Packing problems, tree-width, and lift-and-project

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An old practical problem

Build a directed graph to join a set of nodes

- and in that graph, route a given set of multicommodity demands
- the graph must have degrees at most $\boldsymbol{p}$ (given)

→ Objective: **minimize the maximum flow used on any edge**.

This is a difficult problem!

→ Instance **danoint** of MIPLIB III ($\boldsymbol{p = 2}$, **8 nodes**)

- 661 rows, 521 variables (56 0/1)
- Early 1990's: optimization problem can only be solved to within 4 % error, and only if special-purpose algorithms are used
- 2003: problem can now be solved with general-purpose mixed-integer solvers, in about **1 day CPU time**, enumerating about **1 million** nodes

→ Using early 90's LP solvers and computers, **but modern MIP**, this is about **1 year CPU time**.

→ Larger instance **dano3mip**: 3202 constraints, 13873 variables (552 0/1) beyond the reach of any current solver (gap is about 25 %)

Starting point

Balas, Pulleyblank, Barahona, others (pre 1990).

A polyhedron $P \subseteq R^n$ can be the **projection**

of a *simpler* polyhedron $Q \subseteq R^N$ $(N > n)$

More precisely:

There exist polyhedra $P \subseteq R^n$, such that

- P has exponentially (in n) many facets, and
- P is the projection of $Q \subseteq R^N$, where
- N is polynomial in n, and Q has polynomially many facets.

Sherali-Adams operator

Let $\mathcal{F} = \{\, x \in \{0,1\}^n \ :\ Ax \leq b \,\}$

$\rightarrow$ Let $t \geq 1$ be an integer. Consider a "lifted" formulation using variables

$v[Y, N]$, for all pairs of disjoint Y, $N \subseteq \{1, 2, \ldots, n\}$ with $|Y \cup N| \leq t$

Intuition: $v[Y, N] = 1$ if and only if

$$\boldsymbol{x_j = 1}, \quad \text{for all} \quad \boldsymbol{j \in Y}, \quad \text{and} \quad \boldsymbol{x_j = 0}, \quad \text{for all} \quad \boldsymbol{j \in N}.$$

What constraints can we write, using these variables?

$$\boldsymbol{v \geq 0}, \qquad \boldsymbol{v[\emptyset, \emptyset] = 1}$$

$$\boldsymbol{v[Y \cup j, N] + v[Y, N \cup j] \ = \ v[Y, N]},$$

for all $j \notin Y \cup N$ and appropriate Y, N.

$(\mathcal{F} = \{\, x \in \{0,1\}^n \;:\; Ax \le b \,\})$

$\rightarrow$ For every row $\Sigma_j a_{ij} x_j \le b_i$, and disjoint Y, N with $|Y \cup N| \le t$,

$$\boldsymbol{\Sigma_{j \in Y}\, a_{ij}\, v[Y, N] \;+\; \Sigma_{j \notin Y}\, a_{ij}\, v[Y \cup j, N] \;-\; b_i\, v[Y, N] \;\le\; 0} \tag{1}$$

$$\boldsymbol{v[Y \cup j, N] \;+\; v[Y, N \cup j] \;-\; v[Y, N] \;=\; 0 \quad \forall\, j \notin Y \cup N} \tag{2}$$

$$\boldsymbol{0 \le v, \quad v[\emptyset, \emptyset] = 1} \tag{3}$$

$\rightarrow$ A "lift-and-project" formulation: given

$$\min\{c^T x \;:\; x \in \mathcal{F}\}$$

solve min $\{\, \Sigma_j\, c_j\, v[j, \emptyset] \;:\; (1), (2), (3)\}$, with solution v^*

and set $x^*_j = v^*[j, \emptyset]$, $1 \le j \le n$.

Other lift-and-project operators

- Balas (1970s). Disjunctive programming = one-variable convexification. Also see Balas, Ceria, Cornuejols (1990).

- Lovász and Schrijver (1989). N_0, N, N_+ operators.

- Lasserre (2001).

→ Nice recent interpretation and review by Laurent.

- B. and Zuckerberg (2002). Subset-algebra lifting.

$v[Y \cup j, N] + v[Y, N \cup j] \;=\; v[Y, N] \Rightarrow$ the system is **redundant**:

only need $v[Y, \emptyset]$ for all Y with $|Y| \leq t+1$

$\longrightarrow$ Suppose $t = 1$, and consider the matrix $\boldsymbol{M} = \boldsymbol{w}\,\boldsymbol{w}^T$, where $w[Y] \doteq v[Y, \emptyset]$ for $|Y| \leq 1$

has i, j entry equal to $m_{ij} = w[i]\,w[j] = w[\,\{i,j\}\,] = v[\,\{i,j\},\,\emptyset\,]$ $(= m_{ji})$

For any row h of A, and $1 \leq j \leq n$,

$$\boldsymbol{\Sigma_i\, a_{hi}\, v[\{i,j\},\emptyset] \;-\; b_h\, v[j,\emptyset] \;\leq\; 0} \quad \text{or} \quad \boldsymbol{\Sigma_i\, a_{hi}\, m_{ij} \;-\; b_h\, m_{0j} \;\leq\; 0.}$$

So each column of $\boldsymbol{M}$ satisfies each constraint of $\boldsymbol{Ax} \leq \boldsymbol{b}$, homogeneized.

Also, given $\boldsymbol{j}$, for any $\boldsymbol{i}$, $\boldsymbol{v[i,j] = v[i,\emptyset] \;-\; v[\{i,j\},\emptyset]}$, and

$$\boldsymbol{\Sigma_i\, a_{hi}\, v[i,j] \;-\; b_h\, v[\emptyset,j] \;\leq\; 0} \quad \text{or}$$

$$\boldsymbol{\Sigma_i\, a_{hi}\,(m_{i0} - m_{ij}) \;-\; b_h\,(m_{00} - m_{j0}) \;\leq\; 0.}$$

the $\boldsymbol{0^{th}}$ minus the $\boldsymbol{j^{th}}$ column of $\boldsymbol{M}$ also satisfies each constraint of $\boldsymbol{Ax} \leq \boldsymbol{b}$, homogeneized.

$$x = (1, 0, 1, 1)^T$$

$$w = (1, 1, 0, 1, 1)^T$$

$$M = w\,w^T = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

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$v[Y \cup j, N] + v[Y, N \cup j] \ = \ v[Y, N] \Rightarrow$ the system is **redundant**:

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Same example satisfies $\boldsymbol{x_1 + x_3 \geq 2}$

$$x = (1, 0, 1, 1)^T$$

$$w = (1, 1, 0, 1, 1)^T$$

$$\mathbf{M} = w\, w^T = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

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Matrix satisfies $\boldsymbol{m_{1j} + m_{3j} \geq 2m_{0j}}$ for all columns j.

$v[Y \cup j, N] + v[Y, N \cup j] \;=\; v[Y, N] \Rightarrow$ the system is **redundant**:

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Summary:

→ When $t = 1$, the Sherali-Adams operator is the same as the Lovász-Schrijver N operator (without symmetricity $= N_0$)

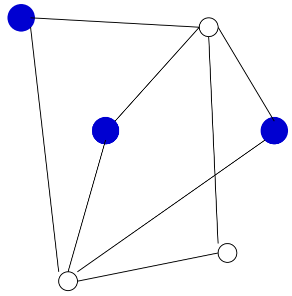
→ For $t > 1$, could also "lift" to the matrix ww^T, but this will require sets of cardinality $2t$,

→ Could also impose ww^T symmetric positive-semidefinite

→ The Lasserre lifting, and the subset-algebra lifting, provide better generalizations

The stable set problem

Def: A stable set in a graph



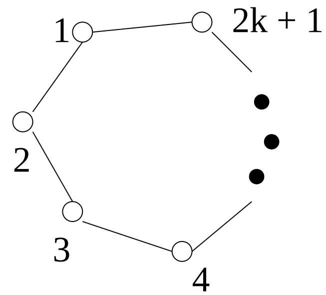
is a set of pairwise non-adjacent vertices

Formulation: Given graph G,

$$x_i + x_j \leq 1, \quad \forall \{i, j\} \in E(G),$$

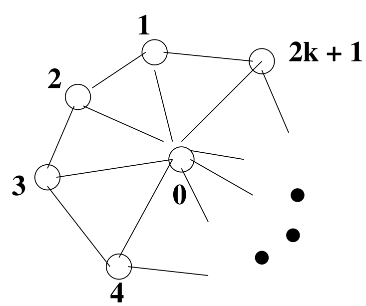
$$x_i = 0 \text{ or } 1, \quad \forall i \in V(G).$$

Classical inequalities



Def: An odd hole is a cycle of odd length with no chords.
Odd-hole inequality (facet-defining, polynomially separable):

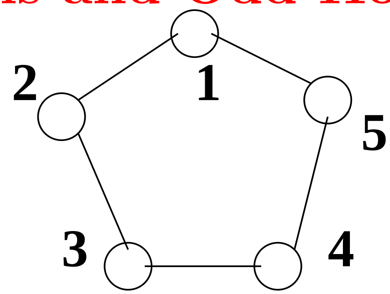
$$\Sigma_{i=1}^{2k+1} x_i \leq k$$



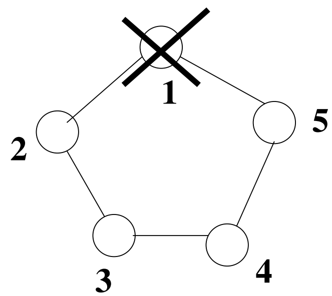
Def: An odd wheel is a cycle of odd length (with no chords) plus an additional vertex adjacent to all wheel vertices.
Odd-wheel inequality (facet-defining, polynomially separable):

$$kx_0 + \Sigma_{i=1}^{2k+1} x_i \leq k$$

Sherali-Adams and Odd Holes



"Case" $\boldsymbol{x_1 = 0}$. Look at the variables $v[Y, N]$, with $N = \{1\}$ and $|Y| = 1$.



Consider the inequality $x_2 + x_3 \leq 1$, and use $v[\emptyset, \{1\}]$:

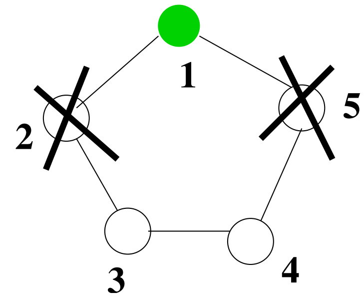
$$v[2,1] \ + \ v[3,1] \ \leq \ v[\emptyset, 1]$$

Similarly, using $x_4 + x_5 \leq 1$,

$$v[4,1] \ + \ v[5,1] \ \leq \ v[\emptyset, 1]$$

and $v[1,1] = 0$, **so:** $\quad \Sigma_i v[i,1] \ \leq \ 2v[\emptyset, 1]$.

"Case" $\boldsymbol{x_1 = 1}$. Now look at the variables $v[\{j, 1\}, \emptyset]$.



Consider $x_1 + x_2 \leq 1$, and use $v[1, \emptyset]$:

$$v[1, \emptyset] \;+\; v[\{1, 2\}, \emptyset] \;\leq\; v[1, \emptyset] \quad \textbf{so} \quad v[\{1, 2\}, \emptyset] = 0.$$

Similarly, $v[\{1, 5\}, \emptyset] = 0$, and using $x_3 + x_4 \leq 1$,

$$v[\{1, 3\}, \emptyset] \;+\; v[\{1, 4\}, \emptyset] \;\leq\; v[1, \emptyset].$$

So:

$$\Sigma_i v[\{1, i\}, \emptyset] \;\leq\; 2v[1, \emptyset]$$

Summary:

$$\Sigma_i\, v[i,1] \;\leq\; 2v[\emptyset,1],$$

and

$$\Sigma_i\, v[\{1,i\},\emptyset] \;\leq\; 2v[1,\emptyset].$$

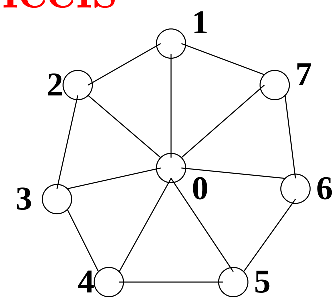
Note:

$$v[i,1] + v[\{1,i\},\emptyset] = x_i \quad \text{and} \quad v[\emptyset,1] + v[1,\emptyset] = 1.$$

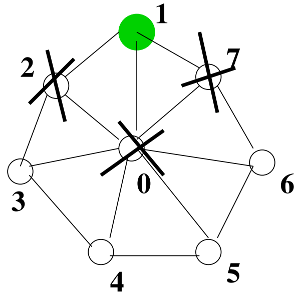
Conclusion:

$$\Sigma_i\, x_i \;\leq\; 2$$

Odd Wheels



"Case" $\boldsymbol{x_0 = 0}$ and $\boldsymbol{x_1 = 1}$.



$$x_3 + x_4 \le 1, \text{ so } v[\{1,3\},0] + v[\{1,4\},0] \le v[1,0].$$

Similarly, $v[\{1,5\},0] + v[\{1,6\},0] \le v[1,0]$,

So, $\Sigma_{i \ge 1} v[\{1,i\},0] \le 3v[1,0]$.

Conclusion: $3x_0 + \Sigma_{i \ge 1} x_i \le 3$.

Cheng and Cunningham (1997):

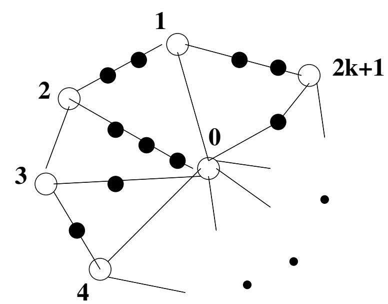
Generalizations of odd-hole and odd-wheel inequalities.

Given a graph G, inequalities of the form

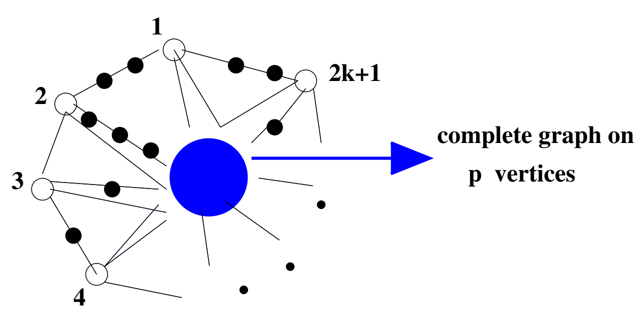
$$\sum_{j \in V(H)} \alpha_j x_j \leq \beta,$$

where H is a subgraph of G.

- Subdivisions of wheels

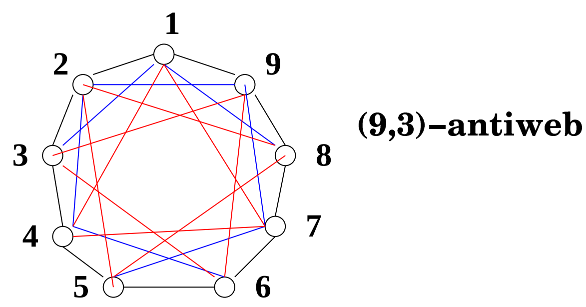


- Subdivisions of $\boldsymbol{p}$—wheels

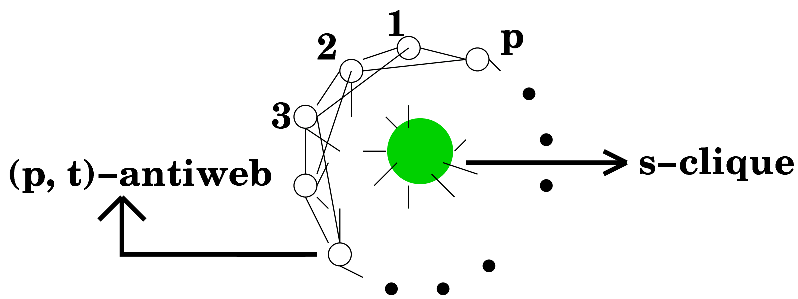


Cheng and de Vries (2002):

$\boldsymbol{(p,t)-antiwebs}$ for $p \geq 2t-1$:



$\boldsymbol{(p,t)-antiwebs-s-wheels}$:



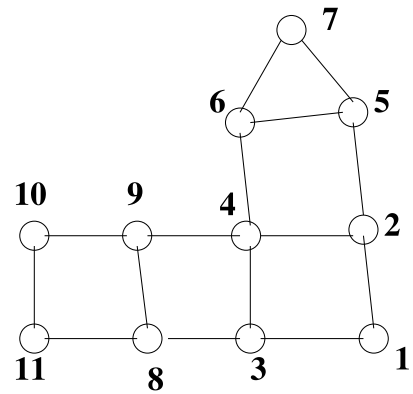
subdivisions of the above.

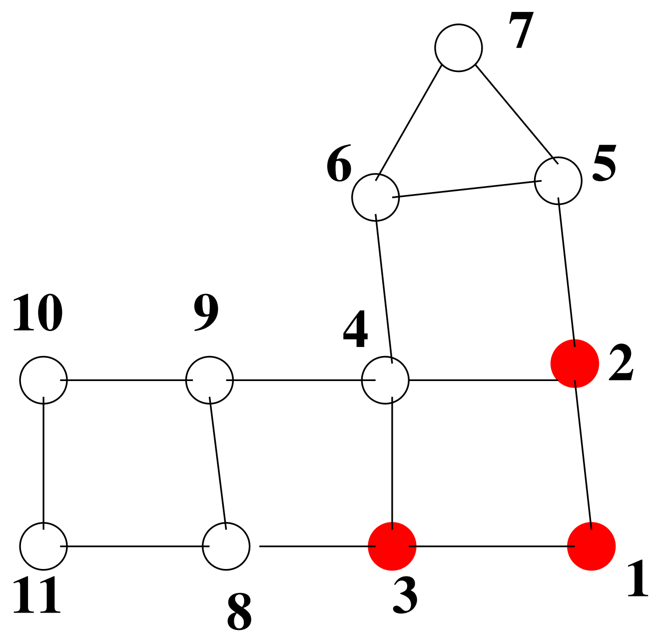
→ polynomially separable for fixed t and s

Robertson and Seymour (1980s - 1990s) : tree-width

Def: A **tree-decomposition** of a graph G consists of a tree T and a family of sets X_t (for each node t of T), such that:

- Each X_t is a subset of vertices of G.
- For each vertex v of G, the collection of sets X_t containing v forms a subtree of T.
- For each edge $\{u, v\}$ of G there is a subset X_t with $\{u, v\} \subseteq X_t$.

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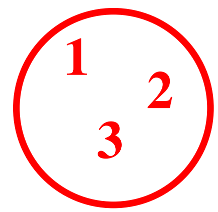
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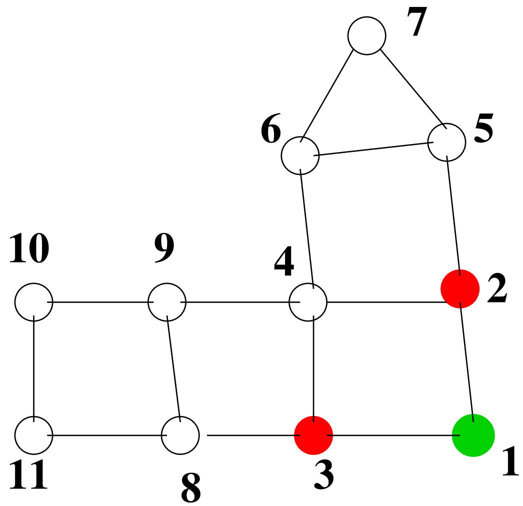
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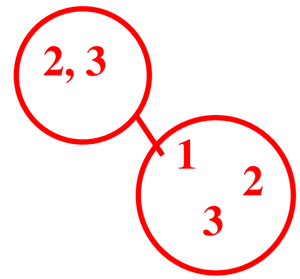
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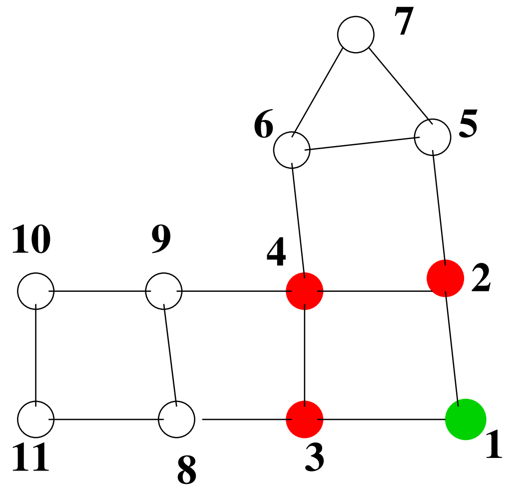
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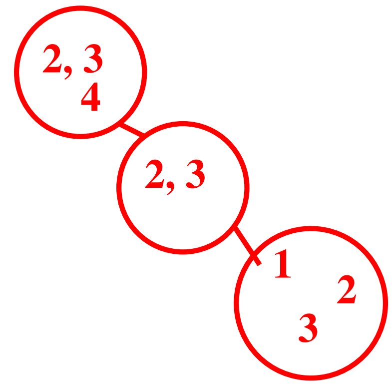
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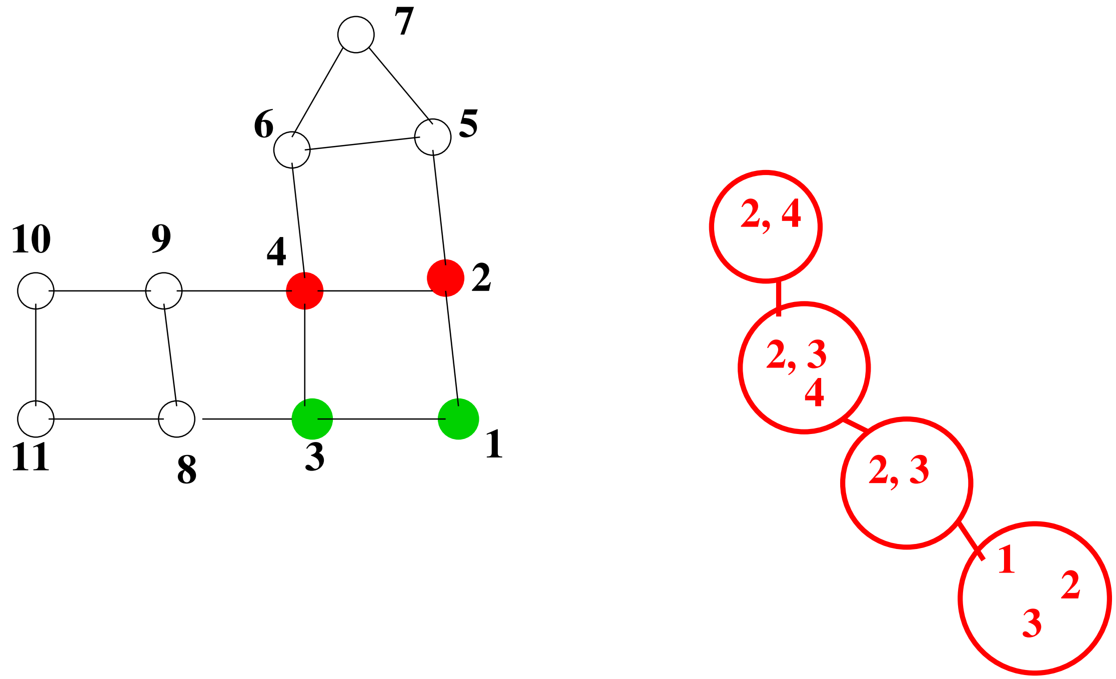
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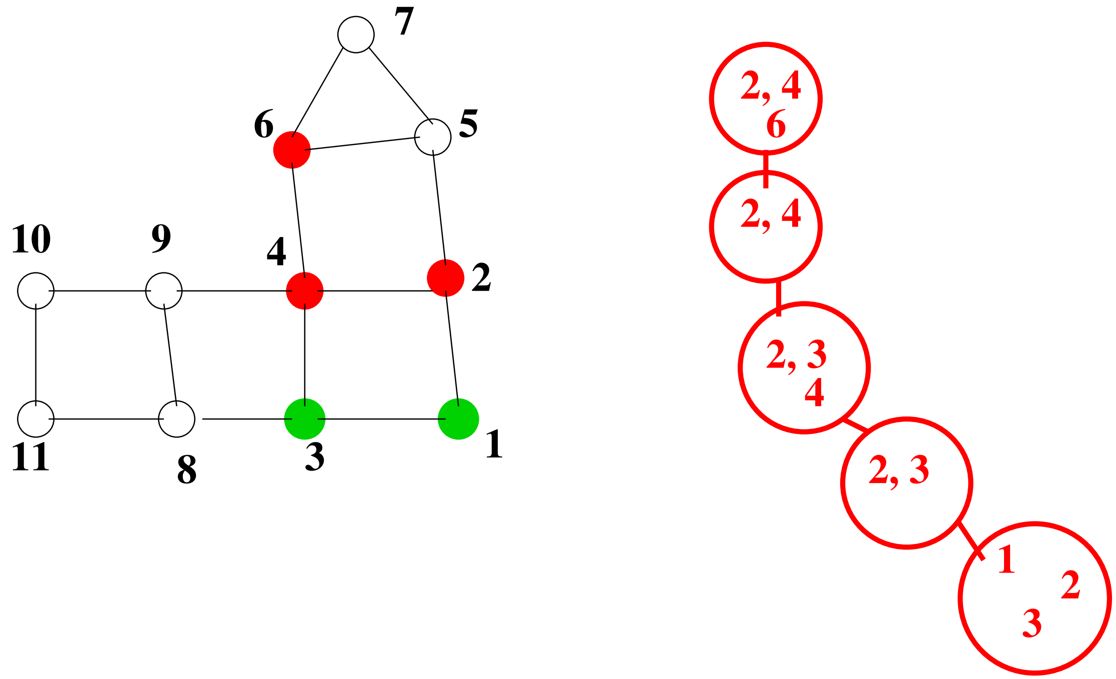
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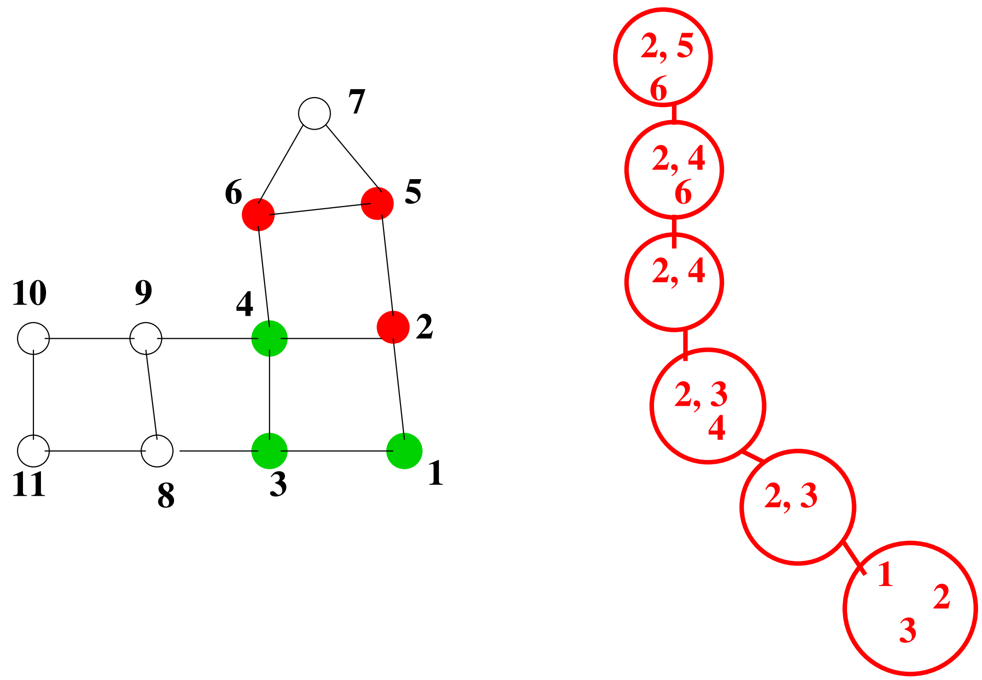

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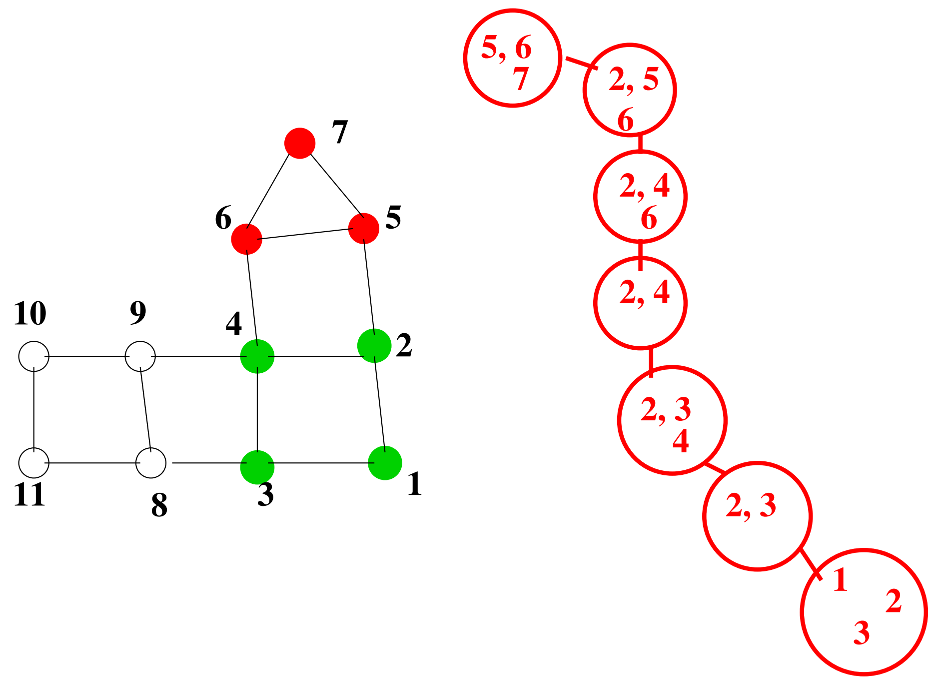
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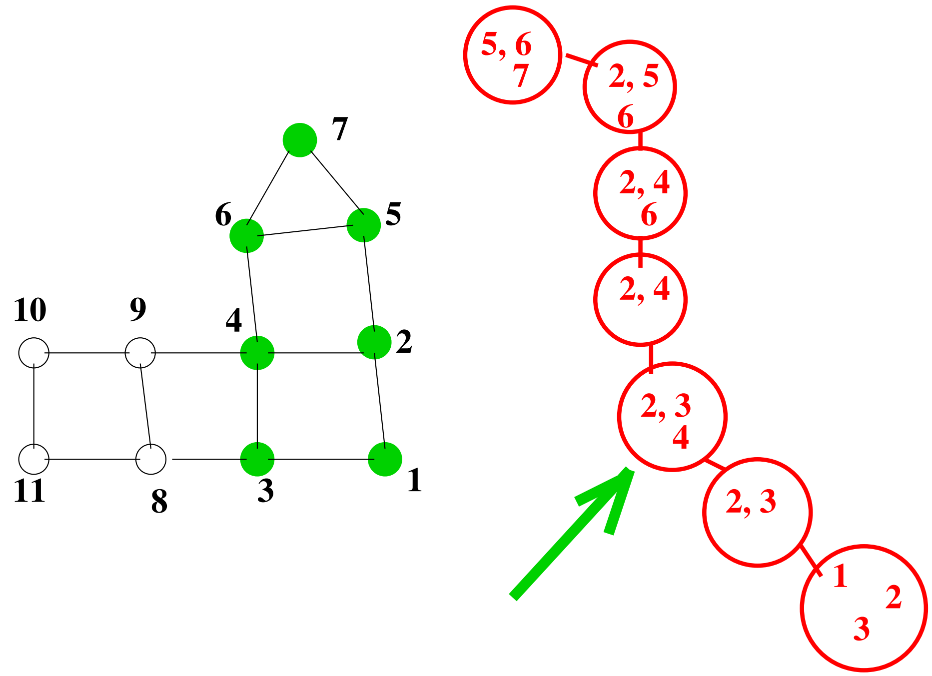
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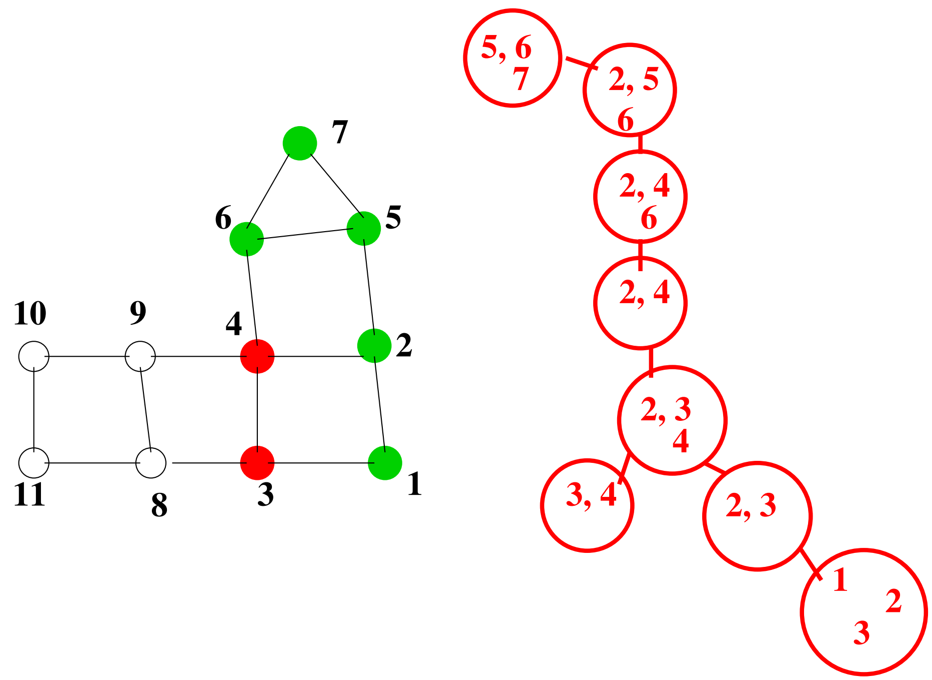

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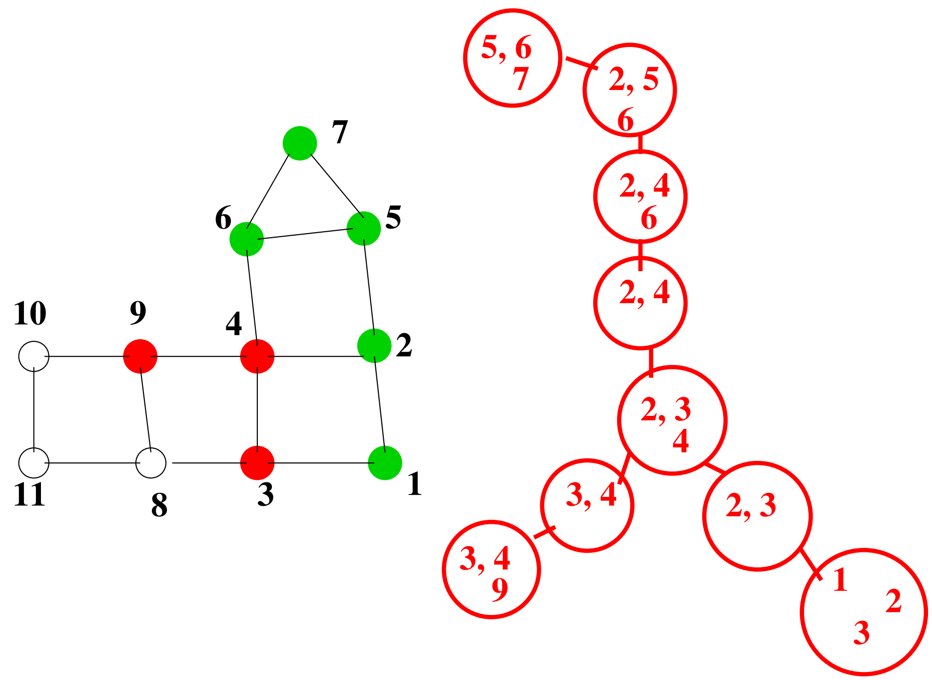
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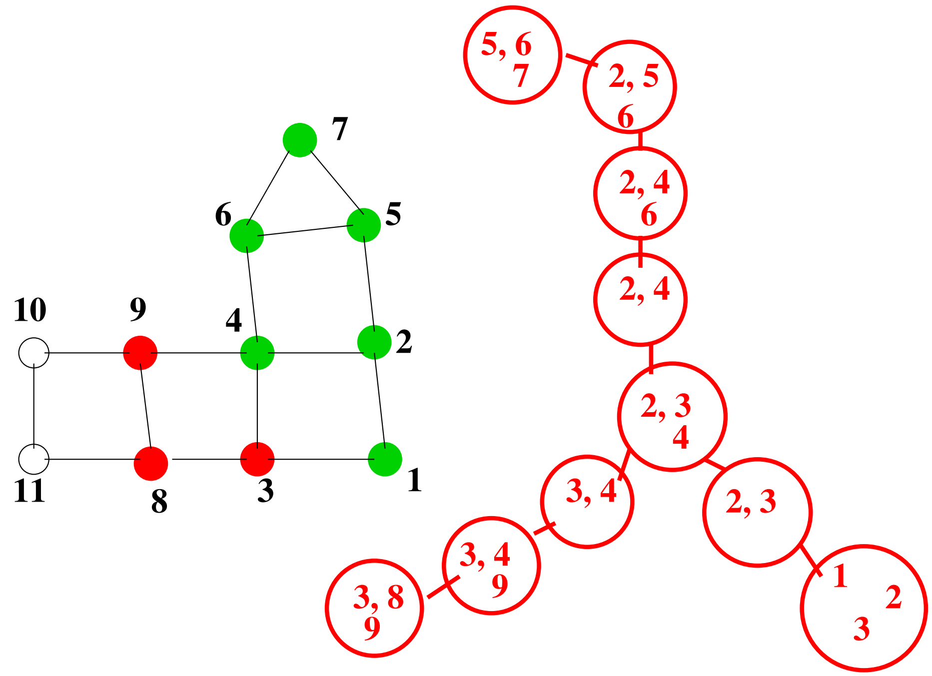
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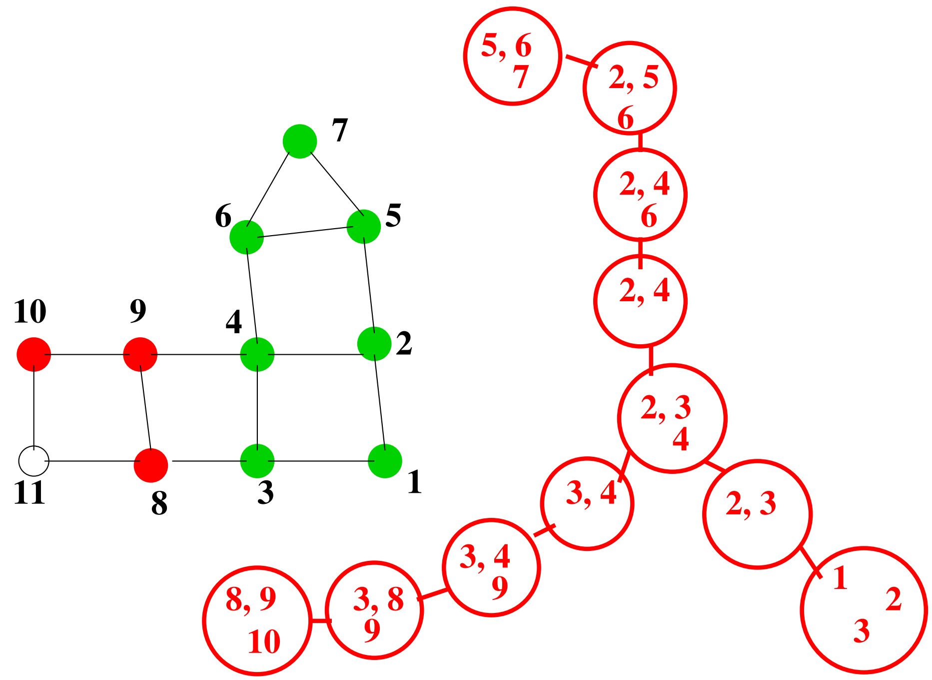

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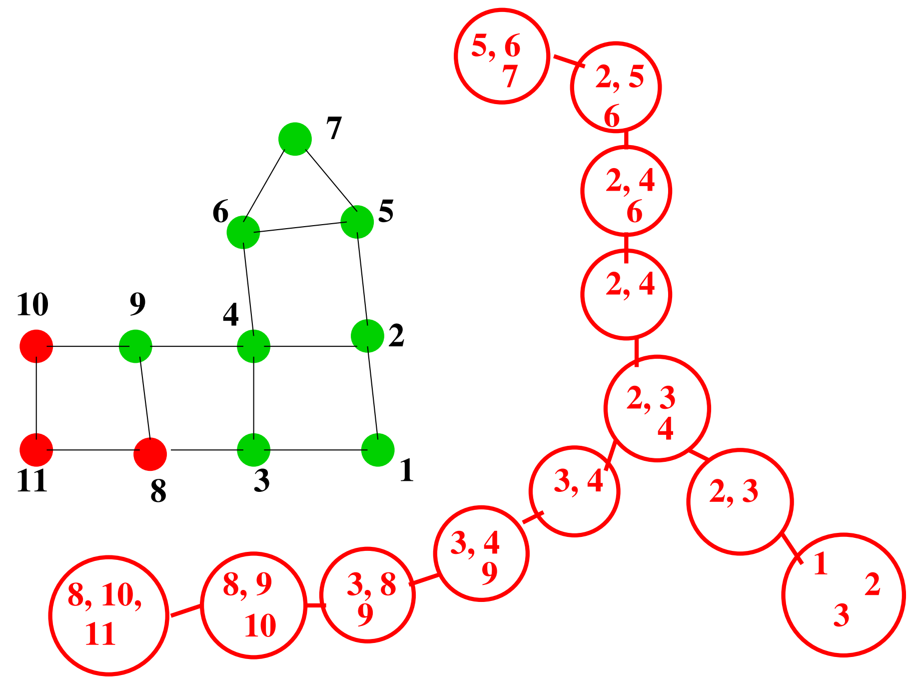
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2
11
8
3
1
5, 6
7
2, 5
6
2, 4
6
2, 4
2, 3
4
3, 4
2, 3
1
2
3
3, 4
9
3, 8
9
8, 9
10
8, 10,
11

Def: A **tree-decomposition** of a graph $\boldsymbol{G}$ consists of a tree $\boldsymbol{T}$ and a family of sets $\boldsymbol{X_t}$ (for each node $\boldsymbol{t}$ of $\boldsymbol{T}$), such that:

- Each $\boldsymbol{X_t}$ is a subset of vertices of $\boldsymbol{G}$.
- For each vertex $\boldsymbol{v}$ of $\boldsymbol{G}$, the collection of sets $\boldsymbol{X_t}$ containing $\boldsymbol{v}$ forms a subtree of $\boldsymbol{T}$.
- For each edge $\boldsymbol{\{u,v\}}$ of $\boldsymbol{G}$ there is a subset $\boldsymbol{X_t}$ with $\boldsymbol{\{u,v\} \subseteq X_t}$.

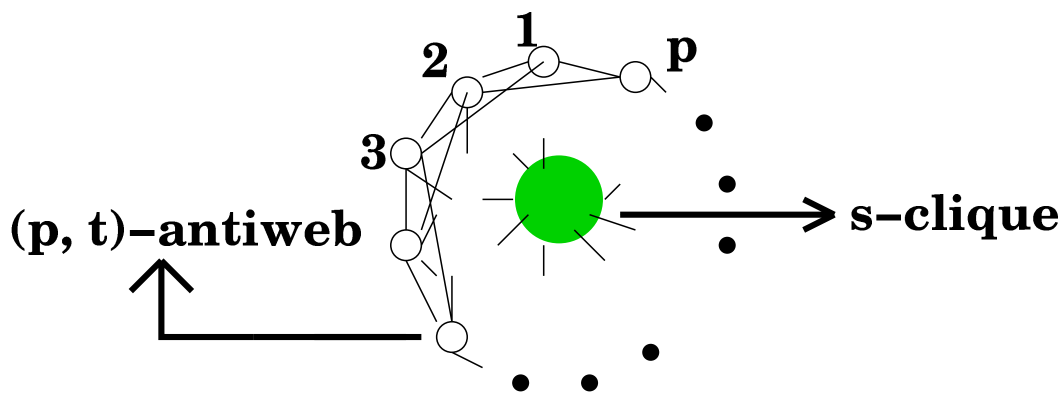
The **width** of the decomposition is $\mathbf{max}_t \{|X_t|\} - 1$.

$\longrightarrow$ $\boldsymbol{G}$ has **tree-width** $\leq \boldsymbol{k}$ if it has a tree-decomposition of width $\leq \boldsymbol{k}$.

... equivalently
if there is a **chordal supergraph** of $\boldsymbol{G}$ of **clique number** $\leq \boldsymbol{k}$

Robertson and Seymour: $\boldsymbol{G}$ has large tree-width if and only if $\boldsymbol{G}$ contains a large square grid minor

$(p,t) - antiwebs - s - wheels$:



Lemma: A subdivision of a $(p,t) - antiwebs - s - wheel$ has tree-width $\leq 2t + s - 2$.

... What can we say about inequalities of the form

$$\sum_{j \in V(H)} \alpha_j x_j \leq \beta,$$

where H has small tree-width ?

Theorem

Suppose we construct the Sherali-Adams level- $\boldsymbol{k}$ formulation $\boldsymbol{SA^k}$ to the vertex packing problem for a graph $\boldsymbol{G}$.

Then, any vector $\hat{\boldsymbol{x}}$ that satisfies the constraints of $\boldsymbol{SA^k}$ also satisfies **all valid inequalites** of the form

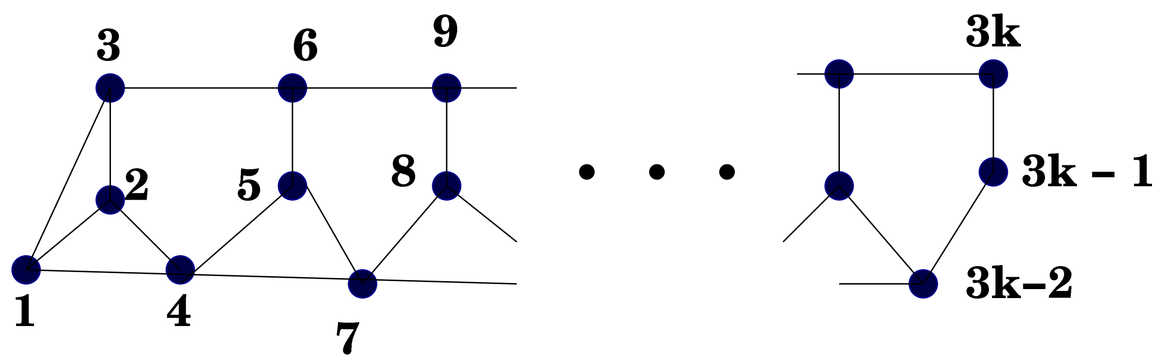
$$\boldsymbol{\Sigma_{j \in V(H)} \alpha_j x_j \le \beta}$$

for every subgraph $\boldsymbol{H}$ of $\boldsymbol{G}$ of tree-width $\le \boldsymbol{k}$. ∎

Corollary: For $\boldsymbol{k \ge 2t + s - 2}$, we are guaranteed to satisfy all (subdivided) $(p, t) - antiweb - s - wheel$ inequalities, etc.

Lipták and Tunçel (2003).
Also see Goemans and Tunçel (2001), Cook and Dash (2001)

Consider the graph:



(= the line-graph of an iterated blossom)

→ Theorem (L & T): the $\boldsymbol{N_0}$-rank of this graph is $\sim \; \mathbf{log_2}\, \boldsymbol{k}$.

→ Conjecture (L & T): the $\boldsymbol{N_0}$- and $\boldsymbol{N}$-ranks of any graph are equal.

→ The graph has tree-width **3**.

Corollary. Its Sherali-Adams rank is $\leq \mathbf{3}$.

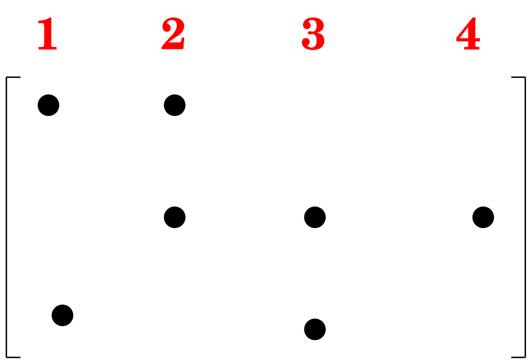
General packing polyhedra

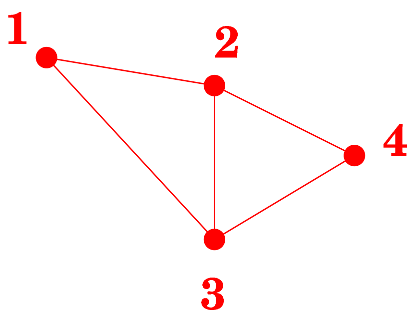
$$Ax \le b, \quad x \in \{0,1\}^n,$$

where $A \geq 0$.

Def: (Balas et al) The clique graph of A has

- A vertex for each **column** of A,
- An edge between k and j if there is a row i with $\boldsymbol{a_{ik} > 0}$ and $\boldsymbol{a_{ij} > 0}$.





Definition: The **tree-width** of an inequality $\boldsymbol{\alpha}^T \boldsymbol{x} \leq \boldsymbol{\beta}$ valid for

$\{\boldsymbol{Ax} \leq \boldsymbol{b}, \;\; \boldsymbol{x} \in \{0,1\}^n\}$,

is the minimum tree-width of the clique graph of any submatrix $\boldsymbol{A'}$ of $\boldsymbol{A}$,

such that $\boldsymbol{\alpha}^T \boldsymbol{x} \leq \boldsymbol{\beta}$ is valid for $\left\{\boldsymbol{A'x} \leq \boldsymbol{b'}, \;\; \boldsymbol{x} \in \{0,1\}^{n'}\right\}$.

Theorem: For any $\boldsymbol{k} \geq \boldsymbol{1}$, the level-$\boldsymbol{k}$ Sherali-Adams formulation for

$\{\boldsymbol{Ax} \leq \boldsymbol{b}, \;\; \boldsymbol{x} \in \{0,1\}^n\}$

is guaranteed to satisfy all inequalities with tree-width $\leq \boldsymbol{k} - \boldsymbol{1}$. ∎

Covering problems?

Set-covering polyhedra: $\{\boldsymbol{Ax} \geq \mathbf{1}, \;\; \boldsymbol{x} \in \{0,1\}^n\}$, A a $0-1$ matrix.

Theorem (B. and Zuckerberg, 2002). Given a set-covering problem, for any fixed $\boldsymbol{k} \geq \mathbf{1}$ we can generate in polynomial time a relaxation that is guaranteed to satisfy all valid inequalities with coefficients in $\{\mathbf{0}, \mathbf{1}, \ldots, \boldsymbol{k}\}$.

Theorem (B. and Zuckerberg, 2002). Let $\boldsymbol{r} \geq \mathbf{1}$ and $\mathbf{0} < \boldsymbol{\epsilon} < \mathbf{1}$ be fixed. Given a set-covering problem

$$\boldsymbol{\tau} = \min\left\{\boldsymbol{c}^T\boldsymbol{x} \;:\; \boldsymbol{Ax} \geq \mathbf{1}, \;\; \boldsymbol{x} \in \{0,1\}^n\right\},$$

in polynomial time we can compute a value $\boldsymbol{v}$ with

$$(\mathbf{1}-\boldsymbol{\epsilon}) \min\left\{\boldsymbol{c}^T\boldsymbol{x} \;: \boldsymbol{x} \in \boldsymbol{K}^r(\boldsymbol{A}, \boldsymbol{b})\right\} \;\leq\; \boldsymbol{v} \;\leq\; \boldsymbol{\tau},$$

where $\boldsymbol{K}^r(\boldsymbol{A}, \boldsymbol{b})$ is the rank-k Chvátal-Gomory closure of $\{\boldsymbol{Ax} \geq \mathbf{1}, \; \mathbf{0} \leq \boldsymbol{x} \leq \mathbf{1}\}$.