LABORATOIRE d'ANALYSE et d'ARCHITECTURE des SYSTEMES

DUALITY AND INTEGER PROGRAMMING

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Current solvers (CPLEX, XPRESS-MP) are rather efficient and can solve many large size problems with thousands of variables. However, the very small 5-variables knapsack problem

$$\max \quad 213x_1 - 1928x_2 - 11111x_3 - 2345x_4 + 9123x_5$$

$$12223x_1 + 12224x_2 + 36674x_3 + 61119x_4 + 85569x_5 \;=\; 89643482$$

x_1, x_2, x_3, x_4, x_5 nonnegative integers

still resists all efficient solvers (takes **HOURS** on CPLEX 8.1 and XPRESS-MP !) Optimal solution $x^* = (7334, 0, 0, 0, 0)$ ...!

Any insight on integer problems is welcome ....

The integer program

$$\max\{ c'x \mid \quad Ax = b; \quad x \in \mathbf{N}^n\}$$

with $A = [A_1, \ldots, A_n] \in \mathbf{Z}^{m\times n}, b \in \mathbf{Z}^m$ has a dual problem

$$\min_{f\in\Gamma} \{ f(b) \mid \quad f(A_j) \geq c_j, \quad j = 1, \ldots, n\}$$

where Γ is the space of functions $f : \mathbf{R}^m \to \mathbf{R}$ that are super-additive $(f(a+b) \geq f(a) + f(b))$, and with $f(0) = 0$. (Jeroslow, Wolsey, etc ...)

Elegant but rather abstract and not very practical; however, used to derive valid inequalities. Moreover, the celebrated Gomory cuts used in Solvers (CPLEX, XPRESS-MP, ..) can be interpreted as such superadditive functions.

CONTINUOUS OPTIM. | **DISCRETE OPTIM.**

$$f(b,c) := \max c'x \quad \text{s.t.} \left[\begin{array}{rcl} Ax & \leq & b \\ x & \in & \mathbf{R}^n_+ \end{array} \right. \quad \longleftrightarrow \quad f_d(b,c) := \max c'x \quad \text{s.t.} \left[\begin{array}{rcl} Ax & \leq & b \\ x & \in & \mathbf{N}^n \end{array} \right.$$

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

INTEGRATION | **SUMMATION**

$$\widehat{f}(b,c) := \int_\Omega e^{c'x} dx \quad \longleftrightarrow \quad \widehat{f}_d(b,c) := \sum_\Omega e^{c'x}$$

$$\Omega := \left[\begin{array}{rcl} Ax & \leq & b \\ x & \in & \mathbf{R}^n_+ \end{array} \right. \qquad \Omega := \left[\begin{array}{rcl} Ax & \leq & b \\ x & \in & \mathbf{N}^n \end{array} \right.$$

$$e^{f(b,c)} = \lim_{r\to\infty} \widehat{f}(b, rc)^{1/r}; \quad e^{f_d(b,c)} = \lim_{r\to\infty} \widehat{f}_d(b, rc)^{1/r}.$$

or, equivalently

$$f(b,c) = \lim_{r\to\infty} \frac{1}{r} \ln \widehat{f}(b, rc); \quad f_d(b,c) = \lim_{r\to\infty} \frac{1}{r} \ln \widehat{f}_d(b, rc).$$

Note in passing that we have the **mean-value** theorem

$$\widehat{f}(b,c) = e^{c'x^*} \times \lambda(\Omega) = e^{c'x^*} \times \mathrm{vol}(\Omega).$$

and a **Maslov "max" mean-value theorem**

$$f(b,c) = c'x^* = c'x^* \times \mu_{maslov}(\Omega) = c'x^*.$$

Maslov duality for "linear" functionals in the Max-Plus algebra

$f \mapsto L(f)$ a "linear" functional in the "max $+$" algebra, i.e.,

$$L(f \vee h) = L(f) \vee L(h) \quad \forall f, h \Rightarrow L(f) = \int f \, d\mu_{maslov}.$$

with

$$\mu_{maslov}(A \cup B) = \mu_{maslov}(A) \vee \mu_{maslov}(B) \quad \forall A, B \in \mathcal{B}.$$

CONTINUOUS OPTIM.		DISCRETE OPTIM.
—	|	—
Legendre-Fenchel Duality	$\longleftrightarrow$	??
$\updownarrow$	—	$\updownarrow$
INTEGRATION		SUMMATION
—	|	—
Laplace-Transform Duality	$\longleftrightarrow$	Z-Transform Duality

Legendre-Fenchel duality : $f : \mathbf{R}^n \to \mathbf{R}$ convex; $f^* : \mathbf{R}^n \to \mathbf{R}$.

$$\lambda \mapsto f^*(\lambda) = \mathcal{F}(f)(\lambda) := \sup_y \{\lambda' y - f(y)\}.$$

(One-sided) Laplace-Transform: $f : \mathbf{R}^n_+ \to \mathbf{R}$; $F : \mathbf{C}^n \to \mathbf{C}$.

$$\lambda \mapsto F(\lambda) = \mathcal{L}(f)(\lambda) := \int_{\mathbf{R}^n_+} \mathrm{e}^{-\lambda' y} f(y)\, dy.$$

(One-sided) $\mathcal{Z}$-Transform: $f : \mathbf{Z}^n_+ \to \mathbf{R}$; $F : \mathbf{C}^n \to \mathbf{C}$.

$$\lambda \mapsto F(z) = \mathcal{Z}(f)(z) := \sum_{m \in \mathbf{Z}^n_+} z^{-m} f(m).$$

Observe :

$$\mathcal{L}(\mathrm{e}^{f})(\lambda) = \int_{\mathbf{R}^n_+} \mathrm{e}^{-(\lambda' y - f(y))}\, dy$$

and

$$\mathrm{e}^{\mathcal{F}(f)(\lambda)} = \begin{cases} \mathrm{e}^{\sup\limits_y (\lambda' y - f(y))} = \sup\limits_y \{\mathrm{e}^{(\lambda' y - f(y))}\} \\ \\ \oint_{\mathrm{Maslov}} \mathrm{e}^{(\lambda' y - f(y))}\, dy \end{cases}$$

exponential (Fenchel (f)) = Laplace (exponential (f))

in “max ,× ” algebra

with $\Omega := \{x \in \mathbf{R}^n \mid \quad Ax \le b; \quad x \ge 0\}$

Fenchel-duality:

$$f(b,c) := \max_{x \in \Omega} c'x$$

$$f^*(\lambda, c) := \min_{y} \{\lambda' y - f(y,c)\}$$

with : $A'\lambda - c \ge 0; \ \lambda \ge 0$

Laplace-duality

$$\widehat{f}(b,c) := \int_\Omega e^{c'x} \, dx$$

$$\widehat{F}(\lambda, c) := \int e^{-\lambda y} \, \widehat{f}(y,c) \, dy$$

$$= \int_{x \ge 0} e^{c'x} \left[\int_{Ax \le y} e^{-\lambda y} \, dy \right] \, dx$$

$$= \frac{1}{\prod_{j=1}^m \lambda_j \prod_{k=1}^n (A'\lambda - c)_k}$$

with : $\Re(A'\lambda - c) > 0; \ \Re(\lambda) > 0$

Fenchel-duality:

$$f(b,c) = \min_{\lambda}\{\lambda' b - f^*(\lambda, c)\}$$

$$= \min_{\lambda}\{b'\lambda \,|\, A'\lambda \geq c;\ \lambda \geq 0\}$$

Laplace-duality

$$\widehat{f}(b,c) = \frac{1}{(2i\pi)^m}\int_{\Gamma} e^{\lambda' b}\widehat{F}(\lambda, c)\, d\lambda$$

$$= \int_{\Gamma} \frac{(2i\pi)^{-m} e^{\lambda' b}}{\prod_{j=1}^m \lambda_j \prod_{k=1}^n (A'\lambda - c)_k} d\lambda$$

where $\Gamma \subset \mathbf{C}^m$

and $\displaystyle\int_{\Gamma} = \int_{u_1 - i\infty}^{u_1 + i\infty} \cdots \int_{u_m - i\infty}^{u_m + i\infty}$

with $u \in \mathbf{R}^m; \quad A'u > c;\ u > 0.$

Polytope volume:$->$ Brion, Brion and Vergnes, Barvinok and Pommersheim, Lasserre and Zeron, ...

Multidimensional integration over a simplex, ellipsoid $->$ Lasserre and Zeron.

Remark:

Logarithmic Barrier function of $\min\{\lambda' b \,|\, A'\lambda - c) \geq 0;\ \lambda \geq 0\}$

$$\lambda \mapsto \varphi(r,\lambda) := r\lambda' b - \sum_{k=1}^{n} \ln\,(A'\lambda - c)_k - \sum_{j=1}^{m} \ln(\lambda_j).$$

$$\begin{aligned}\widehat{f}(b,rc) = \frac{1}{r^n}\widehat{f}(rb,c) &= \frac{1}{r^n(2i\pi)^m}\int_\Gamma \frac{e^{r\lambda' b}}{\prod_{j=1}^m \lambda_j \prod_{k=1}^n (A'\lambda - c)_k}\,d\lambda \\ &= \frac{1}{r^n(2i\pi)^m}\int_\Gamma e^{\varphi(r,\lambda)}\,d\lambda\end{aligned}$$

Hence,

$$\lim_{r\to\infty} e^{\min_\lambda \varphi(r,\lambda)} = \lim_{r\to\infty} \min_\lambda e^{\varphi(r,\lambda)} = \lim_{r\to\infty} \frac{1}{r^n(2i\pi)^m}\int_\Gamma e^{\varphi(r,\lambda)}\,d\lambda$$

Fenchel-duality:

$$f(b,c) = \min_{\lambda}\{\lambda' b - f^*(\lambda,c)\}$$

$$= \min_{\lambda}\{b'\lambda \,|\, A'\lambda \geq c;\ \lambda \geq 0\}$$

Laplace-duality

$$\widehat{f}(b,c) = \frac{1}{(2i\pi)^m}\int_{\Gamma} \mathrm{e}^{\lambda' b}\widehat{F}(\lambda,c)\,d\lambda$$

$$= \int_{\Gamma} \frac{(2i\pi)^{-m}\mathrm{e}^{\lambda' b}}{\prod_{j=1}^{m}\lambda_j \prod_{k=1}^{n}(A'\lambda - c)_k} d\lambda$$

where $\int_{\Gamma} = \int_{u_1 - i\infty}^{u_1 + i\infty} \cdots \int_{u_m - i\infty}^{u_m + i\infty}$
with $u \in \mathbf{R}^m;\quad A'u > c;\ u > 0.$

Simplex Algorithm
Vertices of the polyhedron

$$\{Ax \leq b;\ x \geq 0\}$$

Cauchy 's Residue Technique
Poles of the function

$$\frac{1}{\prod_{j=1}^{m}\lambda_j \prod_{k=1}^{n}(A'\lambda - c)_k}$$

THE SAME!

Laplace-Duality algorithm: Cauchy 's Residue Technique

$$\widehat{f}(b,c) = \frac{1}{(2i\pi)^m} \int_{\Gamma} \frac{e^{\lambda' b}}{\prod_{j=1}^{m} \lambda_j \prod_{k=1}^{n} (A'\lambda - c)_k} \, d\lambda.$$

Integration w.r.t. $\lambda_1, \lambda_2, \ldots, \lambda_m$ (one variable at a time) by repeated application of Cauchy 's residue theorem.

Yields a tree of depth m whose level k is integration w.r.t. λ_k on the integration path $\{u_k - i\infty, u_k + i\infty\}$ while the other variables $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_m$ are fixed, on their respective integration path $\{u_j - i\infty, u_j + i\infty\}$. Application of Cauchy 's Residue Theorem yields a rational fraction in the variables $\lambda_{k+1}, \ldots, \lambda_m$, and so on, until level $m-1$ where one obtains a rational fraction in the single variable λ_m.

Brion and Vergne 's continuous formula

Terminology of LP in standard form:

Let $A_\sigma := [A_{\sigma_1}|\ldots|A_{\sigma_m}]$ be a **basis** of $\max\{c'x | Ax = b; x \geq 0\}$, with $x(\sigma)$ the corresponding **vertex**, $\pi^\sigma := c'_\sigma A_\sigma^{-1}$ the associated **dual variable** and the **reduced cost** vector $c_k - \pi^\sigma A_k$, $k \notin \sigma$. Then :

$$\widehat{f}(b,c) = \sum_{x(\sigma):\ \text{vertex of } \Omega(b)} \frac{e^{c'x(\sigma)}}{\det(A_\sigma)\prod_{k\notin\sigma}(-c_k + \pi_\sigma A_k)}$$

from which it easily follows that

$$\log\left[\lim_{r\to\infty} \widehat{f}(b,rc)^r\right]^{1/r} = \max_{\text{vertex of } \Omega(b)} c'x(\sigma).$$

Let $\Omega(y) := \{x \in \mathbf{R}^n \mid \quad Ax = y; \quad x \geq 0\}$.

Continuous Laplace-duality **Discrete Z-duality**

$$\widehat{f}(y,c) := \int_{\Omega(y)} e^{c'x}\, d\mu \qquad \widehat{f}_d(y,c) := \sum_{x \in \Omega(y) \cap \mathbf{Z}^n} e^{c'x}$$

$$\widehat{F}(\lambda,c) := \int_{\mathbf{R}^m_+} e^{-\lambda y}\, \widehat{f}(y,c)\, dy \qquad \widehat{F}_d(z,c) := \sum_{y \in \mathbf{Z}^m} z^{-y}\, \widehat{f}_d(y,c)$$

$$= \frac{1}{\prod_{k=1}^n (A'\lambda - c)_k} \qquad = \prod_{k=1}^n \frac{1}{1 - e^{c_k} z_1^{-A_{1k}} \cdots z_m^{-A_{mk}}}$$

with $\Re((A'\lambda)_k) > c_k \quad \forall k$ with $|z_1^{A_{1k}} \cdots z_m^{A_{mk}}| > e^{c_k} \quad \forall k$

Continuous Laplace-duality	Discrete $\mathbb{Z}$-duality
I. $\widehat{F}(\lambda, c) \longrightarrow f(b, c)$	I. $\widehat{F}_d(z, c) \longrightarrow f_d(b, c)$
by Inverse Laplace Transform and Cauchy 's Residue Th.	by Inverse $\mathbb{Z}$-Transform and Cauchy 's Residue Th.
II. Data appear as	II. Data appear as
COEFFICIENTS of dual variables λ in $\widehat{F}(\lambda, c)$	EXPONENTS ! of dual variables z in $\widehat{F}_d(z, c)$.
III. The poles of $\widehat{F}(\lambda, c)$ are	III. The poles of $\widehat{F}_d(z, c)$ are
REAL	COMPLEX and much more numerous !

Let $\sigma := [A_{\sigma_1}|\cdots|A_{\sigma_m}]$ be a feasible basis of $\{Ax = b,\ x \geq 0\}$, with $\mu(\sigma) := \det(A_\sigma)$, and "dual" variable $\pi^\sigma A_\sigma = c_\sigma$.

Continuous Laplace-duality	**Discrete $\mathbf{Z}$-duality**
basis $\sigma \rightarrow$ poles : $A'_\sigma \lambda = c_\sigma$	basis $\sigma \rightarrow$ poles : $z^{A_\sigma} = e^{c_\sigma}$
a **single** real pole λ in $\mathbf{R}^m$	$\mu(\sigma)$ complex poles $z = e^\lambda$ in $\mathbf{C}^m$
$\lambda = \pi^\sigma$	$\lambda = \pi^\sigma + i\,2\pi\frac{v}{\mu(\sigma)};\ v \in V_\sigma \subset \mathbf{Z}^m$
	$V_\sigma = \{v \in \mathbf{Z}^m \,\vert\, v' A_\sigma = 0 \bmod \mu(\sigma)\}$

So both $\widehat{f}_d(b,c)$ and $\widehat{f}(b,c)$ are computed with the **same Cauchy's residue technique** but the resulting contribution of each vertex for $\widehat{f}_d(b,c)$ is **more complicated to evaluate** because of the additional **complex zeros**.

Brion and Vergne 's discrete formula

Let $\sigma := [A_{\sigma_1}|\cdots|A_{\sigma_m}]$ be a feasible basis of $\{Ax = b,\ x \geq 0\}$, with $\mu(\sigma) := \det(A_\sigma)$, and $\mu(\sigma)$ "dual" variables $z = e^{\lambda} e^{2i\pi v/\mu(\sigma)} \in \mathbf{C}^m$ with

$$\lambda = \pi^\sigma + i\,2\pi \frac{v}{\mu(\sigma)}; \quad v \in V_\sigma := \{v \in \mathbf{Z}^m | \quad v' A_\sigma = 0 \bmod \mu(\sigma)\}.$$

Re-interpreted with these data, Brion and Vergne 's original discrete formula reads (Lasserre)

$$\widehat{f}_d(b,c) = \sum_{x(\sigma):\ \text{vertex of } \Omega(b)} e^{c'x(\sigma)} \times \frac{1}{\mu(\sigma)} \left[\sum_{v \in V_\sigma} \frac{e^{2i\pi v' b/\mu(\sigma)}}{\prod_{k \notin \sigma} (1 - e^{-(2i\pi v' A_k/\mu(\sigma))}\, e^{(c_k - \pi^\sigma A_k)})} \right]$$

Back to **optimization :** $f_d(b,c) = \max\{c'x \mid Ax = b; \quad x \in \mathbf{N}^n\}$.

Theorem (Lasserre).Assume that

$$\max_\sigma \mathrm{e}^{c'x(\sigma)} \times \lim_{r\to\infty} \left[\sum_{v\in V_\sigma} \frac{\mathrm{e}^{2i\pi v'b/\mu(\sigma)}}{\prod_{k\notin\sigma} (1 - \mathrm{e}^{-(2i\pi v' A_k/\mu(\sigma))}\, \mathrm{e}^{r(c_k - \pi^\sigma A_k)})} \} \right]^{1/r}$$

is attained at a unique basis σ^*. Then :

$$f_d(b,c) = c'x(\sigma^*) + \sum_{k\notin\sigma^*} (c_k - \pi^{\sigma^*} A_k) x_k^* = \sum_j c_j\, x_j^*.$$

σ^* is an optimal basis of the linear program, and

$x(\sigma^*)$ (resp. x^*) is an optimal solution of the linear (resp. integer) program.

In this case :

$$f_d(b,c) = c'x(\sigma^*) + \begin{cases} \max & \displaystyle\sum_{k \notin \sigma^*} (c_k - \pi^{\sigma^*} A_k) x_k \\ & A_{\sigma^*} u + \displaystyle\sum_{k \notin \sigma^*} A_k x_k = b \\ & u \in \mathbf{Z}^m; \quad x_k \in \mathbf{N} \quad \forall k \notin \sigma^* \end{cases}$$

$f_d(b,c) = c'x(\sigma^*) +$ **opt. value of GOMORY relaxation!**

As a consequence,

Corollary For $t \in \mathbf{N}$ sufficiently large, the gap $f(tb,c) - f_d(tb,c)$ between the *continuous* and *discrete* optimal values is a (constant) *periodic function* with period $\mu(\sigma^*) = \det(A_{\sigma^*})$, where σ^* is the optimal basis of the continous LP.

The periodicity is due to the complex poles $z = e^{\lambda^*}$ with

$$\lambda^* = \pi^{\sigma^*} + i2\pi \frac{v}{\mu(\sigma^*)}, \quad v \in V_{\sigma^*} \subset \mathbf{Z}^m;\ v' A_{\sigma^*} = 0 \text{ mod } \mu(\sigma^*).$$

KNAPSACK REVISITED

Let $a \in \mathbf{N}^{n+1}$, $0 \leq c \in \mathbf{R}^{n+1}$, $b \in \mathbf{N}$, and $s := \sum_i a_i$.

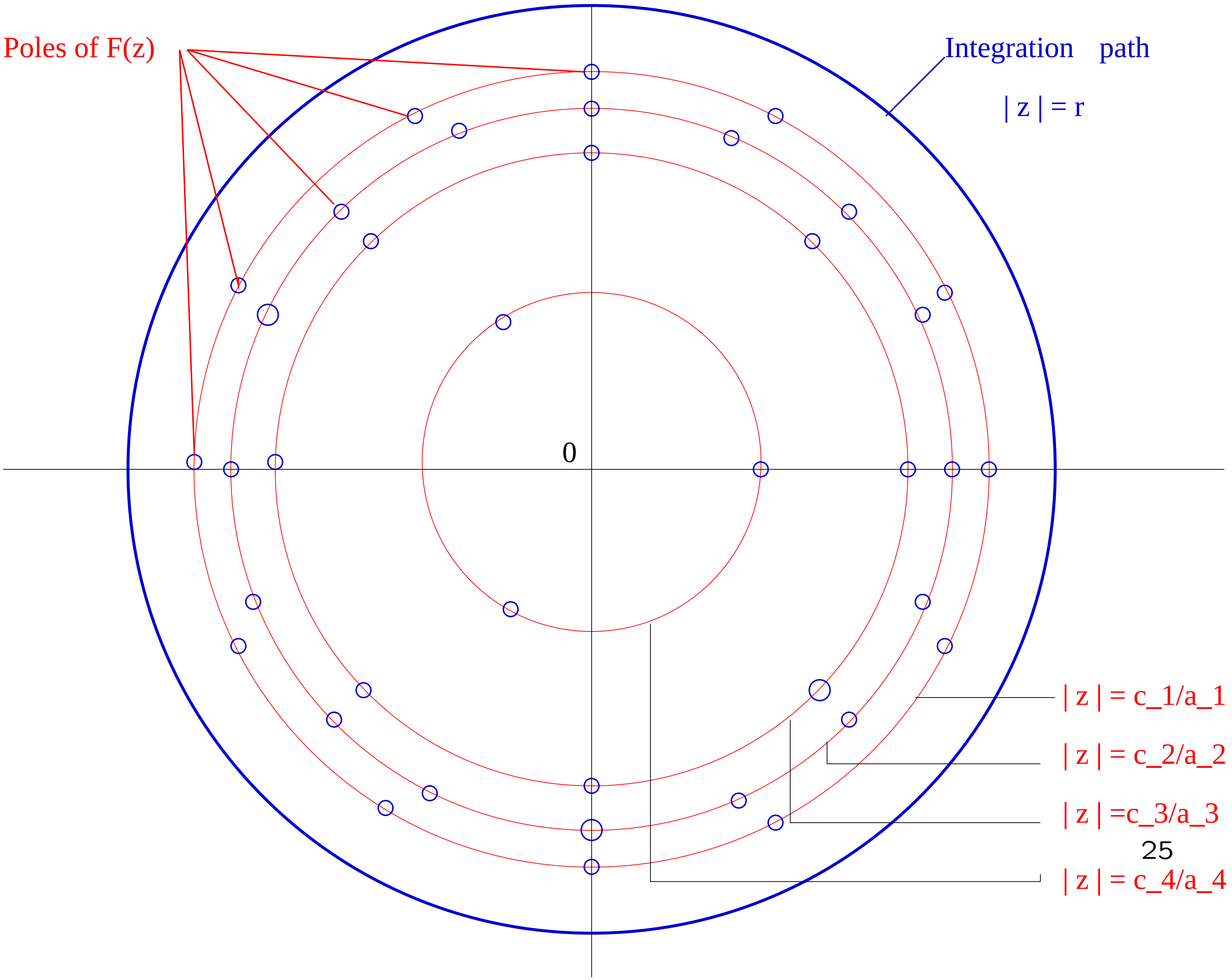
$$f_d(b,c) \;=\; \max_{x \in \mathbf{N}^n} \{c'x \,|\, a'x = b\}; \quad \widehat{f}_d(b,c) \;=\; \sum_{x \in \mathbf{N}^n} \{e^{c'x} \,|\, a'x = b\}.$$

$$\widehat{F}_d(z,c) \;=\; \frac{z^s}{\prod_{j=1}^n (z^{a_j} - e^{c_j})}; \quad \widehat{f}_d(b,c) \;=\; \frac{1}{2i\pi} \int_{|z|=\gamma} z^{b-1} \widehat{F}_d(z,c)\, dz$$

with $\gamma \in \mathbf{R}^+$ and $\ln \gamma > \max_{i=1}^n c_i/a_i$.

The poles $\{z_{jk}\}$ of $\widehat{F}$ satisfy :

$$\ln z_{jk} \;=\; c_j/a_j + 2i\pi k/a_j\,; \qquad k = 1, \ldots, a_j;\; j = 1, \ldots, n$$



Poles of F(z)
Integration path
| z | = r
0
| z | = c_1/a_1
| z | = c_2/a_2
| z | =c_3/a_3
| z | = c_4/a_4

When the ratios $\{c_k/a_k\}$ are close to each other, the integration of $z^{b-1}\widehat{F}_d(z)$ on the circle $|z| = \gamma$ becomes more difficult because nearly all the poles of $\widehat{F}_d(z,c)$ contribute.

Similarly, it is also known that the corresponding knapsack problems are difficult to solve ...

Let $r \in \mathbf{N}$, $c_1/a_1 < c_2/a_2 < \cdots < c_n/a_n$ and $c_j \in \mathbf{N}$ for all j.

$$\frac{\widehat{F}_d(z, rc)}{z} = \frac{z^{s-1}}{\prod_{j=1}^{n}(z^{a_j} - \mathrm{e}^{rc_j})} = \sum_{j=1}^{n} \frac{Q_j(z)}{(z^{a_j} - \mathrm{e}^{rc_j})},$$

for some polynomials

$$z \mapsto Q_j(z) := \sum_{k=0}^{a_j-1} Q_{jk} z^k = \sum_{k=0}^{a_j-1} \left[\frac{M_{jk}(\mathrm{e}^r)}{R_{jk}(\mathrm{e}^r)}\right] z^k,$$

for some polynomials $\{M_{jk}, R_{jk}\}$ of the variable $y := \mathrm{e}^r$.

In principle they can be obtained by symbolic calculation. Therefore, with $b = s_j \bmod a_j$, for all $j = 1, \ldots, n$,

$$\widehat{f}(b, rc) = \sum_{j=1}^{n} Q_{j(a_j-s_j-1)} y^{\frac{(b-s_j)c_j}{a_j}} = \sum_{j=1}^{n} \frac{M_{j(a_j-s_j-1)}(y)}{R_{j(a_j-s_j-1)}(y)} y^{\frac{(b-s_j)c_j}{a_j}}$$

Hence

$$\mathrm{e}^{f_d(b,c)} = \lim_{r\to\infty} \widehat{f}_d(b,rc)^{1/r}$$

$$= \lim_{r\to\infty} \left[\sum_{j=1}^{n} \frac{M_{j(a_j-s_j-1)}(\mathrm{e}^r)}{R_{j(a_j-s_j-1)}(\mathrm{e}^r)} \, \mathrm{e}^{r\frac{(b-s_j)c_j}{a_j}} \right]^{1/r}$$

Thus, for sufficiently large b,

$$\mathrm{e}^{f_d(b,c)} = \max_{j=1,\ldots,n} \lim_{r\to\infty} \left[\frac{M_{j(a_j-s_j-1)}(\mathrm{e}^r)}{R_{j(a_j-s_j-1)}(\mathrm{e}^r)} \, \mathrm{e}^{r\frac{(b-s_j)c_j}{a_j}} \right]^{1/r}$$

$$f_d(b,c) = \frac{c_1 b}{a_1} - \frac{c_1 s_1}{a_1} + \deg M_{1(a_1-s_1-1)}(y) - \deg R_{1(a_1-s_1-1)}(y)$$

Example : Consider the problem

$$\widehat{f}(b,c) \;:=\; \max_{x\in\mathbf{N}^2}\{3x_1+5x_2 \,|\, 2x_1+3x_2 \le b\}.$$

Let $y = \mathrm{e}^r$. We obtain

$$\begin{aligned}
\frac{F(z)}{z} &= \frac{z^5}{(z-1)(z^2-y^3)(z^3-y^5)} \\
&= \frac{(y^9+y^7+y^6)+z\,(y^7+y^6+y^4)+z^2(y^6+y^4+y^2)}{(y^5-1)(y-1)(z^3-y^5)} \\
&- \frac{(y^5+y^3)+z\,(y^3+y^2)}{(y^3-1)(y-1)(z^2-y^3)} + \frac{1}{(y^5-1)(y^3-1)(z-1)}
\end{aligned}$$

Hence, $\widehat{f}_d(b,c) = f(b,c)$ if $b = 0 \bmod 3$ and for large b,

$$\begin{aligned}
\widehat{f}_d(b,c) &= f_d(b,c) - 5/3 + 1 \qquad \text{whenever } b = 1 \bmod 3 \\
\widehat{f}_d(b,c) &= f_d(b,c) - 10/3 + 3 \qquad \text{whenever } b = 2 \bmod 3
\end{aligned}$$

A Discrete Farkas Lemma

Let $A \in \mathbf{N}^{m \times n}, b \in \mathbf{N}^m$ and consider the problem deciding **whether or not** $Ax = b$ has a solution $x \in \mathbf{N}^n$.

Theorem: (i) *$Ax = b$ has a solution $x \in \mathbf{N}^n$ if and only if the polynomial $b \mapsto z^b - 1$ in $\mathbf{R}[z_1, \ldots, z_m]$ can be written*

$$z_1^{b_1} \cdots z_m^{b_m} - 1 = \sum_{j=1}^{n} Q_j(z)(z^{A_j} - 1) = \sum_{j=1}^{n} Q_j(z)(z_1^{A_{1j}} \cdots z_m^{A_{mj}} - 1)$$

for some polynomials $Q_j(z)$, all with **nonnegative coefficients.**

(ii) *The degree of the Q_j's is bounded by* $b^* := \sum_{j=1}^{m} b_j - \max_k \sum_{j=1}^{m} A_{jk}$.

A single LP to solve with $n \times \binom{b^*+m}{b^*}$ variables, $\binom{b^*+m}{m}$ constraints and a (sparse) matrix of coefficients in $\{0, \pm 1\}$

One also retrieves the classical **Farkas Lemma** in $\mathbf{R}^n$, that is,

$$\{x \in \mathbf{R}^n \,|\, Ax = b;\; x \geq 0\} \neq \emptyset \qquad \Leftrightarrow \qquad A'u \geq 0 \Rightarrow \quad b'u \geq 0.$$

Indeed, if $Ax = b$ has a solution $x \in \mathbf{N}^n$, then with $u = \ln z$,

$$\mathrm{e}^{b'u} - 1 = \sum_{j=1}^{n} Q_j(e^{u_1}, \ldots, e^{u_m})(\mathrm{e}^{(A'u)_j} - 1).$$

Therefore,

$$A'u \geq 0 \Rightarrow \mathrm{e}^{(A'u)_j} - 1 \geq 0 \Rightarrow \mathrm{e}^{b'u} - 1 \geq 0 \quad \Rightarrow b'u \geq 0,$$

and one retrieves (*), i.e., $Ax = b$ has a solution $x \in \mathbf{R}^n$.

The general case $A \in \mathbf{Z}^{m\times n}, b \in \mathbf{Z}^m$.

Let $\Omega := \{x \in \mathbf{R}^n | Ax = b; \quad x \geq 0\}$ be a **polytope.**

Let $\alpha \in \mathbf{N}^n$ be such that for every column A_j of A,

$A_{kj} + \alpha_j \geq 0 \; \forall\, k = 1, \ldots, m$; let $\mathbf{N} \ni \beta \geq \rho(\alpha) := \max\{\alpha' x \,|\, x \in \Omega\}$.

Theorem: (i) *$Ax = b$ has a solution $x \in \mathbf{N}^n$ if and only if the polynomial $z \mapsto z^b (zy)^\beta - 1$ in $\mathbf{R}[z_1, \ldots, z_m]$ can be written*

$$z^b (zy)^\beta - 1 \;=\; Q_0(z, y)(zy - 1) + \sum_{j=1}^{n} Q_j(z, y)(z^{A_j}(zy)^{\alpha_j} - 1),$$

for some polynomials $Q_j(z)$, all with **nonnegative coefficients.**

(ii) *The degree of the Q_j's is bounded by $b^* := (m+1)\beta + \sum_{j=1}^{m} b_j$.*

Back to standard Farkas lemma

$$\{x \in \mathbf{R}^n \mid \quad Ax = b,\ x \geq 0\} \neq \emptyset \quad \Leftrightarrow \quad \left[A'\lambda \geq 0\right] \Rightarrow b'\lambda \geq 0.$$

But, **equivalently** $\{x \in \mathbf{R}^n \mid \quad Ax = b,\ x \geq 0\} \neq \emptyset$ if and only if the polynomial $\lambda \mapsto b'\lambda$ can be written

$$b'\lambda = \sum_{j=1}^{n} Q_j(\lambda)(A'\lambda)_j,$$

for some polynomials $\{Q_j\} \subset \mathbf{R}[\lambda_1, \ldots, \lambda_m]$, all with **nonnegative** coefficients.

In this case, each Q_j is necessarily a constant, that is, $Q_j \equiv Q_j(0) = x_j \geq 0$, and $Ax = b$!

$\mathsf{P} = \{x \in \mathbf{R}^n \mid Ax = b,\ x \geq 0\}$	$\mathsf{P} \cap \mathbf{Z}^n$
$x \in \mathsf{P}$ $\Leftrightarrow x = Q(0,\ldots,0)$ with $Q \in \mathbf{R}[\lambda_1,\ldots,\lambda_m]$ $b'\lambda = \langle Q, A'\lambda\rangle$ $Q \succeq 0$	$x \in$ integer hull (P) $\Leftrightarrow x = Q(1,\ldots,1)$ with $Q \in \mathbf{R}[\mathrm{e}^{\lambda_1},\ldots,\mathrm{e}^{\lambda_m}]$ $\mathrm{e}^{b'\lambda} - 1 = \langle Q, \mathrm{e}^{A'\lambda} - 1_n\rangle$ $Q \succeq 0$

Comparing continuous and discrete Farkas lemma

An equivalent Linear program

Let $0 \le q = \{q_{j\alpha}\} \in \mathbf{R}^{ns}$ be the coefficients of the Q_j's in

$$z^b - 1 = \sum_{j=1}^{n} Q_j(z)(z^{A_j} - 1)$$

They are solutions of a linear system

$$\mathsf{M}\, q = r, \quad q \ge 0$$

for some matrix M and vector r, both with $0, \pm 1$ coefficients.

** M and r are easily obtained from A, b with **no** computation

Write $q = (q_1, q_2, \ldots, q_n)$ with each $q_j = \{q_{j\alpha}\} \in \mathbf{R}^s$, and let $\widehat{c}_{j\alpha} := c_j$ for all α

Theorem : *Let* $A \in \mathbf{N}^{m\times n}, b \in \mathbf{N}^m, c \in \mathbf{R}^n$.

(i) *The integer program* $\mathbf{P} \rightarrow \max\{c'x | Ax = b,\ x \in \mathbf{N}^n\}$ *has same value as the linear program*

$$\mathbf{Q} \rightarrow \max\{\sum_{j=1}^{n} \hat{c}'_j\, q_j \mid \quad \mathrm{M}\, q = r; \quad q \geq 0\}.$$

(ii) *Let* q^* *be an optimal vertex, and let*

$$x_j^* := \sum_\alpha q_{j\alpha}^* \quad j = 1, \ldots, n.$$

Then $x^* \in \mathbf{N}^n$ *and* x^* *is an optimal solution of* $\mathbf{P}$.

The link with superadditive functions

The LP-dual $\mathbf{Q}^*$ of the linear program $\mathbf{Q}$ reads

$$\mathbf{Q}^* \to \min_{\pi}\{\pi' r \mid \quad \mathsf{M}'\pi \geq \widehat{c}\}.$$

More precisely, with $\mathcal{D} := \prod_{j=1}^{n}\{0,1,\ldots,b_j\} \subset \mathbf{N}^m$,

$$\mathbf{Q}^* \to \begin{cases} \qquad \min_{\pi} \quad \pi(b) - \pi(0) \\ \\ \text{s.t.} \quad \pi(\alpha + A_j) - \pi(\alpha) \;\geq c_j, \quad \alpha \in \mathcal{D},\ j = 1,\ldots,n \end{cases}$$

Let $\Pi := \{\pi : \mathbf{N}^m \to \mathbf{R} \cup \{+\infty\} \mid \quad \pi(x) = \infty \text{ if and only if } x \notin \mathcal{D}\}$.

For every $\pi \in \Pi$, let $f_\pi : \mathbf{N}^m \to \mathbf{R} \cup \{+\infty\}$ be the function

$$f_\pi(x) := \inf_{\alpha \in \mathcal{D}} \pi(\alpha + x) - \pi(\alpha), \qquad x \in \mathbf{N}^m$$

For every $\pi \in \Pi$, the function f_π is superadditive and $f_\pi(0) = 0$.

The LP dual $\mathbf{Q}^*$ reads

$$\mathbf{Q}^* \to \begin{cases} \quad \min_{\pi \in \Pi} \quad f_\pi(b) \\ \text{s.t.} \quad f_\pi(A_j) \geq c_j, \quad j = 1, \dots, n. \end{cases}$$

Thus, $\mathbf{Q}^*$ is a *simplified and explicit form* of the abstract dual of Jeroslow, Wolsey, stated in terms of superadditive functions.

$\to$ In the abstract dual one may restrict to the subclass of superadditive functions derived from the representation

$$z^b - 1 = \sum_{j=1}^{n} Q_j(z^{A_j} - 1).$$

And, with $\mathsf{P} = \{x \in \mathbf{R}^n_+ \,|\, Ax = b\}$ the integer hull $\mathsf{co}(\mathsf{P} \cap \mathbf{Z}^n)$ reads

$$\mathsf{co}(\mathsf{P} \cap \mathbf{Z}^n) = \{x \in \mathbf{R}^n \,|\, \quad \sum_{j=1}^{n} f_\pi(A_j)x_j \le f_\pi(b)\},$$

for finitely many π, generators of the convex cone

$$(\pi, \lambda): \quad \pi(\alpha + A_j) - \pi(\alpha) + \lambda_j \ge 0, \quad \alpha + A_j \in \mathcal{D},\ j = 1, \ldots n$$

CONCLUSION

Generating functions permit to exhibit a natural **duality** for integer programming, an IP-analogue of **LP duality**.

This duality also shows which kind of **superadditive functions** are useful in the abstract dual of Jeroslow, Wolsey.

This might help providing **efficient Gomory cuts** in MIP solvers like CPLEX, or XPRESS-MP.

Another dual problem

Let $A \in \mathbf{Z}^{m\times n}, b \in \mathbf{Z}^m, c \in \mathbf{R}^n$. Let $y \mapsto f(y,c) = \max\{c'x \mid Ax = y;\ x \geq 0\}$. The Fenchel transform of the convex function $-f(.,c)$ is the convex function

$$\lambda \mapsto \quad (-f)^*(\lambda, c) = \sup_{y\in\mathbf{R}^m} \lambda' y + f(y,c).$$

The dual problem of the linear program is obtained from Fenchel duality as

$$\begin{aligned} f(b,c) &= \inf_{\lambda\in\mathbf{R}^m} b'\lambda + (-f)^*(-\lambda, c) \\ &= \inf_{\lambda\in\mathbf{R}^m} b'\lambda + \sup_{x\in\mathbf{R}^m} (c - A'\lambda)' x = \min\{b'\lambda \mid A'\lambda \geq c\} \end{aligned}$$

Equivalently

$$\mathrm{e}^{f(b,c)} = \inf_{\lambda\in\mathbf{R}^m} \sup_{x\in\mathbf{R}^m} \mathrm{e}^{(b-Ax)'\lambda} \mathrm{e}^{c'x}$$

Define

$$\rho^* := \inf_{z \in \mathbf{C}^m} \sup_{x \in \mathbf{N}^n} \Re\left(z^{b-Ax}\, \mathrm{e}^{c'x}\right) = \inf_{z \in \mathbf{C}^m} f_d^*(z, c).$$

Hence,

$$f_d^*(z, c) = \Re\left(z^b \prod_{j=1}^{n} (z^{-A_j} \mathrm{e}^{c_j})^{x_j}\right) < \infty \quad \text{if } |z^{A_j}| \geq \mathrm{e}^{c_j} \quad \forall j$$

(that is, $A' \ln |z| \geq c$). Next, (writing $z \in \mathbf{C}$ as $\mathrm{e}^{\lambda}\mathrm{e}^{i\theta}$)

$$\rho^* \leq \inf_{z \in \mathbf{R}^m} \sup_{x \in \mathbf{R}^n} \Re\left(z^{b-Ax}\, \mathrm{e}^{c'x}\right) = \inf_{\lambda \in \mathbf{R}^m} \sup_{x \in \mathbf{R}^n} \mathrm{e}^{(b-Ax)'\lambda}\, \mathrm{e}^{c'x} = \mathrm{e}^{f(b,c)}$$

Finally, with $z \in \mathbf{C}^m$ arbitrary fixed

$$\sup_{x \in \mathbf{N}^n} \Re\left(z^{b-Ax}\, \mathrm{e}^{c'x}\right) \geq \mathrm{e}^{c'x^*} = \mathrm{e}^{f_d(b,c)}$$

Hence $\mathbf{f_d(b, c)} \leq \ln \rho^* \leq \mathbf{f(b, c)}$.

Let σ^* be an optimal basis of the linear program. Under uniqueness of the "$\max_\sigma$" in Brion and Vergne 's formula, and an additional technical condition

$$\mathrm{e}^{f_d(b,c)} = \rho^* = \max_{x \in \mathbf{N}^n} \Re\left(\hat{z}^{b-Ax}\mathrm{e}^{c'x}\right) = f_d^*(\hat{z}, c)$$

where $\hat{z}^{A_j} = \gamma\,\mathrm{e}^{c_j} \quad \forall j \in \sigma^*$ for some real $\gamma > 1$.

$\hat{z}$ is an optimal solution of the dual problem

$$\inf_{z \in \mathbf{C}^m} f_d^*(z, c)$$