Cuts for mixed 0-1 conic programs

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Research Goals

What are mixed 0-1 conic programs

Are conic IPs useful?

How to solve mixed 0-1 conic programs?

Gomory cuts and generalizations

Cuts from hierarchies of tighter relaxations

Computational results

Conclusion

What are mixed 0-1 conic programs?

- Mixed 0-1 conic programs (MCP)

$$\begin{array}{rl} \min & \mathbf{c}^T\mathbf{x}, \\ \text{s.t.} & \mathbf{A}\mathbf{x} \succeq_{\mathcal{K}} \mathbf{b} \\ & x_i \in \{0,1\}, \quad i = 1,\dots,p. \end{array}$$

- $\succeq_{\mathcal{K}}$: partial order w.r.t. a cone $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \ldots \times \mathcal{K}_r$
- Each $\mathcal{K}_j$ is one of the following
  - Linear cone (LP): $\mathcal{K} = \{\mathbf{y} : y_i \geq 0, \ \forall i\}$
  - Second-order cone (SOCP): $\mathcal{K} = \{\mathbf{y} = (y_0; \bar{\mathbf{y}}) : y_0 \geq \|\bar{\mathbf{y}}\|\}$
  - Semidefinite cone (SDP): $\mathcal{K} = \{\mathbf{Y} : \mathbf{Y} \text{ sym. pos. semidef.}\}$.
- Each $\mathcal{K}_j = \mathcal{K}_j^*$ is self-dual: $\mathcal{K} = \mathcal{K}^*$ is self-dual

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- Compact formulation: TSP, Scheduling

$$\begin{array}{ll} \min & \mathbf{Tr}(\mathbf{CX}), \\ \text{s. t.} & (2\cos(2\pi/n))\mathbf{I} + \frac{2}{n}(1-\cos(2\pi/n))\mathbf{1}\mathbf{1}^T - (\mathbf{X}+\mathbf{X}^T) \succeq 0 \\ & \mathbf{X1} = \mathbf{X}^T\mathbf{1} = \mathbf{1} \quad \mathbf{diag}(\mathbf{X}) = \mathbf{0} \\ & X_{ij} \in \{0,1\} \quad \forall i,j \end{array}$$

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- Tighter formulations of mixed 0-1 LPs:
  - Sherali-Adams lifting
  - Lovasz-Schrijver lifting
- Naturally arise in many applications:
  - Robust counterparts of uncertain mixed 0-1 LPs
  - Hybrid control

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- Identical to the C-G procedure for integer programs.

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Sub-tour elimination inequalities for TSP are rank-1 C-G cuts.

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Theorem
Triangle inequalities for max-cut are rank-1 C-G cuts.

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- Extends to a special class of integer SDPs
  - SDP constraint: $\sum_i x_i \mathbf{A}_i \succeq \mathbf{A}_0$
  - Each $\mathbf{A}_i$ non-negative integer matrices
  - Simple extension of Lasserre's ideas

Cuts from hierarchies of tighter relaxation

- MCP: $\mathcal{C}^\circ = \mathbf{conv}\left(\mathbf{x} : \mathbf{A}\mathbf{x} \succeq_{\mathcal{K}} \mathbf{b}, x_i \in \{0, 1\}, 1 \le i \le p\right)$

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  - Construct a tighter relaxation $\bar{\mathcal{C}}$ with $\mathbf{x}^{(t)} \notin \mathcal{C}^\circ$
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- Introduce new vars and connect using conic constraints

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- $-\mathbf{Q} \succeq \mathbf{0}$: $\mathbf{x}_B^T \mathbf{Q} \mathbf{x}_B + \boldsymbol{\alpha}^T \mathbf{x} + \beta \geq 0$ valid convex inequality

Traveling salesman problem: Random instance

instance	gap (%)	% closed 25 cuts	% closed 50 cuts	% closed B-and-B	% closed B-and-C
Random $n = 16$	11.27	31.25	31.25	95.07	100.00
$n = 16$	3.11	50.00	100.00	92.23	100.00
$n = 16$	1.33	100.00	100.00	100.00	100.00
Random $n = 20$	6.84	38.46	46.15	93.16	94.22
$n = 20$	6.70	42.86	42.86	81.34	93.30
$n = 20$	6.77	44.45	55.56	93.23	94.74
Random $n = 25$	3.39	50.00	66.67	83.05	100.00
$n = 25$	4.24	60.00	80.00	79.05	100.00
$n = 25$	3.39	80.00	90.00	95.25	97.97
Random $n = 30$	0.69	100.00	100.00	97.24	100.00
$n = 30$	0.86	100.00	100.00	97.41	100.00
$n = 30$	0.77	50.00	100.00	96.93	100.00
Random $n = 35$	1.04	100.00	100.00	92.19	100.00
$n = 35$	0.51	100.00	100.00	96.46	100.00
$n = 35$	1.39	75.00	100.00	78.47	100.00

Traveling salesman problem: TSPLIB problems

instance	gap (%)	% closed		% closed	
		25 cuts	**50 cuts**	**B-and-B**	**B-and-C**
burma14	4.45	20.3	25.0	100.00	100.00
ulysses16	7.30	5.7	6.3	97.08	98.58
gr17	13.2	12.0	13.8	90.12	90.94
ulysses22	9.78	7.70	7.85	92.73	100.00
gr24	3.30	19.05	21.43	76.58	92.80
fri26	5.12	5.26	7.02	82.29	97.55
bayg29	3.54	12.50	13.89	95.47	95.47
bays29	3.56	12.50	13.89	82.98	90.40

- Small size !
- No warm start or dual simplex-like method.

MLPs: Comparison of linear/SDP cuts

- Question: Are cuts from SDP relaxations superior?
- Control the complexity of SDP cut generation
  - At each iteration, project LP into the space of fractional variables
  - Construct an SDP lifting with $|B| \leq 10$
  - Generate disjunctive cut for each index in the set B
  - "Lift" cut to the space of all variables
- One linear cut for each fractional variable: no cut lifting
  - $\mathcal{L}_1$ normalization: cut generation is an LP
  - $\mathcal{L}_2$ normalization: cut generation is an SOCP
- Did not focus on time. Why not? No dual simplex.

Pure 0-1 LPs

problem	semidefinite			linear: $\mathcal{L}_2$			linear : $\mathcal{L}_1$		
	cuts	iter	%	cuts	iter	%	cuts	iter	%
LSB	120	32	100	182	35	100	477	45	100
LSC	520	50	94	627	50	90	767	50	89
PE4	104	25	100	175	31	100	265	42	100
PE5	250	40	100	305	45	100	348	50	98
PE6	342	42	100	442	50	97	540	50	89
PE7	470	50	100	576	50	91	688	50	78
Stein27	430	50	92	546	50	82	646	50	72
Stein45	467	50	84	620	50	73	890	50	57

Mixed 0-1 LPs

problem	semidefinite			linear: $\mathcal{L}_2$			linear: $\mathcal{L}_1$		
	cuts	iter	%	cuts	iter	%	cuts	iter	%
CTN1	212	24	100	254	28	100	540	32	100
CTN2	289	32	100	378	42	100	613	50	95
CTN3	264	36	100	476	45	100	765	50	96
Danoint*	445	50	91	647	50	66	876	50	54

- SDP cuts are superior: Reduce both iters and cuts
- p small: $\mathcal{L}_2$ normalization close to SDP cuts
- SDP cuts are not likely to be facet defining
- Effective because initial relaxation is bad?

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- Mixed 0-1 conic programs are interesting!
- All techniques known for mixed 0-1 LPs extend readily
- "Quality" of conic relaxation is quite good
- But ... computational techniques are lacking