Inverse Linear Programming

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Outline of the talk:

1. Problem formulation
2. Special case: Zero optimal function value
3. Relation to an MPEC
4. A sufficient optimality condition
5. A necessary optimality condition
6. Condition for global minimum

Problem Formulation

Let A be an (m,n)-matrix and $Y \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be polyhedral and closed.

Parametric linear programming problem:

$$\Psi(b,c) := \underset{x}{\operatorname{argmax}} \; \{c^\top x : Ax = b, \; x \geq 0\}.$$

Given $x^0 \in \mathbb{R}^n$:

Inverse Linear Programming Problem:

Find $(x^*, b^*, c^*) \in \mathbb{R}^n \times Y$ solving

$$\min_{x,b,c} \{\|x - x^0\|^2 : x \in \Psi(b,c), \; (b,c) \in Y\} \tag{1}$$

Problem Formulation

Applications:

1. Parameter identification

2. "Best" solutions of multiobjective linear programming problems

3. "Improving" optimal solutions of linear programming problems

Zero optimal function value

R.K.Ahuja & J.B. Orlin: Inverse Optimization, Oper.Res. 2001

Main Assumption (A1):

$$\exists (b^0, c^0) \in Y : x^0 \in \Psi(b^0, c^0).$$

Then: secondary goal

Given (b^*, c^*):

$$\min_{b,c} \{\|(b,c)^\top - (b^*, c^*)^\top\|_r : x^0 \in \Psi(b,c)\}. \quad (2)$$

Zero optimal function value

Definition: $\mathcal{R}(y) := \{(b,c)^\top : y \in \Psi(b,c)\}$ is the *Region of Stability* for the point y.

Theorem: For a linear programming problem, the region of stability is a polyhedral set.

$$\mathcal{R}(x^0) = \{ \ (b,c)^\top : \exists\ u \text{ with } A^\top u \geq c, \\ x^{0\top}(A^\top u - c) = 0,\ Ax^0 = b\}.$$

Corollary (R.K.Ahuja & J.B. Orlin): Problem (2) is a linear programming problem provided that $r \in \{1, \infty\}$.

Assumption (A1) is not longer used!

Relation to an MPEC

From now on:

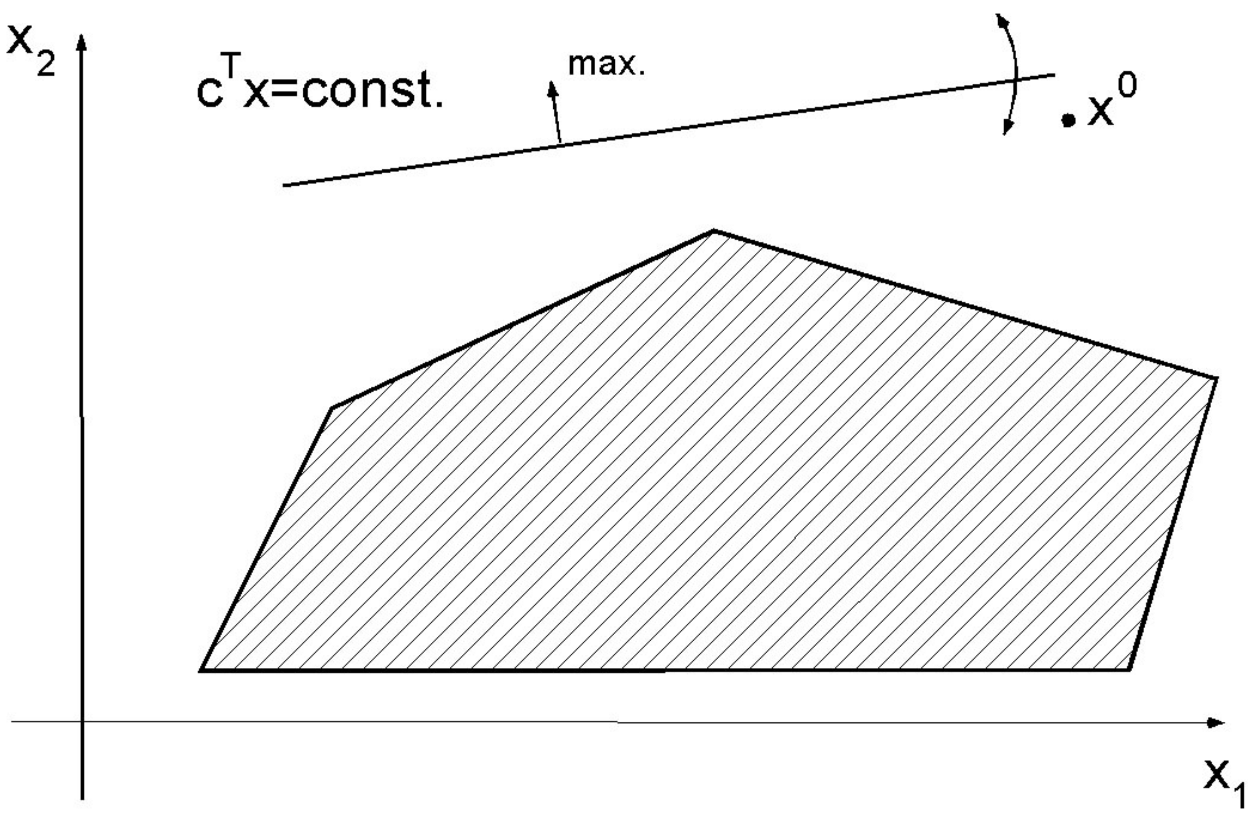
Let b be fixed for simplicity.

Region of stability:

Definition: $\mathcal{R}(y) := \{c : y \in \Psi(c)\}$
is the *Region of Stability* for the point y.

Relation to an MPEC

Without assumption (A1) we get a bilevel programming problem:



The resulting problem is: nonconvex, with implicitly determined feasible set.

Relation to an MPEC

Approaches to obtain optimality conditions for bilevel programs
$\min_{x,y}\{F(x,y) : x \in \Psi(y),\ y \in Y\}$:

1. If $|\Psi(y)| \leq 1 \ \forall\ y \in Y$ then:

$$\min_{x,y}\{F(y(x),y) : y \in Y\}$$

   This is a problem with nondifferentiable objective function; apply nondifferential calculus.
   Bouligand stationary solution, Clarke stationary solution.

   D., 1992

   Here: Not possible.

2. Reformulate (1) using the KKT conditions for the lower level problem:
$\Psi(y) = \underset{x}{\text{argmin}} \{f(x,y) : g(x,y) \le 0\}$
leading to

$$
\begin{aligned}
&F(x,y) \to \min_{x,y,u} \\
&\nabla_x \{f(x,y) + u^\top g(x,y)\} = 0 \\
&u \ge 0, g(x,y) \le 0, u^\top g(x,y) = 0 \\
&y \in Y
\end{aligned}
$$

Use MPEC-MFCQ or MPEC-LICQ to obtain necessary optimality conditions Bouligand stationary solution, Clarke stationary solution.

Scheel, Scholtes, 2000

3. Let
   $T_{\Psi(y)}(x)$ denote the tangent cone to grph$\Psi(y)$ at some point (x, y),
   $T_Y(y)$ – tangent cone to Y at y.
   Then we get the necessary optimality condition

$$\nabla F(x, y)(d, r) \geq 0$$
$$\forall\ (d, r) \in T_{\Psi(y)}(x),\ r \in T_Y(y)$$

This implies Bouligand stationarity

Pang, Fukushima, 1999

Relation to an MPEC

Transform (1):
$\min_{x,c}\{\|x-x^0\|^2 : x \in \Psi(c),\ c \in Y\}$
into an (MPEC):

$$\begin{array}{ll} \min\limits_{x,b,c,u} & \|x-x^0\|^2 \\ \text{subject to} & Ax = b,\ x \geq 0 \\ & A^\top u \geq c \\ & x^\top (A^\top u - c) = 0, \\ & (b,c)^\top \in Y. \end{array} \qquad (3)$$

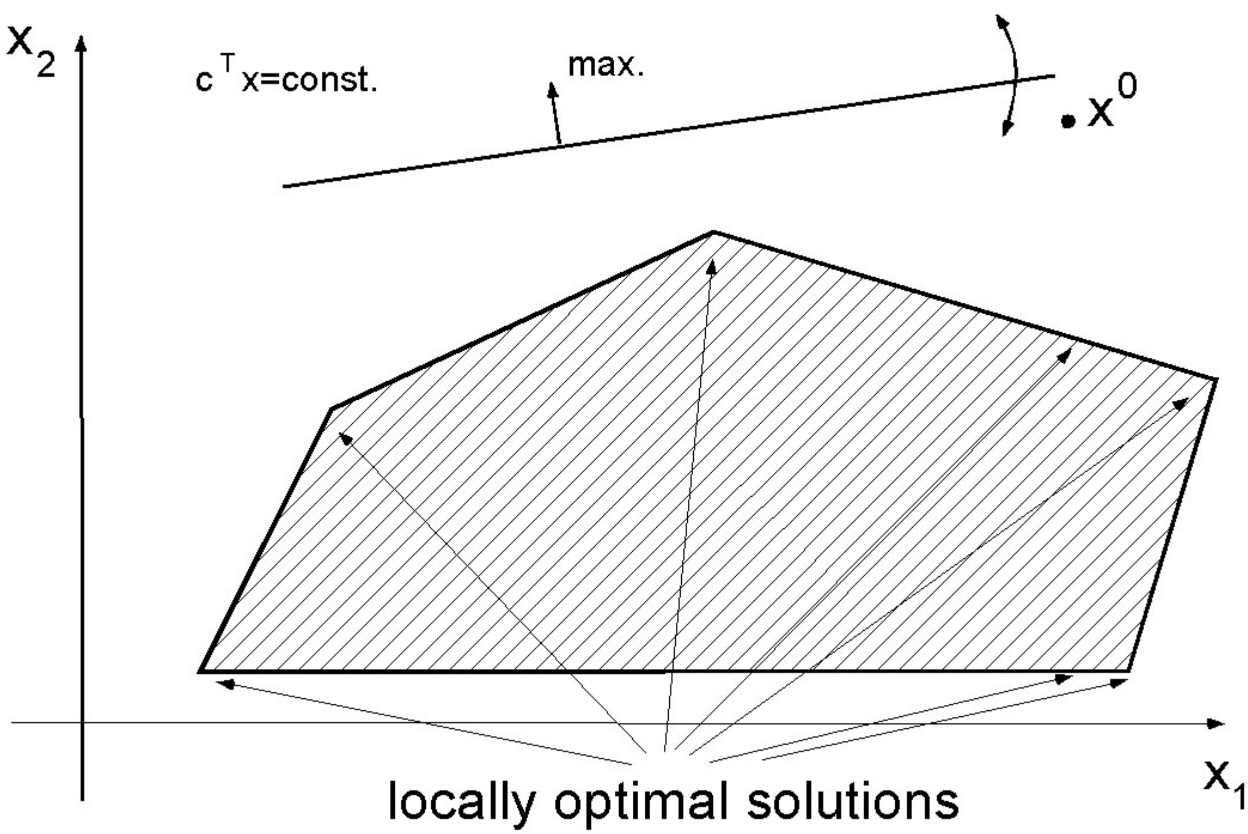
Let $\mathcal{R}(y) := \{c : y \in \Psi(c)\}$.

Theorem: Let $(\overline{x}, \overline{c})$ be such that $\overline{c}^\top \in Y \cap \operatorname{int} \mathcal{R}(\overline{x})$ and $|\Psi(\overline{c})| = 1$.
Then, $(\overline{x}, \overline{c}, \overline{u})$ is a locally optimal solution of (3).

Proof: For all feasible points (x, c, u) for (3) sufficiently close to $(\overline{x}, \overline{c}, \overline{u})$ there is $x = \overline{x}$ by $\overline{c}^\top \in \operatorname{int} \mathcal{R}(\overline{x})$.

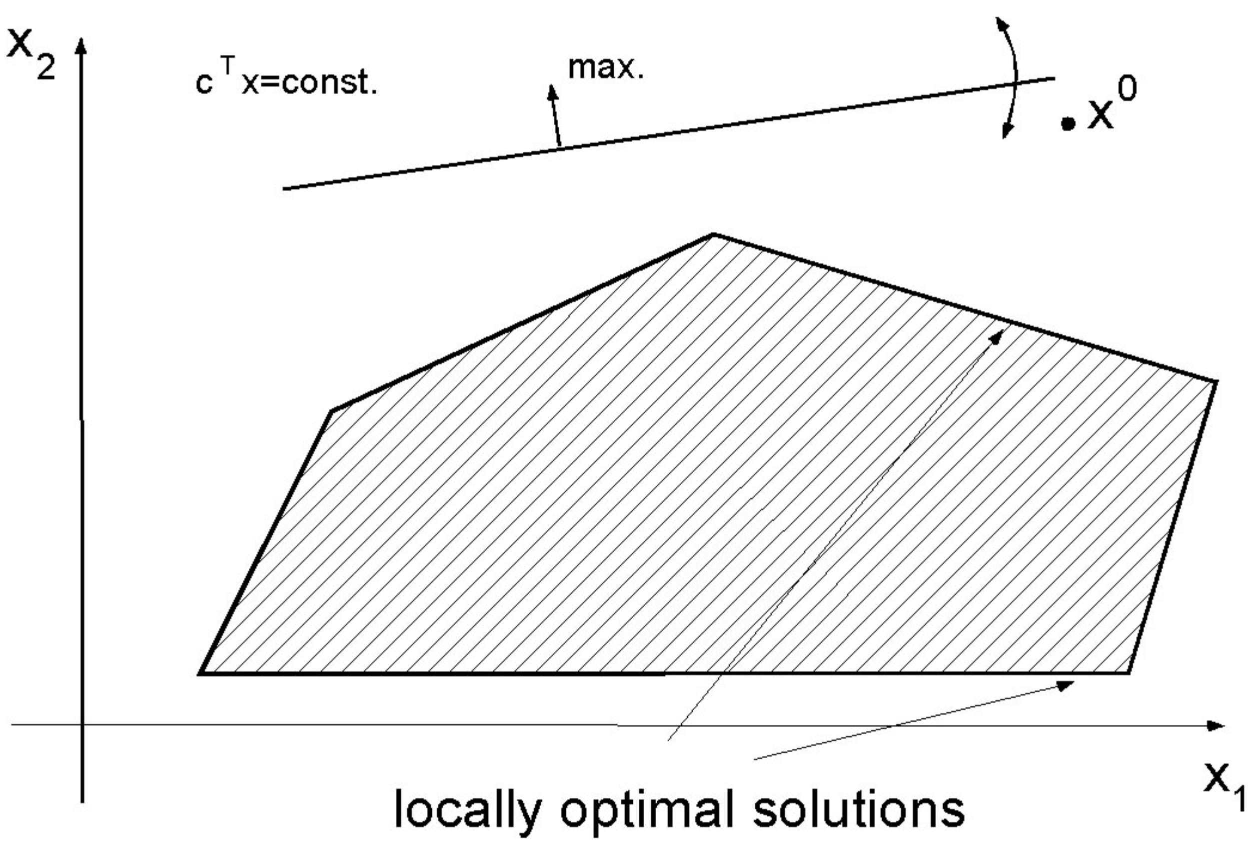
Relation to an MPEC

Corollary: There are an infinite number of locally optimal solutions of (3).



A sufficient optimality condition

Definition: A point (x^*, c^*) is a locally optimal solution of problem (2) if there is an open neighborhood V of x^* such that $\|x - x^0\|^2 \geq \|x^* - x^0\|^2$ for all c, x with $c \in Y$, $x \in \Psi(c)$, $x \in V$.



A sufficient optimality condition

Theorem: Consider a point (x^*, c^*) such that $c^* \in Y,\ x^* \in \Psi(c^*)$ and

$$\mathcal{R}(x^*) = \{c^*\}$$

as well as

$$x^* \in \underset{x}{\operatorname{argmin}} \{\|x - x^0\|^2 : x \in \Psi(c^*)\}.$$

Then, (x^*, c^*) is locally optimal for (2).

Proof: Let $\widehat{x}$ be close to x^* such that $\widehat{x} \in \Psi(\widehat{c})$ for some $\widehat{c} \in Y$. Since the sets $\Psi(\cdot)$ are closed polyhedral sets, $|\mathcal{R}(x^*)| = 1$ implies that $\mathcal{R}(x) = \{c^*\}$ for all x sufficiently close to x^* with $\mathcal{R}(x) \neq \emptyset$. Hence, $\widehat{x} \in \Psi(c^*)$ implying the proof.

A necessary optimality condition

General assumption: Y is bounded.

Lemma: Let $c^1, \ldots, c^t$ be the vertices of $\mathcal{R}(x^*) \cap Y$. (x^*, c^*) is a locally optimal solution of (2) if and only if

$$x^* \in \operatorname*{argmin}_x \{\|x - x^0\|^2 : x \in \Psi(c^i)\},\ i = 1, \ldots, t. \tag{4}$$

Proof: Local optimality of (x^*, c^*) is equivalent to

$$\|\widehat{x} - x^0\|^2 \geq \|x^0 - x^*\|^2 \tag{5}$$

for all $\widehat{x}$ sufficiently close to x^* with $\widehat{x} \in \Psi(\widehat{c})$ for some $\widehat{c} \in Y$. This is equivalent to $\widehat{c} \in \mathcal{R}(\widehat{x}) \cap Y$. Since $\mathcal{R}(\widehat{x}) \cap Y$ is a convex polyhedron, $\widehat{c}$ can be taken as a vertex of $\mathcal{R}(\widehat{x}) \cap Y$.

If (4) is not valid, $x^* \in \Psi(c^i)$ and convexity implies the proof.

If (x^*, c^*) is not locally optimal, (5) is not valid for a sequence $\{(x^s, c^s)\}$ with $\lim_{s\to\infty} x^s = x^*, c^s$ vertex of $\mathcal{R}(x^s) \cap Y$. Upper semicontinuity of $\mathcal{R}(\cdot) \cap Y$ then implies that c^s converges to a vertex of $\mathcal{R}(x^*) \cap Y$. Finiteness of the number of all such vertices implies that c^s is a vertex of $\mathcal{R}(x^*) \cap Y$ for large s. Convexity now proves the Lemma.

A necessary optimality condition

Let $T_{\Psi(c)}(x)$ denote the tangent cone to $\Psi(c)$ at a point $x \in \Psi(c)$.

Theorem: Let (x^*, c^*) be a locally optimal solution of the problem (2). Then, $\forall\ i = 1, \ldots, t$ we have

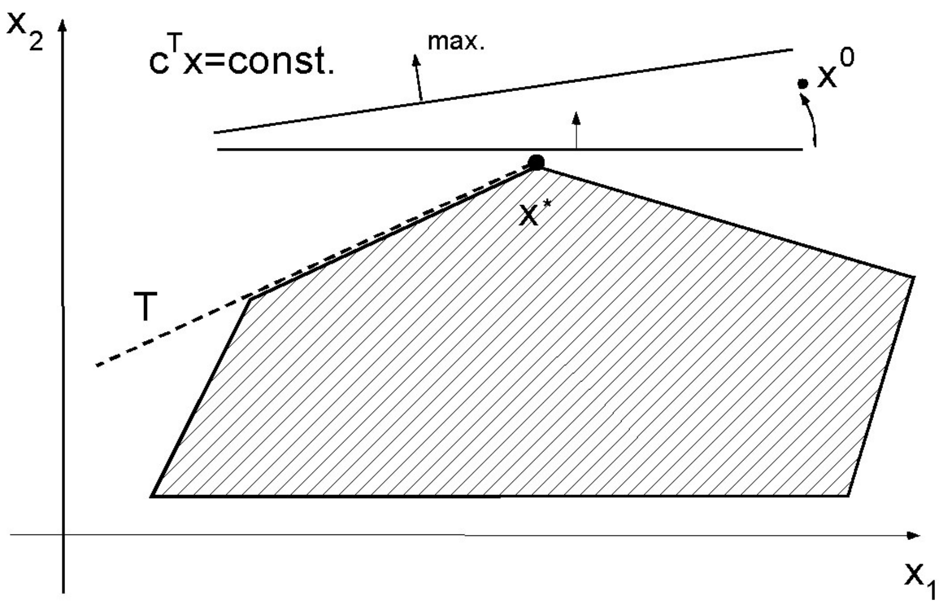
$$(x^* - x^0)^\top d \geq 0$$
$$\forall d \in T_{\Psi(c^i)}(x^*)$$

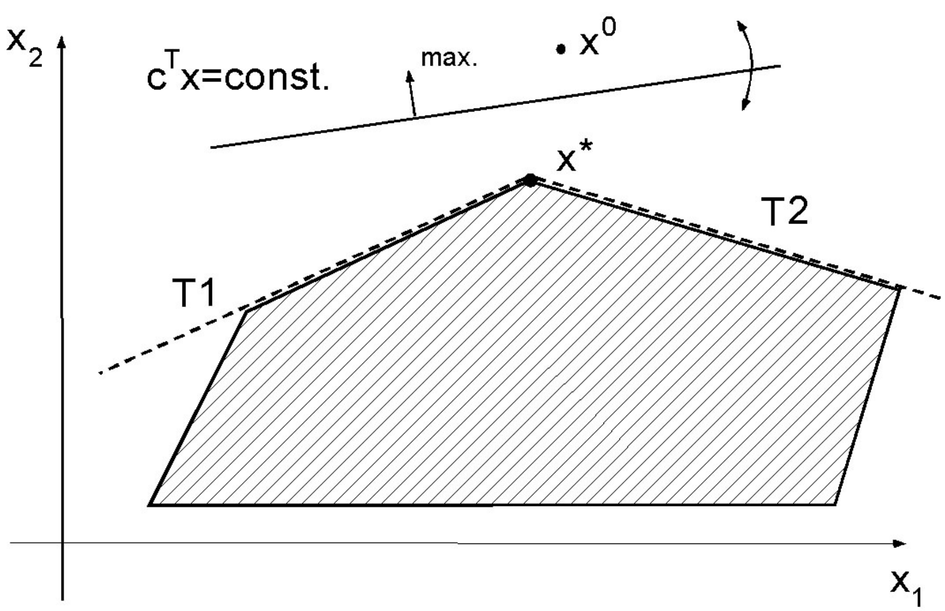
or equivalently

$$(x^* - x^0)^\top d \geq 0$$
$$\forall d \in \text{conv} \bigcup_{i=1}^{t} T_{\Psi(c^i)}(x^*).$$

Remark: If $|\Psi(c)| = 1$ for $c \in \text{int}\, \mathcal{R}(x^*)$ then $T_{\Psi(c^i)}(x^*) = \{0\}$ for $c^i \in \text{bd}\, Y \cap \text{int}\, \mathcal{R}(x^*)$.

A necessary optimality condition





A sufficient optimality condition

Theorem: Let (x^*, c^*) be a feasible solution of the problem (2). Then, (x^*, c^*) is a local minimum if

$$(x^* - x^0)^\top d \geq 0$$
$$\forall d \in \text{conv} \bigcup_{i=1}^{t} T_{\Psi(c^i)}(x^*).$$

Proof: The condition of the Theorem implies

$$(x^* - x^0)^\top d \geq 0 \;\forall\; d \in \bigcup_{i=1}^{t} T_{\Psi(c^i)}(x^*) \Rightarrow$$

$$(x^* - x^0)^\top d \geq 0 \;\forall\; d \in T_{\Psi(c^i)}(x^*) \;\forall\; i = 1, \ldots, t.$$

By strong convexity this shows that x^* is a global optimum of

$$\min\{\|x - x^0\|^2 : x \in \Psi(c^i)\} \;\forall\; i = 1, \ldots, t.$$

Hence, x^* is a global optimal solution of

$$\min\{\|x - x^0\|^2 : x \in \bigcup_{i=1}^{t} \Psi(c^i)\}.$$

Tangent cone

$I(x) = \{i : x_i = 0\}$

$I(y, c) = \{j : (A^\top y - c)_j > 0\}$

$$\mathcal{I}(x) = \{I(y, c) : \begin{array}{l} c \in \{c^1, \ldots, c^t\}, A^\top y - c \geq 0, \\ (A^\top y - c)_j = 0, \;\; j \notin I(x)\} \end{array}$$

(c^i vertex of $\mathcal{R}(x^*) \cap Y$)

$$I^0(x) = \bigcap_{I \in \mathcal{I}(x)} I$$

Then:

$$T_{\Psi(\cdot)}(x) = \bigcup_{I \in \mathcal{I}(x)} T_I(x)$$

with

$T_I(x) =$
$\{d : Ad = 0, d_j \geq 0, j \in I(x) \setminus I, d_j = 0, j \in I\}.$

Tangent cone

$i_0 \notin I^0(x)$ if and only if the following system has a solution:

$$
\begin{aligned}
&A^\top y - c \geq 0 \\
&(A^\top y - c)_j = 0, \; j \notin I(x) \\
&(A^\top y - c)_j = 0, \quad \text{for } j = i_0 \\
&c \in Y
\end{aligned}
$$

Tangent cone

$$T_R(x) = \{d : \ Ad = 0,\ d_j \geq 0,\ j \in I(x) \setminus I^0(x), \\ d_j \geq 0,\ j \in I^0(x)\}.$$

Theorem: If span$\{A_i : i \notin I(\overline{x})\} = \mathbb{R}^m$, then cone $T_{\Psi(\cdot)}(\overline{x}) = T_R(\overline{x})$.

Corollary: For this special problem, verification of the necessary and sufficient optimality conditions belongs to $\mathcal{P}$.

Global optimum

Theorem: Let (x^*, c^*) be local optimal solution for (2), assume that $\mathcal{R}(x^*) \subseteq Y$. If (x^*, c^*) is not a global optimum, then $|\mathcal{R}(x^*)| = 1$.

Proof: Assume, $\mathcal{R}(x^*)$ contains infinitely many elements with vertices $c^1, \ldots, c^t, t > 1$. Then,

$$\{x \geq 0 : Ax = b\} \subseteq \{x^*\} + \text{conv} \bigcup_{i=1}^{t} T_{c^i}(x^*).$$

Local optimality of (x^*, c^*) implies that

$$(x^* - x^0)^\top d \geq 0 \ \forall \ d \in \text{conv} \bigcup_{i=1}^{t} T_{\Psi(c^i)}(x^*). \Rightarrow$$

$$(x^* - x^0)^\top (y - x^*) \geq 0 \ \forall \ y \in \{x^*\} + \text{conv} \bigcup_{i=1}^{t} T_{\Psi(c^i)}(x^*),$$

$$\Rightarrow (x^* - x^0)^\top (y - x^*) \geq 0 \ \forall \ y \in \text{conv} \bigcup_{c \in Y} \Psi(c)$$

$\Rightarrow x^*$ is projection of x^0 on conv $\bigcup_{c \in Y} \Psi(c)$.