# Packing problems, tree-width, and lift-and-project

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## An old practical problem

Build a directed graph to join a set of nodes

- and in that graph, route a given set of multicommodity demands
- the graph must have degrees at most $\boldsymbol{p}$ (given)

$\longrightarrow$ Objective: **minimize the maximum flow used on any edge**.

# This is a difficult problem!

$\longrightarrow$ Instance **danoint** of MIPLIB III ( $\boldsymbol{p = 2}$, **8 nodes**)

- 661 rows, 521 variables (56 0/1)
- Early 1990's: optimization problem can only be solved to within 4 % error, and only if special-purpose algorithms are used
- 2003: problem can now be solved with general-purpose mixed-integer solvers, in about **1 day CPU time**, enumerating about **1 million** nodes

$\longrightarrow$ Using early 90's LP solvers and computers, **but modern MIP**, this is about **1 year CPU time**.

$\longrightarrow$ Larger instance **dano3mip**: 3202 constraints, 13873 variables (552 0/1) beyond the reach of any current solver (gap is about 25 %)

# Starting point

Balas, Pulleyblank, Barahona, others (pre 1990).

A polyhedron $P \subseteq R^n$ can be the **projection**

of a *simpler* polyhedron $Q \subseteq R^N$ $(N > n)$

**More precisely:**

There exist polyhedra $P \subseteq R^n$, such that

- $P$ has exponentially (in $n$) many facets, and
- $P$ is the projection of $Q \subseteq R^N$, where
- $N$ is polynomial in $n$, and $Q$ has polynomially many facets.

## Sherali-Adams operator

Let $\mathcal{F} = \{\, x \in \{0,1\}^n \ : \ Ax \le b \,\}$

$\rightarrow$ Let $t \ge 1$ be an integer. Consider a "lifted" formulation using variables

$v[Y, N]$, for all pairs of disjoint $Y,\ N \subseteq \{1, 2, \ldots, n\}$ with $|Y \cup N| \le t$

Intuition: $v[Y, N] = 1$ if and only if

$$\boldsymbol{x_j = 1}, \quad \text{for all} \quad \boldsymbol{j \in Y}, \quad \text{and} \quad \boldsymbol{x_j = 0}, \quad \text{for all} \quad \boldsymbol{j \in N}.$$

What constraints can we write, using these variables?

$$\boldsymbol{v \ge 0}, \qquad \boldsymbol{v[\emptyset, \emptyset] = 1}$$

$$\boldsymbol{v[Y \cup j, N] + v[Y, N \cup j] \ = \ v[Y, N]},$$

for all $j \notin Y \cup N$ and appropriate $Y, N$.

$(\mathcal{F} = \{\, x \in \{0,1\}^n \ : \ Ax \le b \,\})$

$\rightarrow$ For every row $\Sigma_j a_{ij} x_j \le b_i$, and disjoint $Y, N$ with $|Y \cup N| \le t$,

$$\boldsymbol{\Sigma_{j \in Y}\, a_{ij}\, v[Y, N] \ + \ \Sigma_{j \notin Y}\, a_{ij}\, v[Y \cup j, N] \ - \ b_i\, v[Y, N] \ \le \ 0} \tag{1}$$

$$\boldsymbol{v[Y \cup j, N] \ + \ v[Y, N \cup j] \ - \ v[Y, N] \ = \ 0 \ \ \forall\, j \notin Y \cup N} \tag{2}$$

$$\boldsymbol{0 \le v, \quad v[\emptyset, \emptyset] = 1} \tag{3}$$

$\rightarrow$ A "lift-and-project" formulation: given

$$\min\{c^T x \ : \ x \in \mathcal{F}\}$$

solve $\min \{\, \Sigma_j\, c_j\, v[j, \emptyset] \ : (1), (2), (3)\}$, with solution $v^*$

and set $x^*_j = v^*[j, \emptyset]$, $1 \le j \le n$.

## Other lift-and-project operators

- Balas (1970s). Disjunctive programming = one-variable convexification. Also see Balas, Ceria, Cornuejols (1990).

- Lovász and Schrijver (1989). $N_0$, $N$, $N_+$ operators.

- Lasserre (2001).

$\rightarrow$ Nice recent interpretation and review by Laurent.

- B. and Zuckerberg (2002). Subset-algebra lifting.

$v[Y \cup j, N] + v[Y, N \cup j] \ = \ v[Y, N] \Rightarrow$ the system is **redundant**:

only need $v[Y, \emptyset]$ for all $Y$ with $|Y| \leq t+1$

$\longrightarrow$ Suppose $t = 1$, and consider the matrix $\boldsymbol{M} \ = \ \boldsymbol{w}\,\boldsymbol{w}^T$, where $w[Y] \doteq v[Y, \emptyset]$ for $|Y| \leq 1$

has $i, j$ entry equal to $m_{ij} = w[i]\,w[j] = w[\,\{i, j\}\,] = v[\,\{i, j\}, \,\emptyset\,] \ \ (= m_{ji})$

For any row $h$ of $A$, and $1 \leq j \leq n$,

$$\boldsymbol{\Sigma_i\, a_{hi}\, v[\{i, j\}, \emptyset] \ - \ b_h\, v[j, \emptyset] \ \leq \ 0} \quad \text{or} \quad \boldsymbol{\Sigma_i\, a_{hi}\, m_{ij} \ - \ b_h\, m_{0j} \ \leq \ 0.}$$

So each column of $\boldsymbol{M}$ satisfies each constraint of $\boldsymbol{Ax} \leq \boldsymbol{b}$, homogeneized.

Also, given $\boldsymbol{j}$, for any $\boldsymbol{i}$, $\boldsymbol{v[i, j] = v[i, \emptyset] \ - \ v[\{i, j\}, \emptyset]}$, and

$$\boldsymbol{\Sigma_i\, a_{hi}\, v[i, j] \ - \ b_h\, v[\emptyset, j] \ \leq \ 0} \quad \text{or}$$

$$\boldsymbol{\Sigma_i\, a_{hi}\, (m_{i0} - m_{ij}) \ - \ b_h\, (m_{00} - m_{j0}) \ \leq \ 0.}$$

the $\boldsymbol{0}^{th}$ minus the $\boldsymbol{j}^{th}$ column of $\boldsymbol{M}$ also satisfies each constraint of $\boldsymbol{Ax} \leq \boldsymbol{b}$, homogeneized.

$$x = (1, 0, 1, 1)^T$$

$$w = (1, 1, 0, 1, 1)^T$$

$$M = w\,w^T = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$v[Y \cup j, N] + v[Y, N \cup j] \;=\; v[Y, N] \Rightarrow$ the system is **redundant**:

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Same example satisfies $\boldsymbol{x_1 + x_3 \geq 2}$

$$x = (1, 0, 1, 1)^T$$
$$w = (1, 1, 0, 1, 1)^T$$

$$\mathbf{M} = w\, w^T = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

\

Matrix satisfies $\boldsymbol{m_{1j} + m_{3j} \geq 2m_{0j}}$ for all columns $j$.

$v[Y \cup j, N] + v[Y, N \cup j] \;=\; v[Y, N] \Rightarrow$ the system is **redundant**:

only need $v[Y, \emptyset]$ for all $Y$ with $|Y| \leq t+1$

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**Summary:**

$\longrightarrow$ When $t = 1$, the Sherali-Adams operator is the same as the Lovász-Schrijver $N$ operator (without symmetricity $= N_0$)

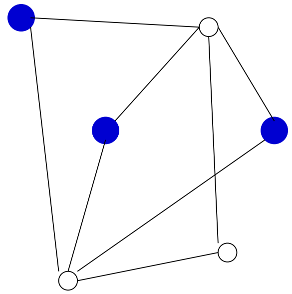
$\longrightarrow$ For $t > 1$, could also "lift" to the matrix $ww^T$, but this will require sets of cardinality $2t$,

$\longrightarrow$ Could also impose $ww^T$ symmetric positive-semidefinite

$\longrightarrow$ The Lasserre lifting, and the subset-algebra lifting, provide better generalizations

# The stable set problem

Def: A stable set in a graph



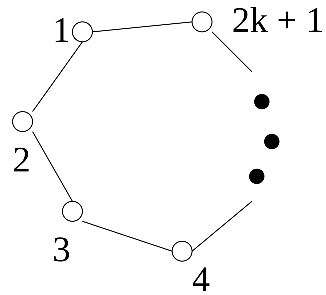
is a set of pairwise non-adjacent vertices

**Formulation:** Given graph $G$,

$$x_i + x_j \leq 1, \quad \forall \{i, j\} \in E(G),$$

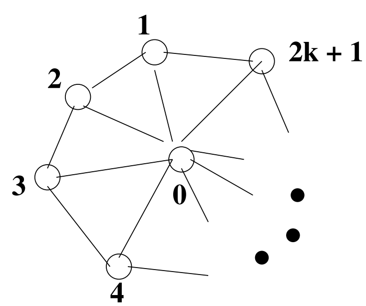
$$x_i = 0 \text{ or } 1, \quad \forall i \in V(G).$$

# Classical inequalities




**Def:** An odd hole is a cycle of odd length with no chords.
Odd-hole inequality (facet-defining, polynomially separable):

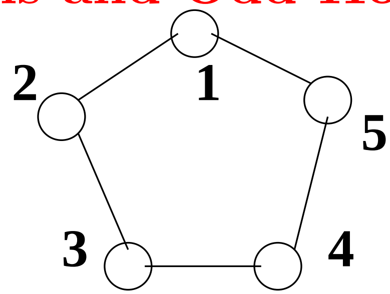
$$\Sigma_{i=1}^{2k+1} x_i \leq k$$



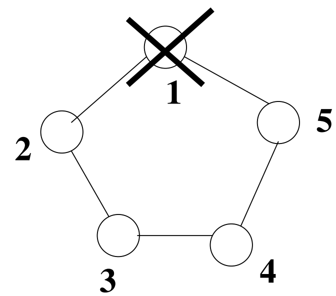

**Def:** An odd wheel is a cycle of odd length (with no chords) plus an additional vertex adjacent to all wheel vertices.
Odd-wheel inequality (facet-defining, polynomially separable):

$$kx_0 + \Sigma_{i=1}^{2k+1} x_i \leq k$$

# Sherali-Adams and Odd Holes




**"Case"** $\boldsymbol{x_1 = 0}$. Look at the variables $v[Y, N]$, with $N = \{1\}$ and $|Y| = 1$.




Consider the inequality $x_2 + x_3 \leq 1$, and use $v[\emptyset, \{1\}]$:

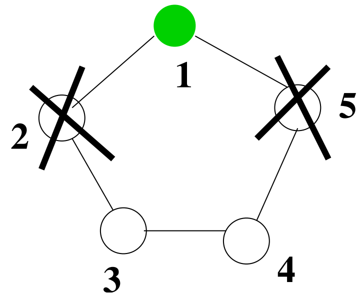
$$v[2,1] + v[3,1] \leq v[\emptyset, 1]$$

Similarly, using $x_4 + x_5 \leq 1$,

$$v[4,1] + v[5,1] \leq v[\emptyset, 1]$$

**and** $v[1,1] = 0$, **so:** $\Sigma_i v[i,1] \leq 2v[\emptyset, 1]$.

**"Case"** $\boldsymbol{x_1 = 1}$. Now look at the variables $v[\{j,1\},\emptyset]$.




Consider $x_1 + x_2 \leq 1$, and use $v[1,\emptyset]$:

$$v[1,\emptyset] \; + \; v[\{1,2\},\emptyset] \; \leq \; v[1,\emptyset] \quad \textbf{so} \quad v[\{1,2\},\emptyset] = 0.$$

Similarly, $v[\{1,5\},\emptyset] = 0$, and using $x_3 + x_4 \leq 1$,

$$v[\{1,3\},\emptyset] \; + \; v[\{1,4\},\emptyset] \; \leq \; v[1,\emptyset].$$

**So:**

$$\Sigma_i v[\{1,i\},\emptyset] \; \leq \; 2v[1,\emptyset]$$

Summary:

$$\Sigma_i \, v[i,1] \;\le\; 2v[\emptyset,1],$$

and

$$\Sigma_i \, v[\{1,i\},\emptyset] \;\le\; 2v[1,\emptyset].$$

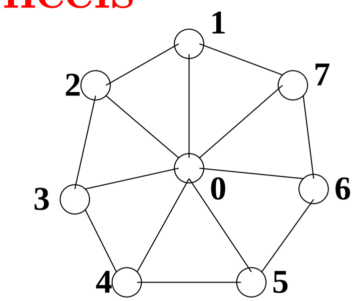
Note:

$$v[i,1] + v[\{1,i\},\emptyset] = x_i \quad \text{and} \quad v[\emptyset,1] + v[1,\emptyset] = 1.$$

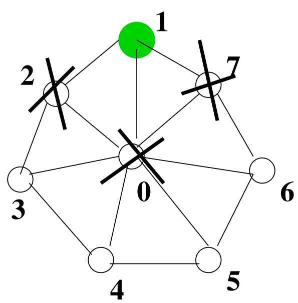
**Conclusion:**

$$\Sigma_i \, x_i \;\le\; 2$$

# Odd Wheels




**"Case"** $\boldsymbol{x_0 = 0}$ and $\boldsymbol{x_1 = 1}$.




$$x_3 + x_4 \leq 1, \text{ so } v[\{1,3\},0] + v[\{1,4\},0] \leq v[1,0].$$

Similarly, $v[\{1,5\},0] + v[\{1,6\},0] \leq v[1,0]$,

So, $\Sigma_{i \geq 1} v[\{1,i\},0] \leq 3v[1,0]$.

**Conclusion:** $3x_0 + \Sigma_{i \geq 1} x_i \leq 3$.

## Cheng and Cunningham (1997):

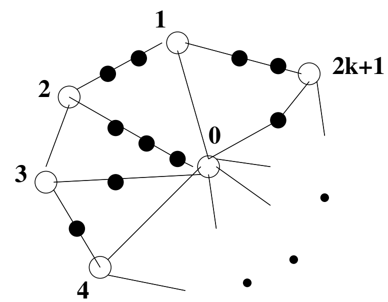
Generalizations of odd-hole and odd-wheel inequalities.

Given a graph $G$, inequalities of the form

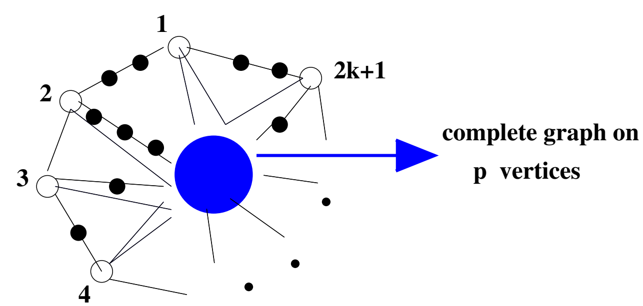
$$\sum_{j \in V(H)} \alpha_j x_j \leq \beta,$$

where $H$ is a subgraph of $G$.

- Subdivisions of wheels

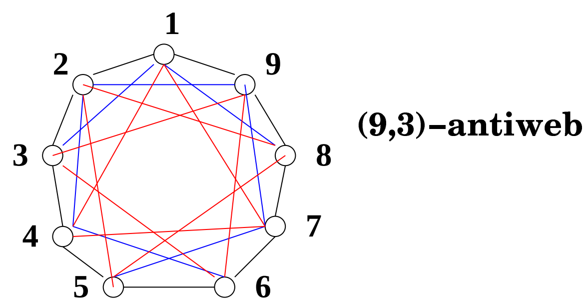



- Subdivisions of $\boldsymbol{p}$—wheels

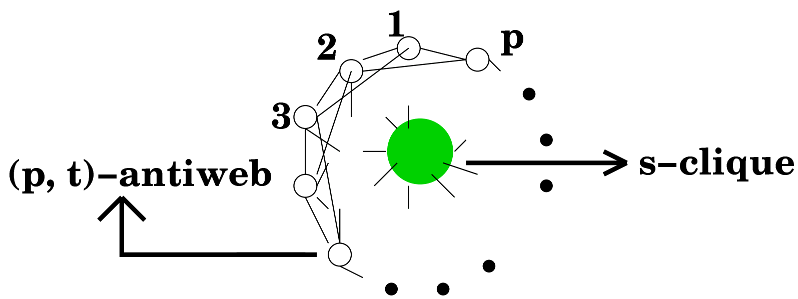


## Cheng and de Vries (2002):

$(p,t) - \boldsymbol{antiwebs}$ for $p \geq 2t-1$:




$(p,t) - \boldsymbol{antiwebs} - s - \boldsymbol{wheels}$:



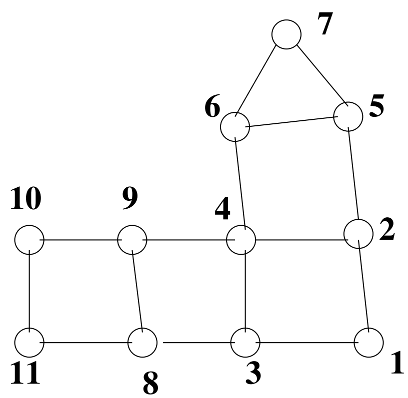

$\boldsymbol{subdivisions}$ of the above.

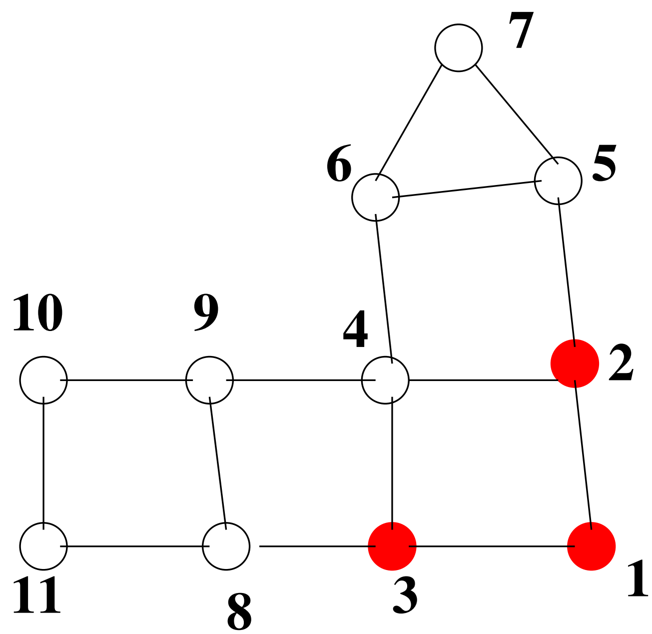
**→ polynomially separable for fixed t and s**

# Robertson and Seymour (1980s - 1990s) : tree-width

**Def:** A **tree-decomposition** of a graph $G$ consists of a tree $T$ and a family of sets $X_t$ (for each node $t$ of $T$), such that:

- Each $X_t$ is a subset of vertices of $G$.
- For each vertex $v$ of $G$, the collection of sets $X_t$ containing $v$ forms a subtree of $T$.
- For each edge $\{u, v\}$ of $G$ there is a subset $X_t$ with $\{u, v\} \subseteq X_t$ .

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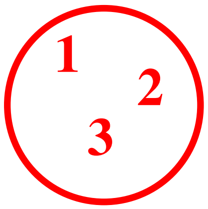
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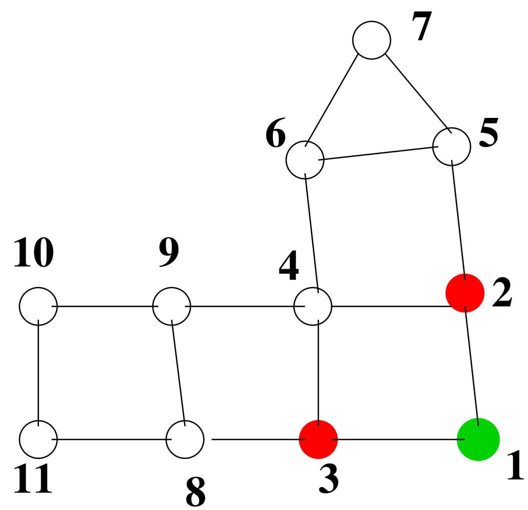
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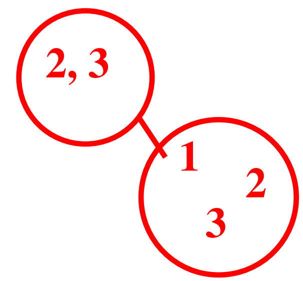
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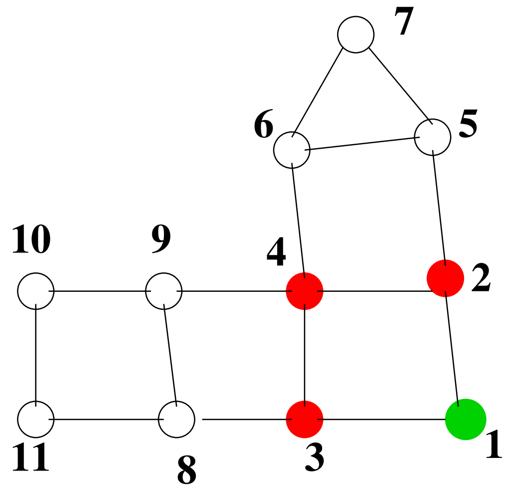
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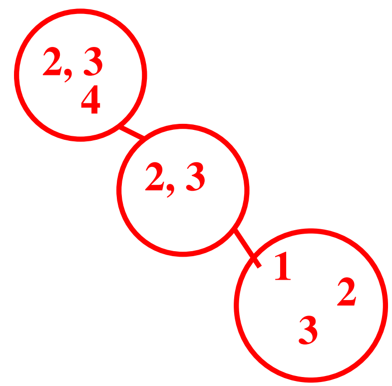
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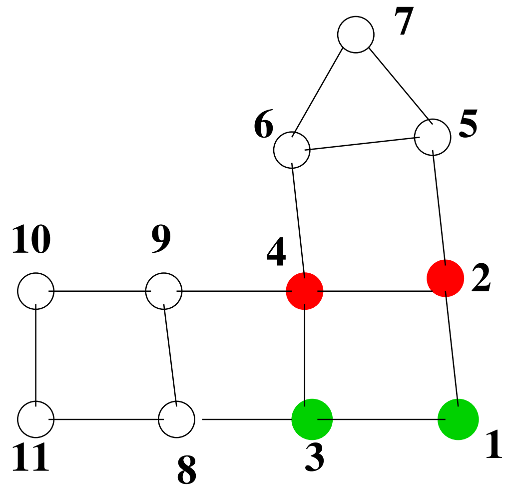
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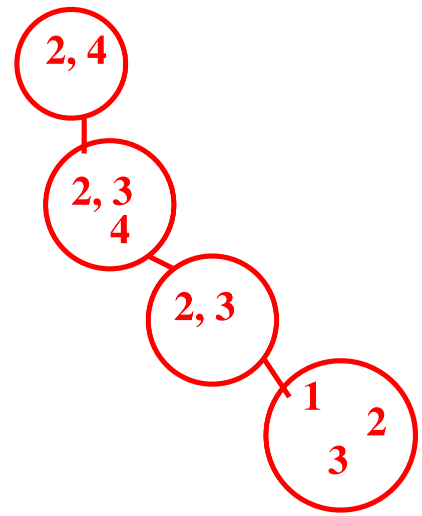
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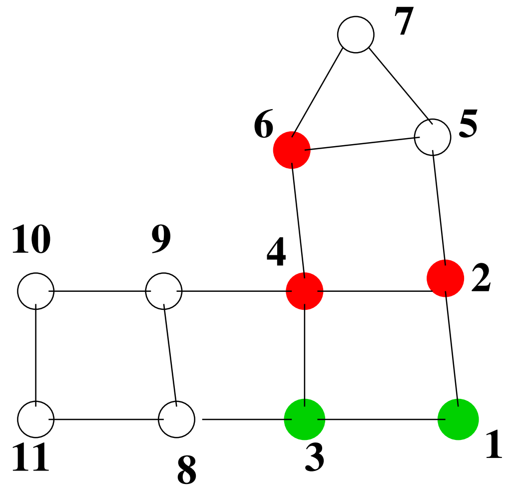

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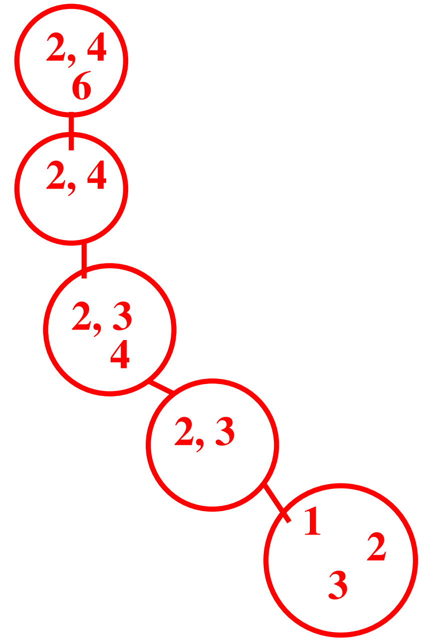

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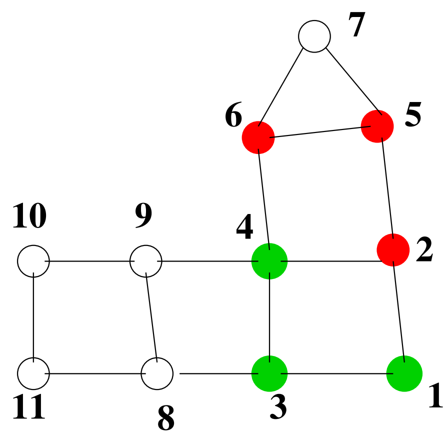
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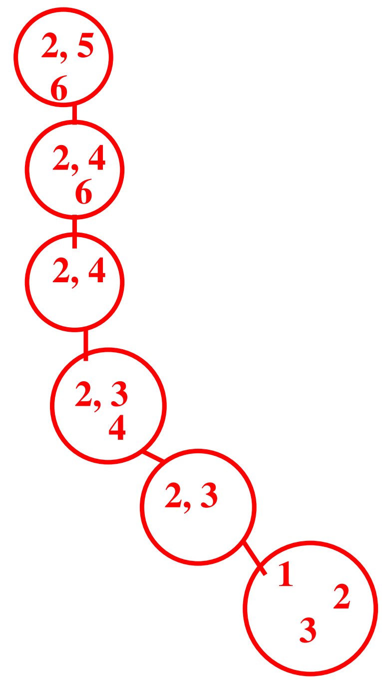
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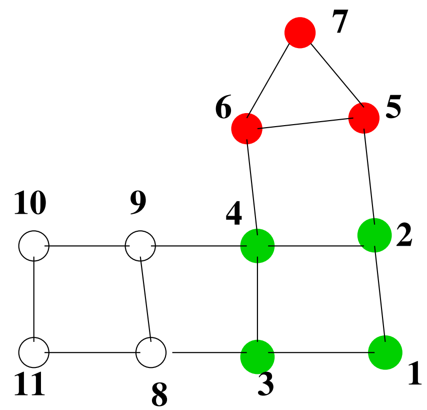
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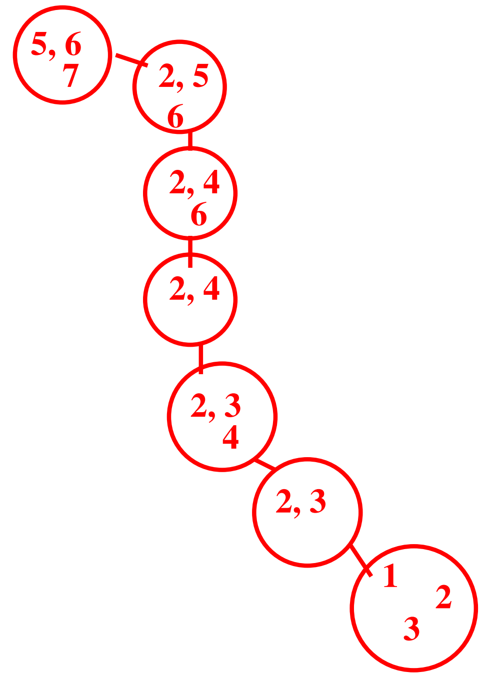
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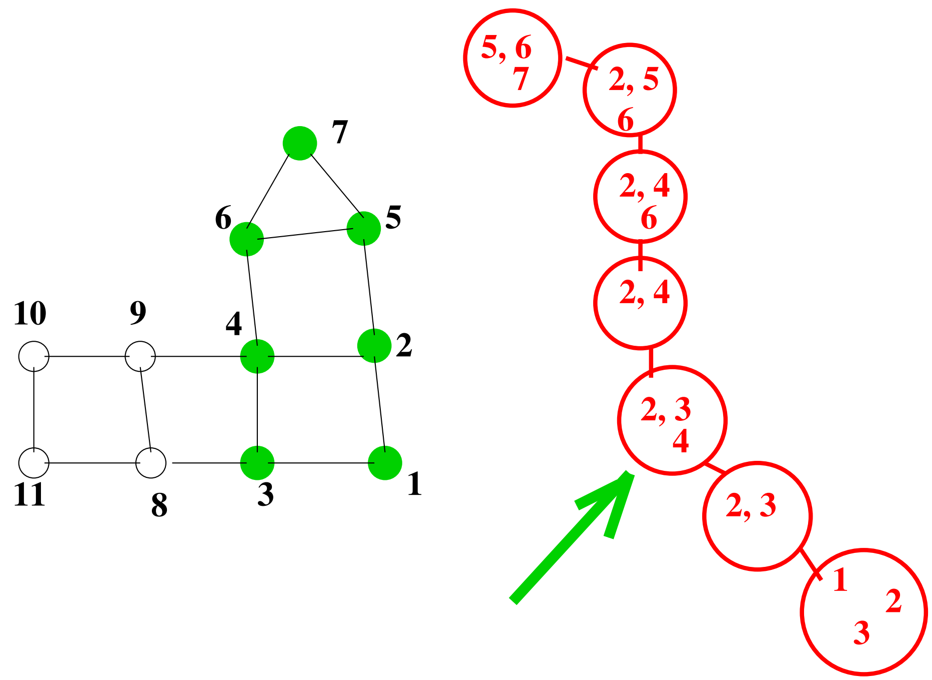

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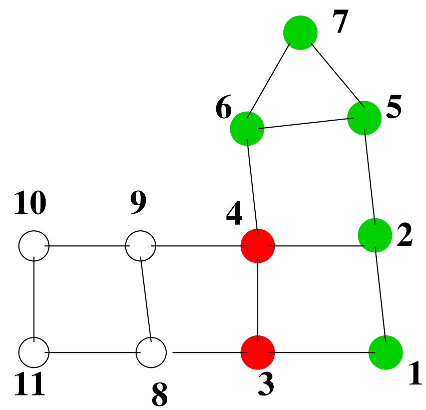
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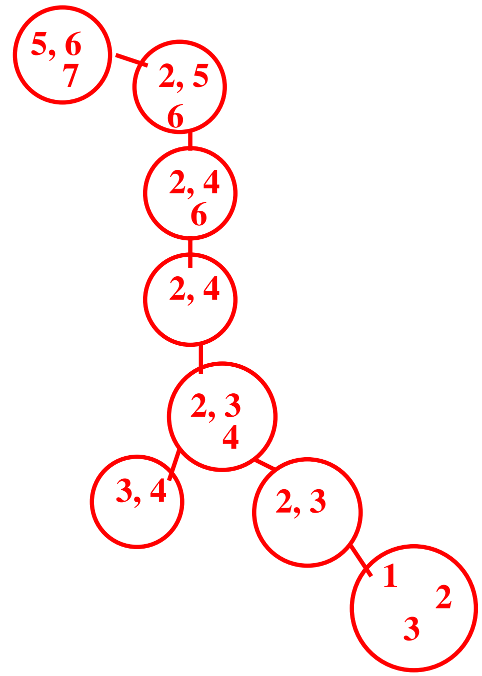
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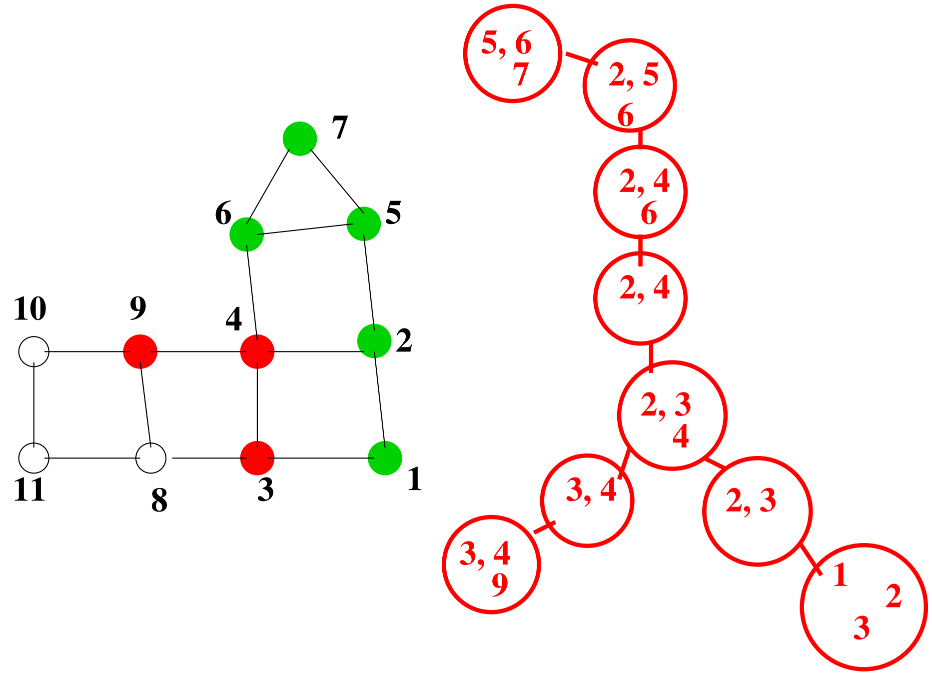
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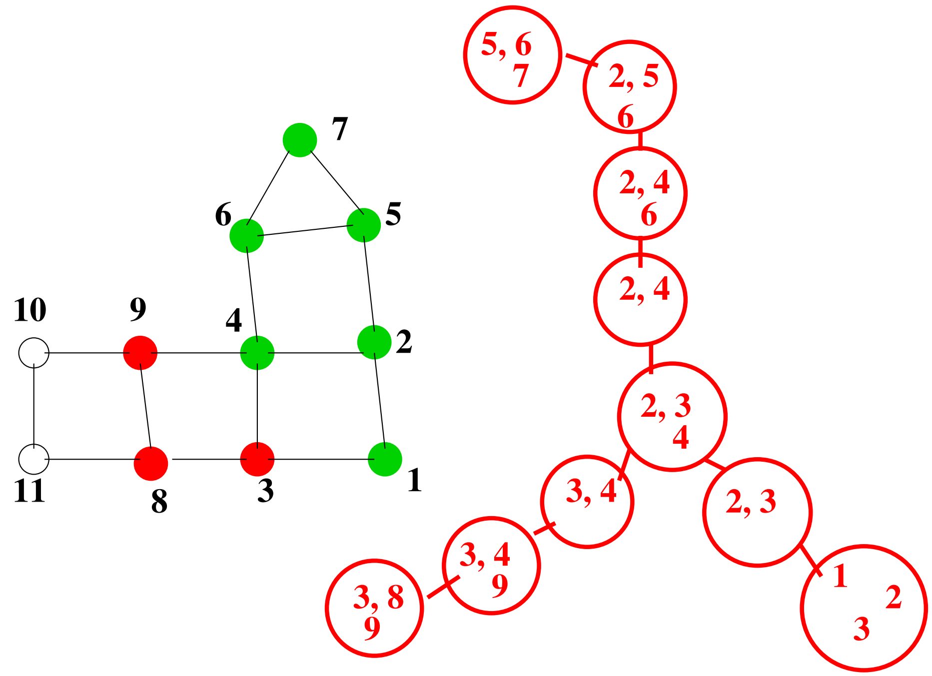
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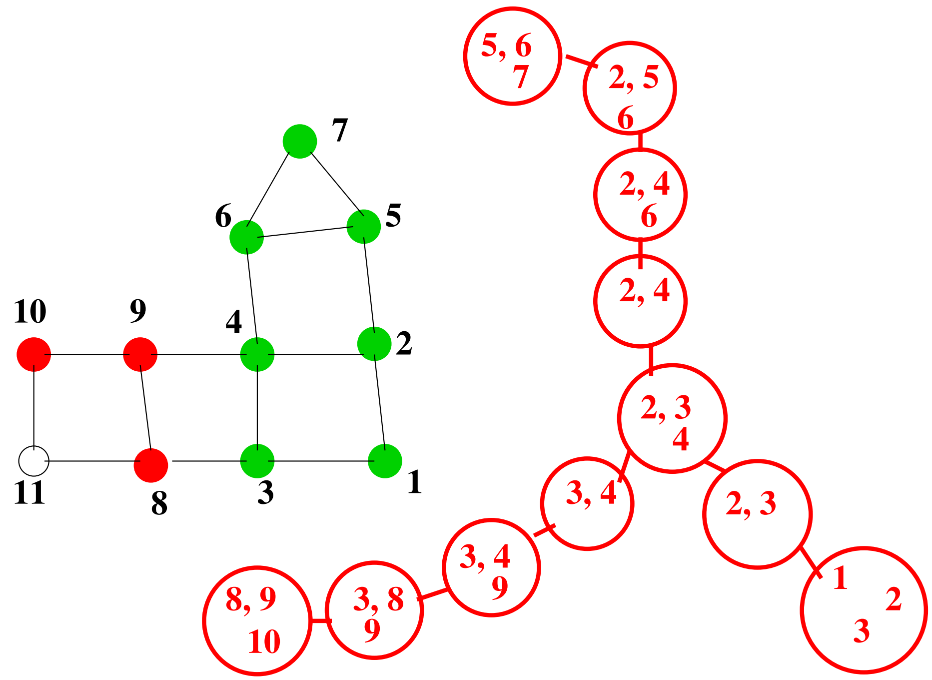

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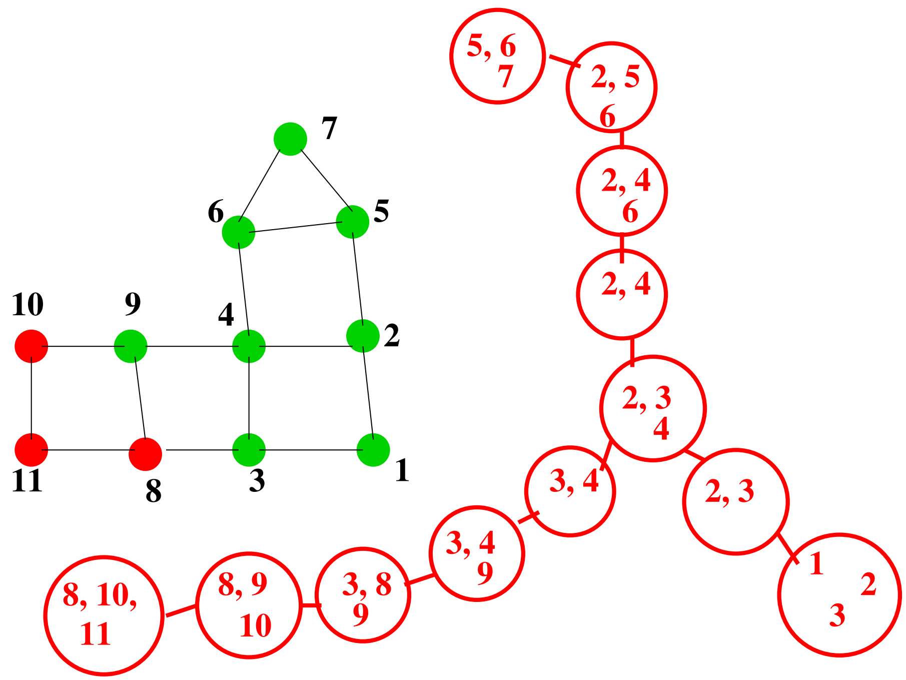

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**Def:** A **tree-decomposition** of a graph $\boldsymbol{G}$ consists of a tree $\boldsymbol{T}$ and a family of sets $\boldsymbol{X_t}$ (for each node $\boldsymbol{t}$ of $\boldsymbol{T}$), such that:

- Each $\boldsymbol{X_t}$ is a subset of vertices of $\boldsymbol{G}$.
- For each vertex $\boldsymbol{v}$ of $\boldsymbol{G}$, the collection of sets $\boldsymbol{X_t}$ containing $\boldsymbol{v}$ forms a subtree of $\boldsymbol{T}$.
- For each edge $\boldsymbol{\{u, v\}}$ of $\boldsymbol{G}$ there is a subset $\boldsymbol{X_t}$ with $\boldsymbol{\{u, v\} \subseteq X_t}$ .

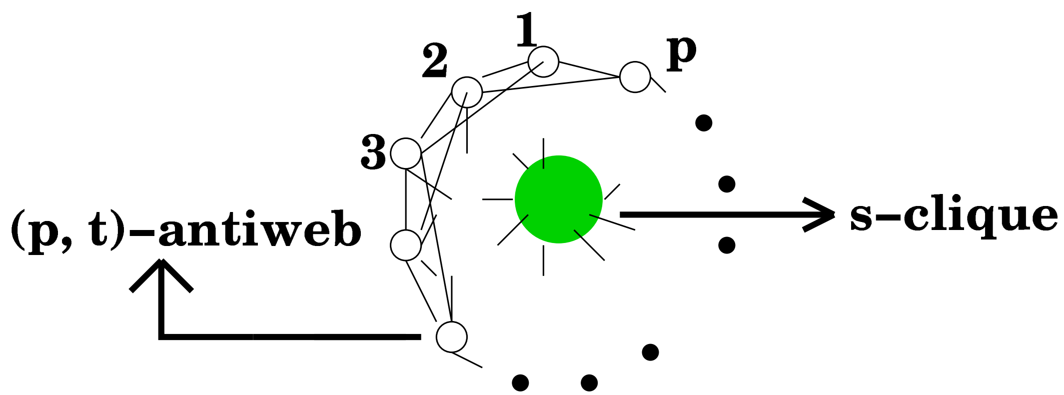
The **width** of the decomposition is $\mathbf{max}_t \boldsymbol{\{|X_t|\} - 1}$.

$\longrightarrow$ $\boldsymbol{G}$ has **tree-width** $\boldsymbol{\leq k}$ if it has a tree-decomposition of width $\boldsymbol{\leq k}$.

**...** equivalently
if there is a **chordal supergraph** of $\boldsymbol{G}$ of **clique number** $\boldsymbol{\leq k}$

Robertson and Seymour: $\boldsymbol{G}$ has large tree-width if and only if $\boldsymbol{G}$ contains a large square grid minor

$(p,t) - antiwebs - s - wheels$:




**Lemma:** A subdivision of a $(p,t) - antiwebs - s - wheel$ has tree-width $\leq 2t + s - 2$.

... What can we say about inequalities of the form

$$\sum_{j \in V(H)} \alpha_j x_j \leq \beta,$$

**where** $H$ has small tree-width ?

## Theorem

Suppose we construct the Sherali-Adams level- $\boldsymbol{k}$ formulation $\boldsymbol{SA^k}$ to the vertex packing problem for a graph $\boldsymbol{G}$.

Then, any vector $\hat{\boldsymbol{x}}$ that satisfies the constraints of $\boldsymbol{SA^k}$ also satisfies **all valid inequalites** of the form

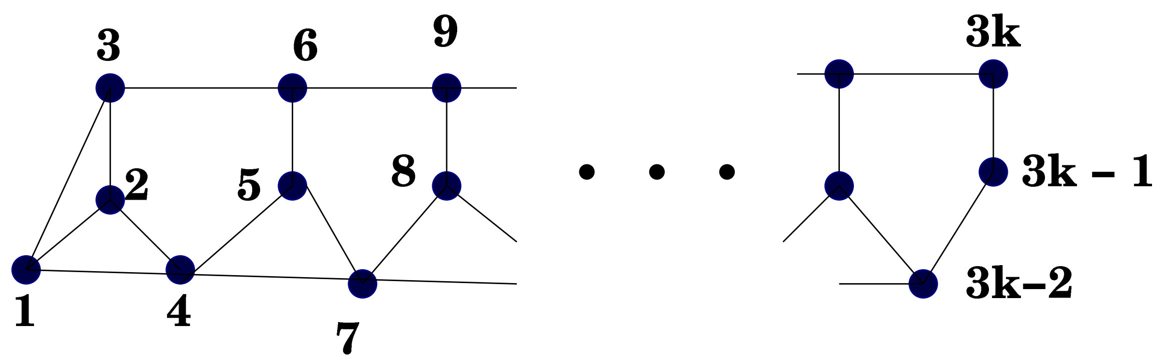
$$\boldsymbol{\Sigma_{j \in V(H)} \alpha_j \, x_j \leq \beta}$$

for every subgraph $\boldsymbol{H}$ of $\boldsymbol{G}$ of tree-width $\leq \boldsymbol{k}$. ■

**Corollary:** For $\boldsymbol{k \geq 2t + s - 2}$, we are guaranteed to satisfy all (subdivided) $(p, t) - antiweb - s - wheel$ inequalities, etc.

**Lipták and Tunçel (2003)**.
Also see Goemans and Tunçel (2001), Cook and Dash (2001)

Consider the graph:




(= the line-graph of an iterated blossom)

$\longrightarrow$ Theorem (L & T): the $\boldsymbol{N_0}$-rank of this graph is $\sim \mathbf{log_2}\, \boldsymbol{k}$.

$\longrightarrow$ Conjecture (L & T): the $\boldsymbol{N_0}$- and $\boldsymbol{N}$-ranks of any graph are equal.

$\longrightarrow$ The graph has tree-width $\mathbf{3}$.

***Corollary.*** Its Sherali-Adams rank is $\leq \mathbf{3}$.

# General packing polyhedra

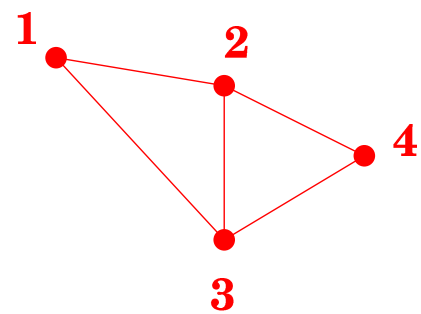
$$Ax \;\leq\; b, \qquad x \in \{0,1\}^n,$$

where $A \geq 0$.

**Def:** (Balas et al) The clique graph of $A$ has

- A vertex for each **column** of $A$,
- An edge between $k$ and $j$ if there is a row $i$ with $\boldsymbol{a_{ik} > 0}$ and $\boldsymbol{a_{ij} > 0}$.

$$\begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array}$$
$$\begin{bmatrix} \bullet & \bullet & & \\ & \bullet & \bullet & \bullet \\ \bullet & & \bullet & \end{bmatrix}$$



**Definition:** The **tree-width** of an inequality $\boldsymbol{\alpha}^T \boldsymbol{x} \leq \boldsymbol{\beta}$ valid for

$$\{\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \ \ \boldsymbol{x} \in \{0,1\}^n\},$$

is the minimum tree-width of the clique graph of any submatrix $\boldsymbol{A'}$ of $\boldsymbol{A}$, such that $\boldsymbol{\alpha}^T \boldsymbol{x} \leq \boldsymbol{\beta}$ is valid for $\left\{\boldsymbol{A'}\boldsymbol{x} \leq \boldsymbol{b'}, \ \ \boldsymbol{x} \in \{0,1\}^{n'}\right\}$.

**Theorem:** For any $\boldsymbol{k} \geq \mathbf{1}$, the level- $\boldsymbol{k}$ Sherali-Adams formulation for

$$\{\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \ \ \boldsymbol{x} \in \{0,1\}^n\}$$

is guaranteed to satisfy all inequalities with tree-width $\leq \boldsymbol{k} - \mathbf{1}$. ∎

# Covering problems?

Set-covering polyhedra: $\{\boldsymbol{Ax} \geq \mathbf{1}, \ \ \boldsymbol{x} \in \{\mathbf{0}, \mathbf{1}\}^n\}$, $A$ a $0-1$ matrix.

**Theorem** (B. and Zuckerberg, 2002). Given a set-covering problem, for any fixed $\boldsymbol{k} \geq \mathbf{1}$ we can generate in polynomial time a relaxation that is guaranteed to satisfy all valid inequalities with coefficients in $\{\mathbf{0}, \mathbf{1}, \ldots, \boldsymbol{k}\}$.

**Theorem** (B. and Zuckerberg, 2002). Let $\boldsymbol{r} \geq \mathbf{1}$ and $\mathbf{0} < \boldsymbol{\epsilon} < \mathbf{1}$ be fixed. Given a set-covering problem

$$\boldsymbol{\tau} = \min \left\{ \boldsymbol{c}^T \boldsymbol{x} \ : \ \boldsymbol{Ax} \geq \mathbf{1}, \ \ \boldsymbol{x} \in \{\mathbf{0}, \mathbf{1}\}^n \right\},$$

in polynomial time we can compute a value $\boldsymbol{v}$ with

$$(\mathbf{1} - \boldsymbol{\epsilon}) \min \left\{ \boldsymbol{c}^T \boldsymbol{x} \ : \boldsymbol{x} \in \boldsymbol{K}^r(\boldsymbol{A}, \boldsymbol{b}) \right\} \ \leq \ \boldsymbol{v} \ \leq \ \boldsymbol{\tau},$$

where $\boldsymbol{K}^r(\boldsymbol{A}, \boldsymbol{b})$ is the rank-k Chvátal-Gomory closure of $\{\boldsymbol{Ax} \geq \mathbf{1}, \ \mathbf{0} \leq \boldsymbol{x} \leq \mathbf{1}\}$.