} q_{ij} + \sum_{j \in N^+ : \alpha_j = 0} q_{ij} > 0$

$\longrightarrow$ Necessary condition to be facet of $S_i|_{y_i^0 = 0}$.

# Lifting (cont.)

## Lifting Problem simplified

$$\theta = \min \frac{\sum_{j \in N^-} \alpha_j (1 - x_j) - \sum_{j \in N^+} \alpha_j x_j}{c_i + \sum_j q_{ij} x_j}$$
$$(0, y_i^0, 0, x) \in S_i, y_i^0 > 0$$

Set $x_j = 1$ if $\alpha_j > 0$ or if $j \in N^+$ and $\alpha_j = 0$

Feasible point if $c_i + \sum_{j \in N^- : \alpha_j > 0} q_{ij} + \sum_{j \in N^+ : \alpha_j = 0} q_{ij} > 0$

$\longrightarrow$ Necessary condition to be facet of $S_i|_{y_i^0 = 0}$.

# Lifting (cont.)

## Lifting Problem simplified

$$\theta = \min \frac{\sum_{j \in N^-} \alpha_j (1 - x_j) - \sum_{j \in N^+} \alpha_j x_j}{c_i + \sum_j q_{ij} x_j}$$

$$(0, y_i^0, 0, x) \in S_i, y_i^0 > 0$$

Set $x_j = 1$ if $\alpha_j > 0$ or if $j \in N^+$ and $\alpha_j = 0$

Feasible point if $c_i + \sum_{j \in N^- : \alpha_j > 0} q_{ij} + \sum_{j \in N^+ : \alpha_j = 0} q_{ij} > 0$

$\longrightarrow$ Necessary condition to be facet of $S_i|_{y_i^0 = 0}$.

# Lifting (cont.)

## Lifting problem solution

$$\theta = 0$$

$$y_i^1 + \sum_j \alpha_j x_j \leq \sum_j \alpha_j^+$$

if original inequality is facet of $S_i|_{y_i^0=0}$

Set $x_j = 1$ if $\alpha_j > 0$ or if $j \in N^+$ and $\alpha_j = 0$

Feasible point if $\boxed{c_i + \sum_{j \in N^- : \alpha_j > 0} q_{ij} + \sum_{j \in N^+ : \alpha_j = 0} q_{ij} > 0}$

$\longrightarrow$ Necessary condition to be facet of $S_i|_{y_i^0=0}$.

# Nontrivial Facets

Arbitrary Valid Inequality:

$$\alpha^{y^1} y_i^1 + \alpha^{y^0} y_i^0 + \alpha^z z_i + \sum_{j \in N} \alpha_j x_j \leq \beta$$

Eliminate $y_i^1$ using equality set:

$$\alpha^{y^0} y_i^0 + \alpha^z z_i + \sum_{j \in N} \alpha_j x_j \leq \beta$$

**Theorem**

*For any nontrivial facet, $\alpha^z \geq 0$.*
*Furthermore, $\alpha^z = 0 \Rightarrow \alpha^{y^0} > 0$ and $\alpha_i = 0$*

Consider two cases

1. $\alpha^z = 0$
2. $\alpha^z > 0$

# Nontrivial Facets

Arbitrary Valid Inequality:

$$\alpha^{y^1} y_i^1 + \alpha^{y^0} y_i^0 + \alpha^z z_i + \sum_{j \in N} \alpha_j x_j \leq \beta$$

Eliminate $y_i^1$ using equality set:

$$\alpha^{y^0} y_i^0 + \alpha^z z_i + \sum_{j \in N} \alpha_j x_j \leq \beta$$

**Theorem**

*For any nontrivial facet, $\alpha^z \geq 0$.*
*Furthermore, $\alpha^z = 0 \Rightarrow \alpha^{y^0} > 0$ and $\alpha_i = 0$*

Consider two cases

1. $\alpha^z = 0$
2. $\alpha^z > 0$

# Nontrivial Facets

Arbitrary Valid Inequality:

$$\alpha^{y^1} y_i^1 + \alpha^{y^0} y_i^0 + \alpha^z z_i + \sum_{j \in N} \alpha_j x_j \leq \beta$$

Eliminate $y_i^1$ using equality set:

$$\alpha^{y^0} y_i^0 + \alpha^z z_i + \sum_{j \in N} \alpha_j x_j \leq \beta$$

**Theorem**

*For any nontrivial facet, $\alpha^z \geq 0$.*
*Furthermore, $\alpha^z = 0 \Rightarrow \alpha^{y^0} > 0$ and $\alpha_i = 0$*

Consider two cases

1. $\alpha^z = 0$
2. $\alpha^z > 0$

# $\alpha^z = 0$

$$y_i^0 + \sum_{j \neq i} \alpha_j \leq \sum_{j \neq i} \alpha_j^+$$

$$SEP^{y^0} := \left\{ \begin{array}{ll} \sum_{j \neq i} |\alpha_j| = \bar{y}_i^0 & \\ -q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^+ \\ 0 \leq \alpha_j \leq -q_{ij} & \forall j \in N^- \end{array} \right\}$$

**Theorem**

*Suppose $\alpha$ is a vertex of $SEP^{y^0}$ and*

$$c_i + q_{ii} + \sum_{j \in N^+ : \alpha_j = 0} q_{ij} + \sum_{j \in N^- : \alpha_j > 0} q_{ij} > 0,$$

*then inequality is a facet of* $\text{conv}(S_i)$.

# $\alpha^z = 0$

$$y_i^0 + \sum_{j \neq i} \alpha_j \leq \sum_{j \neq i} \alpha_j^+$$

$$SEP^{y^0} := \left\{ \begin{array}{ll} \displaystyle\sum_{j \neq i} |\alpha_j| = \bar{y}_i^0 & \\ -q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^+ \\ 0 \leq \alpha_j \leq -q_{ij} & \forall j \in N^- \end{array} \right\}$$

**Theorem**

*Suppose $\alpha$ is a vertex of $SEP^{y^0}$ and*

$$c_i + q_{ii} + \sum_{j \in N^+ : \alpha_j = 0} q_{ij} + \sum_{j \in N^- : \alpha_j > 0} q_{ij} > 0,$$

*then inequality is a facet of* $\mathrm{conv}(S_i)$.

# $\alpha^z > 0, \alpha_i < 0$

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

**Conjecture**

$\alpha_i < 0 \Rightarrow \alpha^{y^0} = -1$

Use equality set to rewrite inequality as

$$y_i^1 + \sum_j \alpha_j x_j \leq \beta$$

- Consistent with lifting result
- Equivalent to $y_i$-inequalities from $S_i|_{y_i^0=0}$
- Can use $SEP^y$ to separate
- Nothing new and hence no fun

# $\alpha^z > 0, \alpha_i < 0$

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

**Conjecture**

$\alpha_i < 0 \Rightarrow \alpha^{y^0} = -1$

Use equality set to rewrite inequality as

$$y_i^1 + \sum_j \alpha_j x_j \leq \beta$$

- Consistent with lifting result
- Equivalent to $y_i$-inequalities from $S_i|_{y_i^0=0}$
- Can use $SEP^y$ to separate
- Nothing new and hence no fun

# $\alpha^z > 0, \alpha_i < 0$

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \le \beta$$

**Conjecture**

$\alpha_i < 0 \Rightarrow \alpha^{y^0} = -1$

Use equality set to rewrite inequality as

$$y_i^1 + \sum_j \alpha_j x_j \le \beta$$

- Consistent with lifting result
- Equivalent to $y_i$-inequalities from $S_i|_{y_i^0=0}$
- Can use $SEP^y$ to separate
- Nothing new and hence no fun

# $\alpha^z > 0, \alpha_i < 0$

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

## Conjecture

$\alpha_i < 0 \Rightarrow \alpha^{y^0} = -1$

Use equality set to rewrite inequality as

$$y_i^1 + \sum_j \alpha_j x_j \leq \beta$$

- Consistent with lifting result
- Equivalent to $y_i$-inequalities from $S_i|_{y_i^0=0}$
- Can use $SEP^y$ to separate
- Nothing new and hence no fun

# $\alpha^z > 0, \alpha_i < 0$

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

**Conjecture**

$\alpha_i < 0 \Rightarrow \alpha^{y^0} = -1$

Use equality set to rewrite inequality as

$$y_i^1 + \sum_j \alpha_j x_j \leq \beta$$

- Consistent with lifting result
- Equivalent to $y_i$-inequalities from $S_i|_{y_i^0=0}$
- Can use $SEP^y$ to separate
- Nothing new and hence no fun

# $\alpha^z > 0, \alpha_i < 0$

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \le \beta$$

## Conjecture

$\alpha_i < 0 \Rightarrow \alpha^{y^0} = -1$

Use equality set to rewrite inequality as

$$y_i^1 + \sum_j \alpha_j x_j \le \beta$$

- Consistent with lifting result
- Equivalent to $y_i$-inequalities from $S_i|_{y_i^0=0}$
- Can use $SEP^y$ to separate
- Nothing new and hence no fun

# $\alpha^z > 0, \alpha_i < 0$

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

**Conjecture**

$\alpha_i < 0 \Rightarrow \alpha^{y^0} = -1$

Use equality set to rewrite inequality as

$$y_i^1 + \sum_j \alpha_j x_j \leq \beta$$

- Consistent with lifting result
- Equivalent to $y_i$-inequalities from $S_i|_{y_i^0=0}$
- Can use $SEP^y$ to separate
- Nothing new and hence no fun

# $\alpha^z > 0, \alpha_i \geq 0$

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

**Theorem**

$\alpha_i \geq 0 \Rightarrow$

- $\alpha_j \leq q_{ij}\ \forall j \in N^+$
- $\alpha_j \geq q_{ij}\ \forall j \in N^-$
- $\alpha^{y^0} \geq 0$

# $\alpha^z > 0, \alpha_i \geq 0$

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

**Theorem**

$\alpha_i \geq 0 \Rightarrow$

- $\alpha_j \leq q_{ij} \; \forall j \in N^+$
- $\alpha_j \geq q_{ij} \; \forall j \in N^-$
- $\alpha^{y^0} \geq 0$

# $\alpha^z > 0, \alpha_i \geq 0$

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

**Theorem**

$\alpha_i \geq 0 \Rightarrow$

- $\alpha_j \leq q_{ij} \ \forall j \in N^+$
- $\alpha_j \geq q_{ij} \ \forall j \in N^-$
- $\alpha^{y^0} \geq 0$

Unlike with $S_i|_{y_i^0=0}$, we cannot assume

- $\alpha_j \geq 0 \ \forall j \in N^+$
- $\alpha_j \leq 0 \ \forall j \in N^-$

# $\alpha^z > 0, \alpha_i \geq 0$

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

## Theorem

$\alpha_i \geq 0 \Rightarrow$

- $\alpha_j \leq q_{ij} \; \forall j \in N^+$
- $\alpha_j \geq q_{ij} \; \forall j \in N^-$
- $\alpha^{y^0} \geq 0$

## Definition

$B^+ := \{j \in N^+ : \alpha_j < 0\}$ $B^- := \{j \in N^- : \alpha_j > 0\}$

$$B = B^+ \cup B^-$$

# Necessary conditions

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

$$\beta = \alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j + \sum_{j \in B^-} \alpha_j$$

$$\alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j - \sum_{j \in N^- \setminus B} \alpha_j = \bar{z}_i$$

$$0 \leq \alpha_j \leq q_{ij} \; \forall j \in N^+ \setminus B^+$$

$$q_{ij} \leq \alpha_j \leq 0 \; \forall j \in N^- \setminus B^-$$

$$\alpha_i, \alpha^{y^0} \geq 0$$

$$\alpha_i = \sum_{j \in B^+} \alpha_j - \sum_{j \in B^-} \alpha_j + \bar{y}_i^0 \alpha^{y^0}$$

$$\alpha_j = -\alpha^{y^0} q_{ij} \; \forall j \in B$$

Previously $SEP^z =$

$$\left\{ \begin{array}{cc} \sum_j |\alpha_j| = \bar{z}_i & \\ \alpha_i \geq 0 & \\ 0 \leq \alpha_j \leq q_{ij} & \forall j \in N^+ \\ q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^- \end{array} \right\}$$

- $B = \emptyset$ same as lifting.
- For fixed $B$, separation still $O(n \log n)$

# Necessary conditions

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

$$\beta = \alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j + \sum_{j \in B^-} \alpha_j$$

$$\alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j - \sum_{j \in N^- \setminus B} \alpha_j = \bar{z}_i$$

$$0 \leq \alpha_j \leq q_{ij} \; \forall j \in N^+ \setminus B^+$$

$$q_{ij} \leq \alpha_j \leq 0 \; \forall j \in N^- \setminus B^-$$

$$\alpha_i, \alpha^{y^0} \geq 0$$

$$\alpha_i = \sum_{j \in B^+} \alpha_j - \sum_{j \in B^-} \alpha_j + \bar{y}_i^0 \alpha^{y^0}$$

$$\alpha_j = -\alpha^{y^0} q_{ij} \; \forall j \in B$$

Previously $SEP^z =$

$$\left\{ \begin{array}{ll} \sum_j |\alpha_j| = \bar{z}_i & \\ \alpha_i \geq 0 & \\ 0 \leq \alpha_j \leq q_{ij} & \forall j \in N^+ \\ q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^- \end{array} \right\}$$

- $B = \emptyset$ same as lifting.
- For fixed $B$, separation still $\mathcal{O}(n \log n)$

# Necessary conditions

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

$$\beta = \alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j + \sum_{j \in B^-} \alpha_j$$

$$\alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j - \sum_{j \in N^- \setminus B} \alpha_j = \bar{z}_i$$

$$0 \leq \alpha_j \leq q_{ij} \; \forall j \in N^+ \setminus B^+$$

$$q_{ij} \leq \alpha_j \leq 0 \; \forall j \in N^- \setminus B^-$$

$$\alpha_i, \alpha^{y^0} \geq 0$$

$$\alpha_i = \sum_{j \in B^+} \alpha_j - \sum_{j \in B^-} \alpha_j + \bar{y}_i^0 \alpha^{y^0}$$

$$\alpha_j = -\alpha^{y^0} q_{ij} \; \forall j \in B$$

Previously $SEP^z =$

$$\left\{ \begin{array}{ll} \sum_j |\alpha_j| = \bar{z}_i & \\ \alpha_i \geq 0 & \\ 0 \leq \alpha_j \leq q_{ij} & \forall j \in N^+ \\ q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^- \end{array} \right\}$$

- $B = \emptyset$ same as lifting.
- For fixed $B$, separation still $\mathcal{O}(n \log n)$

# Necessary conditions

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

$$\beta = \alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j + \sum_{j \in B^-} \alpha_j$$

$$\alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j - \sum_{j \in N^- \setminus B} \alpha_j = \bar{z}_i$$

$$0 \leq \alpha_j \leq q_{ij} \; \forall j \in N^+ \setminus B^+$$

$$q_{ij} \leq \alpha_j \leq 0 \; \forall j \in N^- \setminus B^-$$

$$\alpha_i, \alpha^{y^0} \geq 0$$

$$\alpha_i = \sum_{j \in B^+} \alpha_j - \sum_{j \in B^-} \alpha_j + \bar{y}_i^0 \alpha^{y^0}$$

$$\alpha_j = -\alpha^{y^0} q_{ij} \; \forall j \in B$$

Previously $SEP^z =$

$$\left\{ \begin{array}{ll} \sum_j |\alpha_j| = \bar{z}_i & \\ \alpha_i \geq 0 & \\ 0 \leq \alpha_j \leq q_{ij} & \forall j \in N^+ \\ q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^- \end{array} \right\}$$

- $B = \emptyset$ same as lifting.
- For fixed $B$, separation still $\mathcal{O}(n \log n)$

# Necessary conditions

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

$$\beta = \alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j + \sum_{j \in B^-} \alpha_j$$

$$\alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j - \sum_{j \in N^- \setminus B} \alpha_j = \bar{z}_i$$

$$0 \leq \alpha_j \leq q_{ij} \ \forall j \in N^+ \setminus B^+$$

$$q_{ij} \leq \alpha_j \leq 0 \ \forall j \in N^- \setminus B^-$$

$$\alpha_i, \alpha^{y^0} \geq 0$$

$$\alpha_i = \sum_{j \in B^+} \alpha_j - \sum_{j \in B^-} \alpha_j + \bar{y}_i^0 \alpha^{y^0}$$

$$\alpha_j = -\alpha^{y^0} q_{ij} \ \forall j \in B$$

Previously $SEP^z =$

$$\left\{ \begin{array}{ll} \sum_j |\alpha_j| = \bar{z}_i & \\ \alpha_i \geq 0 & \\ 0 \leq \alpha_j \leq q_{ij} & \forall j \in N^+ \\ q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^- \end{array} \right\}$$

- $B = \emptyset$ same as lifting.
- For fixed $B$, separation still $\mathcal{O}(n \log n)$

# Necessary conditions

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

$$\beta = \alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j + \sum_{j \in B^-} \alpha_j$$

$$\alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j - \sum_{j \in N^- \setminus B} \alpha_j = \bar{z}_i$$

$$0 \leq \alpha_j \leq q_{ij} \; \forall j \in N^+ \setminus B^+$$

$$q_{ij} \leq \alpha_j \leq 0 \; \forall j \in N^- \setminus B^-$$

$$\alpha_i, \alpha^{y^0} \geq 0$$

$$\alpha_i = \sum_{j \in B^+} \alpha_j - \sum_{j \in B^-} \alpha_j + \bar{y}_i^0 \alpha^{y^0}$$

$$\alpha_j = -\alpha^{y^0} q_{ij} \; \forall j \in B$$

Previously $SEP^z =$

$$\left\{ \begin{array}{ll} \sum_j |\alpha_j| = \bar{z}_i & \\ \alpha_i \geq 0 & \\ 0 \leq \alpha_j \leq q_{ij} & \forall j \in N^+ \\ q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^- \end{array} \right\}$$

- $B = \emptyset$ same as lifting.
- For fixed $B$, separation still $\mathcal{O}(n \log n)$

# Necessary conditions

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \le \beta$$

$$\beta = \alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j + \sum_{j \in B^-} \alpha_j$$

$$\alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j - \sum_{j \in N^- \setminus B} \alpha_j = \bar{z}_i$$

$$0 \le \alpha_j \le q_{ij} \; \forall j \in N^+ \setminus B^+$$

$$q_{ij} \le \alpha_j \le 0 \; \forall j \in N^- \setminus B^-$$

$$\alpha_i, \alpha^{y^0} \ge 0$$

$$\alpha_i = \sum_{j \in B^+} \alpha_j - \sum_{j \in B^-} \alpha_j + \bar{y}_i^0 \alpha^{y^0}$$

$$\alpha_j = -\alpha^{y^0} q_{ij} \; \forall j \in B$$

Previously $SEP^z =$

$$\left\{ \begin{array}{ll} \sum_j |\alpha_j| = \bar{z}_i & \\ \alpha_i \ge 0 & \\ 0 \le \alpha_j \le q_{ij} & \forall j \in N^+ \\ q_{ij} \le \alpha_j \le 0 & \forall j \in N^- \end{array} \right\}$$

- $B = \emptyset$ same as lifting.
- For fixed $B$, separation still $\mathcal{O}(n \log n)$

# Necessary conditions

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

$$\beta = \alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j + \sum_{j \in B^-} \alpha_j$$

$$\alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j - \sum_{j \in N^- \setminus B} \alpha_j = \bar{z}_i$$

$$0 \leq \alpha_j \leq q_{ij} \; \forall j \in N^+ \setminus B^+$$

$$q_{ij} \leq \alpha_j \leq 0 \; \forall j \in N^- \setminus B^-$$

$$\alpha_i, \alpha^{y^0} \geq 0$$

$$\alpha_i = \sum_{j \in B^+} \alpha_j - \sum_{j \in B^-} \alpha_j + \bar{y}_i^0 \alpha^{y^0}$$

$$\alpha_j = -\alpha^{y^0} q_{ij} \; \forall j \in B$$

Previously $SEP^z =$

$$\left\{ \begin{array}{rl} \sum_j |\alpha_j| = \bar{z}_i & \\ \alpha_i \geq 0 & \\ 0 \leq \alpha_j \leq q_{ij} & \forall j \in N^+ \\ q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^- \end{array} \right\}$$

- $B = \emptyset$ same as lifting.
- For fixed $B$, separation still $\mathcal{O}(n \log n)$

# Necessary conditions

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

$$\beta = \alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j + \sum_{j \in B^-} \alpha_j$$

$$\alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j - \sum_{j \in N^- \setminus B} \alpha_j = \bar{z}_i$$

$$0 \leq \alpha_j \leq q_{ij} \; \forall j \in N^+ \setminus B^+$$

$$q_{ij} \leq \alpha_j \leq 0 \; \forall j \in N^- \setminus B^-$$

$$\alpha_i, \alpha^{y^0} \geq 0$$

$$\alpha_i = \sum_{j \in B^+} \alpha_j - \sum_{j \in B^-} \alpha_j + \bar{y}_i^0 \alpha^{y^0}$$

$$\alpha_j = -\alpha^{y^0} q_{ij} \; \forall j \in B$$

Previously $SEP^z =$

$$\left\{ \begin{array}{ll} \sum_j |\alpha_j| = \bar{z}_i & \\ \alpha_i \geq 0 & \\ 0 \leq \alpha_j \leq q_{ij} & \forall j \in N^+ \\ q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^- \end{array} \right\}$$

- $B = \emptyset$ same as lifting.
- For fixed $B$, separation still $\mathcal{O}(n \log n)$

# Necessary conditions

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

$$\beta = \alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j + \sum_{j \in B^-} \alpha_j$$

$$\alpha_i + \sum_{j \in N^+ \setminus B} \alpha_j - \sum_{j \in N^- \setminus B} \alpha_j = \bar{z}_i$$

$$0 \leq \alpha_j \leq q_{ij} \ \forall j \in N^+ \setminus B^+$$

$$q_{ij} \leq \alpha_j \leq 0 \ \forall j \in N^- \setminus B^-$$

$$\alpha_i, \alpha^{y^0} \geq 0$$

$$\alpha_i = \sum_{j \in B^+} \alpha_j - \sum_{j \in B^-} \alpha_j + \bar{y}_i^0 \alpha^{y^0}$$

$$\alpha_j = -\alpha^{y^0} q_{ij} \ \forall j \in B$$

Previously $SEP^z =$

$$\left\{ \begin{array}{cc} \sum_j |\alpha_j| = \bar{z}_i & \\ \alpha_i \geq 0 & \\ 0 \leq \alpha_j \leq q_{ij} & \forall j \in N^+ \\ q_{ij} \leq \alpha_j \leq 0 & \forall j \in N^- \end{array} \right\}$$

- $B = \emptyset$ same as lifting.
- For fixed $B$, separation still $\mathcal{O}(n \log n)$

# Facets

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

**Theorem**

*If* $(\alpha^{y^0}, \alpha, \beta)$ *is an extreme point, and*

- $c_i + \sum_{j \in N^+ \setminus B} q_{ij} + \sum_{j \in B^-} q_{ij} > 0$
- $q_{ii} + c_i + \sum_{j \in N^+ \setminus B} q_{ij} + \sum_{j \in B^-} q_{ij} > 0,$

*then it is a facet of* $S_i$*.*

Some comments:

- Condition is also necessary.
- When $B = \emptyset$, gives facets which are lifted "non-facets".
- How to choose $B$ ?

# Facets

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

**Theorem**

*If* $(\alpha^{y^0}, \alpha, \beta)$ *is an extreme point, and*

- $c_i + \sum_{j \in N^+ \setminus B} q_{ij} + \sum_{j \in B^-} q_{ij} > 0$
- $q_{ii} + c_i + \sum_{j \in N^+ \setminus B} q_{ij} + \sum_{j \in B^-} q_{ij} > 0$,

*then it is a facet of* $S_i$.

Some comments:

- Condition is also necessary.
- When $B = \emptyset$, gives facets which are lifted "non-facets".
- How to choose $B$ ?

# Facets

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \le \beta$$

**Theorem**

*If $(\alpha^{y^0}, \alpha, \beta)$ is an extreme point, and*

- $c_i + \sum_{j \in N^+ \setminus B} q_{ij} + \sum_{j \in B^-} q_{ij} > 0$
- $q_{ii} + c_i + \sum_{j \in N^+ \setminus B} q_{ij} + \sum_{j \in B^-} q_{ij} > 0$,

*then it is a facet of $S_i$.*

Some comments:

- Condition is also necessary.
- When $B = \emptyset$, gives facets which are lifted "non-facets".
- How to choose $B$ ?

# Facets

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

**Theorem**

*If* $(\alpha^{y^0}, \alpha, \beta)$ *is an extreme point, and*

- $c_i + \sum_{j \in N^+ \setminus B} q_{ij} + \sum_{j \in B^-} q_{ij} > 0$
- $q_{ii} + c_i + \sum_{j \in N^+ \setminus B} q_{ij} + \sum_{j \in B^-} q_{ij} > 0$,

*then it is a facet of* $S_i$.

Some comments:

- Condition is also necessary.
- When $B = \emptyset$, gives facets which are lifted "non-facets".
- How to choose $B$ ?

# Facets

$$z_i + \alpha^{y^0} y_i^0 + \sum_j \alpha_j x_j \leq \beta$$

**Theorem**

*If* $(\alpha^{y^0}, \alpha, \beta)$ *is an extreme point, and*

- $c_i + \sum_{j \in N^+ \setminus B} q_{ij} + \sum_{j \in B^-} q_{ij} > 0$
- $q_{ii} + c_i + \sum_{j \in N^+ \setminus B} q_{ij} + \sum_{j \in B^-} q_{ij} > 0,$

*then it is a facet of* $S_i$*.*

Some comments:

- Condition is also necessary.
- When $B = \emptyset$, gives facets which are lifted "non-facets".
- How to choose $B$ ?

# Future Work

- Determine how to "truncate" general LP relaxation
- Develop valid inequalities for general QP
- Determine most general structure of mixed integer QP that allows reformulation
- Develop valid inequalities implied by binary variables and complementarity constraints

# Future Work

- Determine how to "truncate" general LP relaxation
- Develop valid inequalities for general QP
- Determine most general structure of mixed integer QP that allows reformulation
- Develop valid inequalities implied by binary variables and complementarity constraints

# Future Work

- Determine how to "truncate" general LP relaxation
- Develop valid inequalities for general QP
- Determine most general structure of mixed integer QP that allows reformulation
- Develop valid inequalities implied by binary variables and complementarity constraints

# Future Work

- Determine how to "truncate" general LP relaxation
- Develop valid inequalities for general QP
- Determine most general structure of mixed integer QP that allows reformulation
- Develop valid inequalities implied by binary variables and complementarity constraints