SDP versus RLT for Nonconvex QCQP

Kurt M. Anstreicher
Dept. of Management Sciences
University of Iowa

Workshop on Integer Programming and Continuous Optimization

Chemnitz, November 2004

1 The QCQP Problem

We consider a Quadratically Constrained Quadratic Programming problem of the form:

$$
\begin{array}{rll}
\text{QCQP}: \quad & \max & x^T Q_0 x + a_0^T x \\
& \text{s.t.} & x^T Q_i x + a_i^T x \le b_i, \quad i \in \mathcal{I} \\
& & x^T Q_i x + a_i^T x = b_i, \quad i \in \mathcal{E} \\
& & l \le x \le u,
\end{array}
$$

where $x \in \Re^n$ and $\mathcal{I} \cup \mathcal{E} = \{1, \ldots, m\}$. The matrices Q_i are assumed to be symmetric. If $Q_i \preceq 0$ for $i = 0$, $Q_i \succeq 0$ for $i \in \mathcal{I}$ and $Q_i = 0$ for $i \in \mathcal{E}$, then QCQP is a convex optimization problem. In general however QCQP is NP-hard.

QCQP is a well-studied problem in the global optimization literature with many applications, frequently arising from Euclidean distance geometry.

2 RLT and SDP Relaxations

Relaxations of QCQP based on Semidefinite Programming (SDP) and the Reformulation-Linearlization Technique (RLT) both relax product terms $x_i x_j$ to an element X_{ij} of an $n \times n$ matrix X. The two relaxations differ in the form of the constraints on X.

Semidefinite Programming

The SDP relaxation of QCQP may be written

$$
\begin{aligned}
\text{SDP}: \qquad & \min \;\; Q_0 \bullet X + a_0^T x \\
& \text{s.t.} \;\; Q_i \bullet X + a_i^T x \le b_i, \quad i \in \mathcal{I} \\
& \qquad\;\; Q_i \bullet X + a_i^T x = b_i, \quad i \in \mathcal{E} \\
& \qquad\;\; l \le x \le u, \quad X - xx^T \succeq 0.
\end{aligned}
$$

It is very well known that the condition $X - xx^T \succeq 0$ is equivalent to

$$
\tilde{X} := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0,
$$

and therefore SDP may be alternatively written in the form

$$
\begin{aligned}
\text{SDP}: \qquad & \min \;\; \tilde{Q}_0 \bullet \tilde{X} \\
& \text{s.t.} \;\; \tilde{Q}_i \bullet \tilde{X} \le 0, \quad i \in \mathcal{I} \\
& \qquad\;\; \tilde{Q}_i \bullet \tilde{X} = 0, \quad i \in \mathcal{E} \\
& \qquad\;\; l \le x \le u, \quad \tilde{X} \succeq 0,
\end{aligned}
$$

where

$$
\tilde{Q}_i := \begin{pmatrix} -b_i & a_i^T/2 \\ a_i/2 & Q_i \end{pmatrix}.
$$

Reformulation-Linearization Technique

The RLT relaxation of QCQP is based on forming products of the bound constraints $x_i - l_i \geq 0$ and $u_i - x_i \geq 0$, $i = 1, \ldots, n$. Forming all such possible products, and relaxing product terms $x_i x_j$ to X_{ij}, results in the system of constraints

$$
\begin{aligned}
X_{ij} - l_i x_j - l_j x_i &\geq -l_i l_j, \\
X_{ij} - u_i x_j - u_j x_i &\geq -u_i u_j, \\
X_{ij} - l_i x_j - u_j x_i &\leq -l_i u_j, \\
X_{ij} - l_j x_i - u_i x_j &\leq -l_j u_i,
\end{aligned}
$$

$i, j = 1, \ldots, n$. Note that for $i = j$ the two upper bounds on X_{ii} are the same. Using the fact that $X_{ij} = X_{ji}$, the result is an ordinary Linear Programming (LP) problem with $n(n+1)/2$ variables and a total of $m + n(2n+3)$ constraints. For the purpose of interpretation it is helpful to write the constraints on X_{ij} in the alternative form

$$
\begin{aligned}
X_{ij} &\geq x_i x_j - (x_i - l_i)(x_j - l_j), \\
X_{ij} &\geq x_i x_j - (u_i - x_i)(u_j - x_j), \\
X_{ij} &\leq x_i x_j + (x_i - l_i)(u_j - x_j), \\
X_{ij} &\leq x_i x_j + (u_i - x_i)(x_j - l_j).
\end{aligned}
$$

Comparison between SDP and RLT

To compare the SDP and RLT relaxations it is useful to consider the principal submatrix of $\tilde{X}$ corresponding to two variables x_i and x_j. Taking $i = 1$ and $j = 2$ w.l.o.g., let

$$\tilde{X}^{12} = \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & X_{11} & X_{12} \\ x_2 & X_{12} & X_{22} \end{pmatrix}.$$

It is then straightforward to show that the condition $\tilde{X}^{12} \succeq 0$, from SDP, is equivalent to the constraints

$$\begin{aligned} X_{ii} &\geq x_i^2, \quad i = 1, 2, \\ X_{12} &\leq x_1 x_2 + \sqrt{(X_{11} - x_1^2)(X_{22} - x_2^2)}, \\ X_{12} &\geq x_1 x_2 - \sqrt{(X_{11} - x_1^2)(X_{22} - x_2^2)}. \end{aligned}$$

Proposition. Consider the SDP and RLT constraints on X_{11}, X_{22} and X_{12} for values of x_i satisfying $l_i \leq x_i \leq u_i$, $i = 1, 2$. Then:

1. SDP implies no upper bound on X_{ii}, $i = 1, 2$ compared to the RLT upper bounds

$$X_{ii} \leq x_i^2 + (x_i - l_i)(u_i - x_i).$$

2. The SDP lower bounds $X_{ii} \geq x_i^2$, $i = 1, 2$ dominate the RLT lower bounds

$$X_{ii} \geq x_i^2 - (x_i - l_i)^2, \quad X_{ii} \geq x_i^2 - (u_i - x_i)^2.$$

3. The SDP bounds on X_{12} dominate the RLT bounds on X_{12} if for $i = 1, 2$

$$X_{ii} \leq x_i^2 + (x_i - l_i)^2, \quad X_{ii} \leq x_i^2 + (u_i - x_i)^2.$$

Proof of part 3: Assume that $X_{ii} \le x_i^2 + (x_i - l_i)^2$, $i = 1, 2$. Then

$$
\begin{aligned}
(x_i - l_i)^2 &\ge X_{ii} - x_i^2, \quad i = 1, 2 \\
(x_i - l_i) &\ge \sqrt{X_{ii} - x_i^2}, \quad i = 1, 2 \\
(x_1 - l_1)(x_2 - l_2) &\ge \sqrt{(X_{11} - x_1^2)(X_{22} - x_2^2)}.
\end{aligned}
$$

It follows that the SDP lower bound

$$X_{12} \ge x_1 x_2 - \sqrt{(X_{11} - x_1^2)(X_{22} - x_2^2)}$$

can be no worse than the RLT lower bound

$$X_{12} \ge x_1 x_2 - (x_1 - l_1)(x_2 - l_2).$$

The analysis for the other RLT bounds is similar. □

Remark. If $x_i = (l_i + u_i)/2$, $i = 1, 2$ then the SDP bounds on X_{12} dominate the RLT bounds for all X_{ii} that satisfy the RLT upper bounds in part 1. In this case can compute that the 3–dimensional volume of the intersection of the SDP and RLT constraints on X_{11}, X_{22}, X_{12} is $(u_1 - l_1)^3(u_2 - l_2)^3/72$, compared to $(u_1 - l_1)^3(u_2 - l_2)^3/8$ for RLT constraints alone. So for these "midpoint" values of x_i, adding SDP decreases volume by a factor of 9.

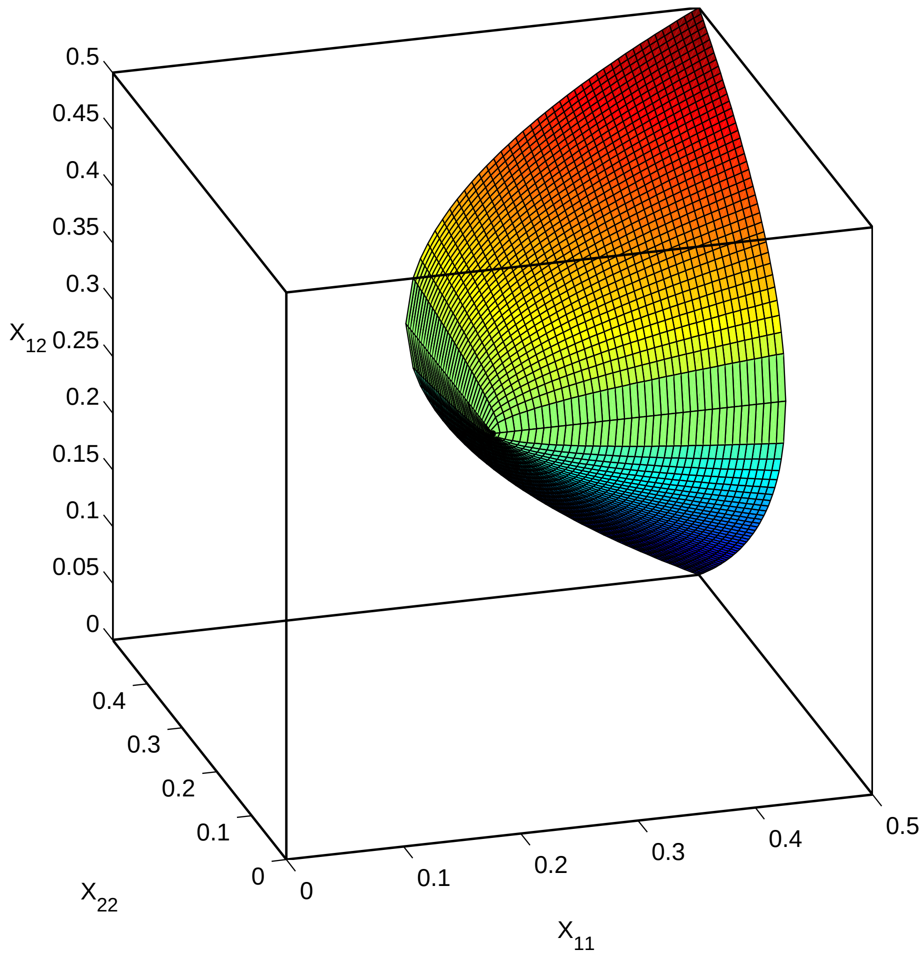


Figure 1: RLT versus SDP+RLT regions, $0 \leq x \leq e$, $x_1 = x_2 = .5$.

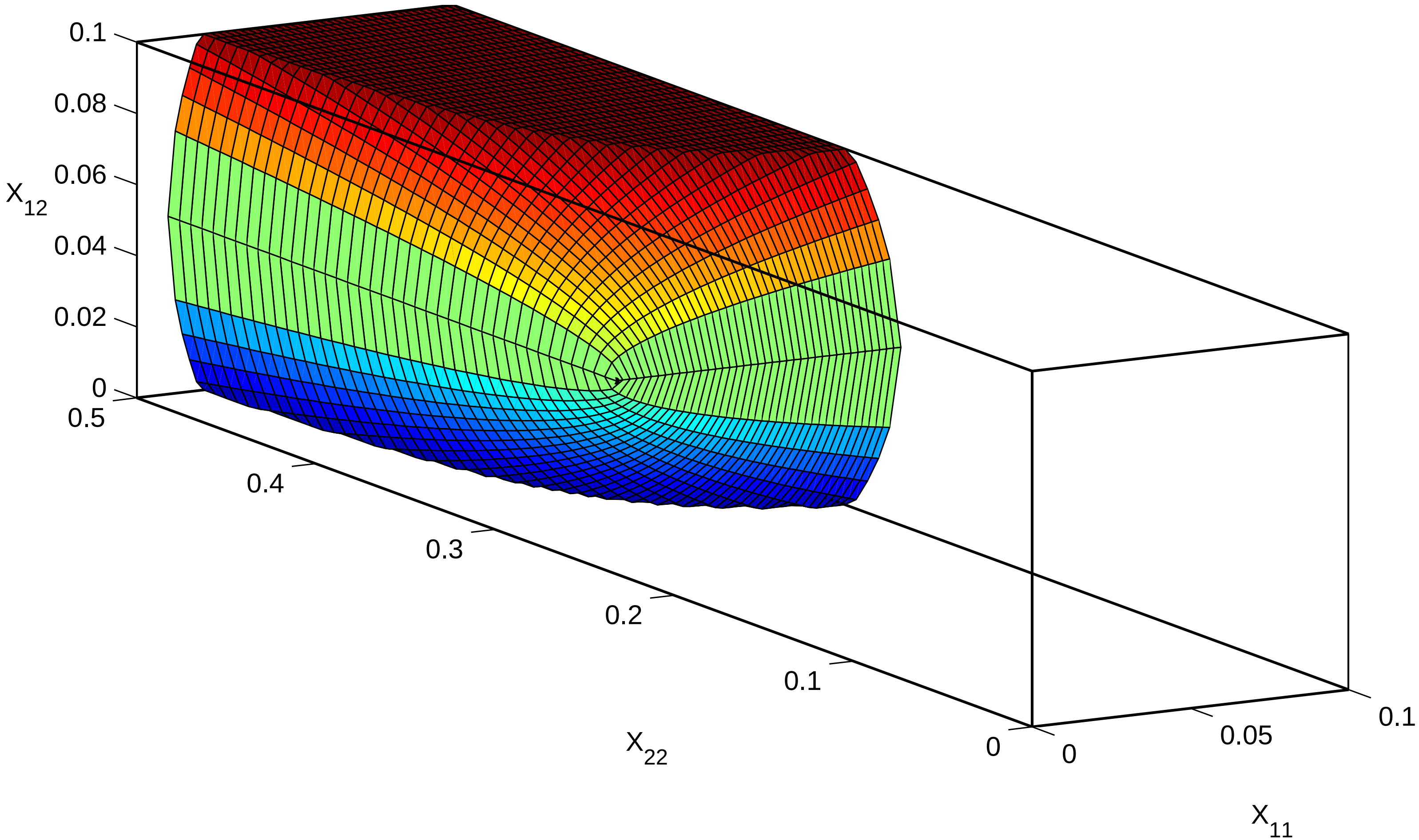


Figure 2: RLT versus SDP+RLT regions, $0 \leq x \leq e$, $x_1 = .1$, $x_2 = .5$.

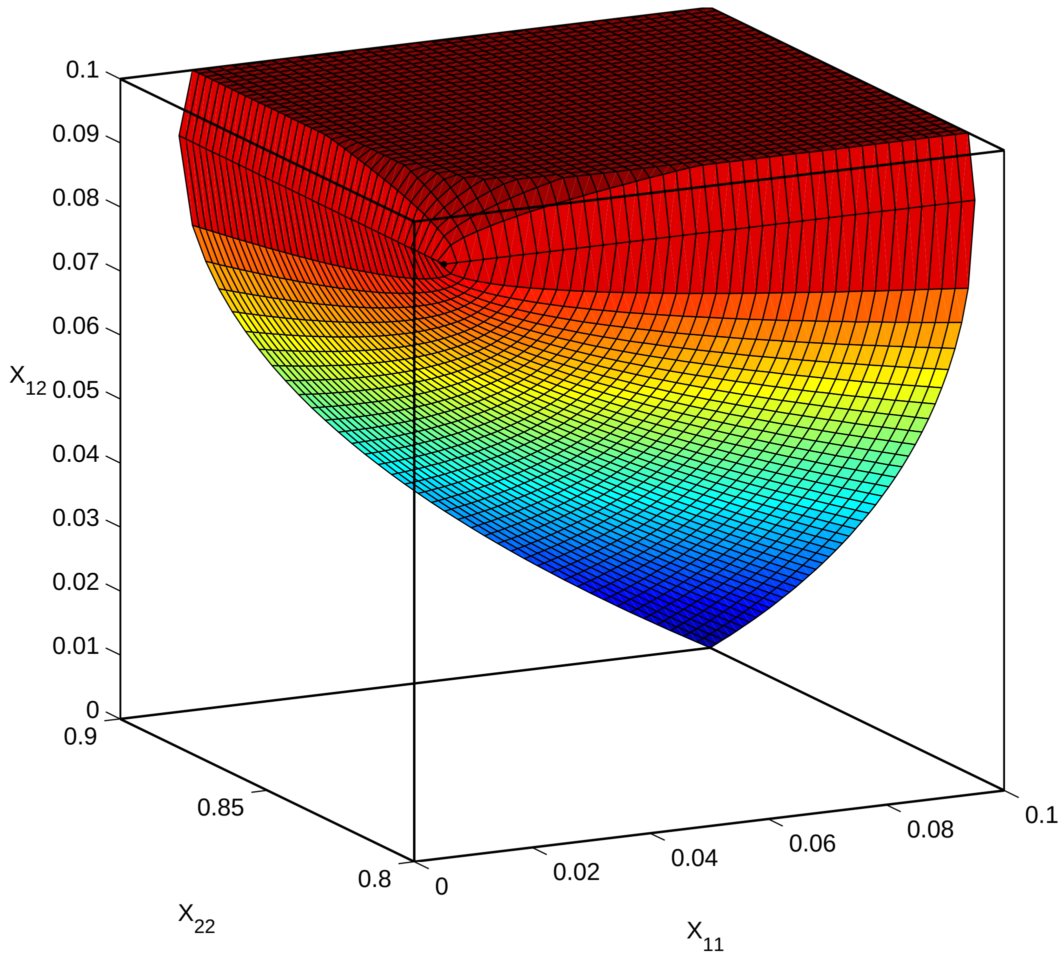


Figure 3: RLT versus SDP+RLT regions, $0 \leq x \leq e$, $x_1 = .1$, $x_2 = .9$.

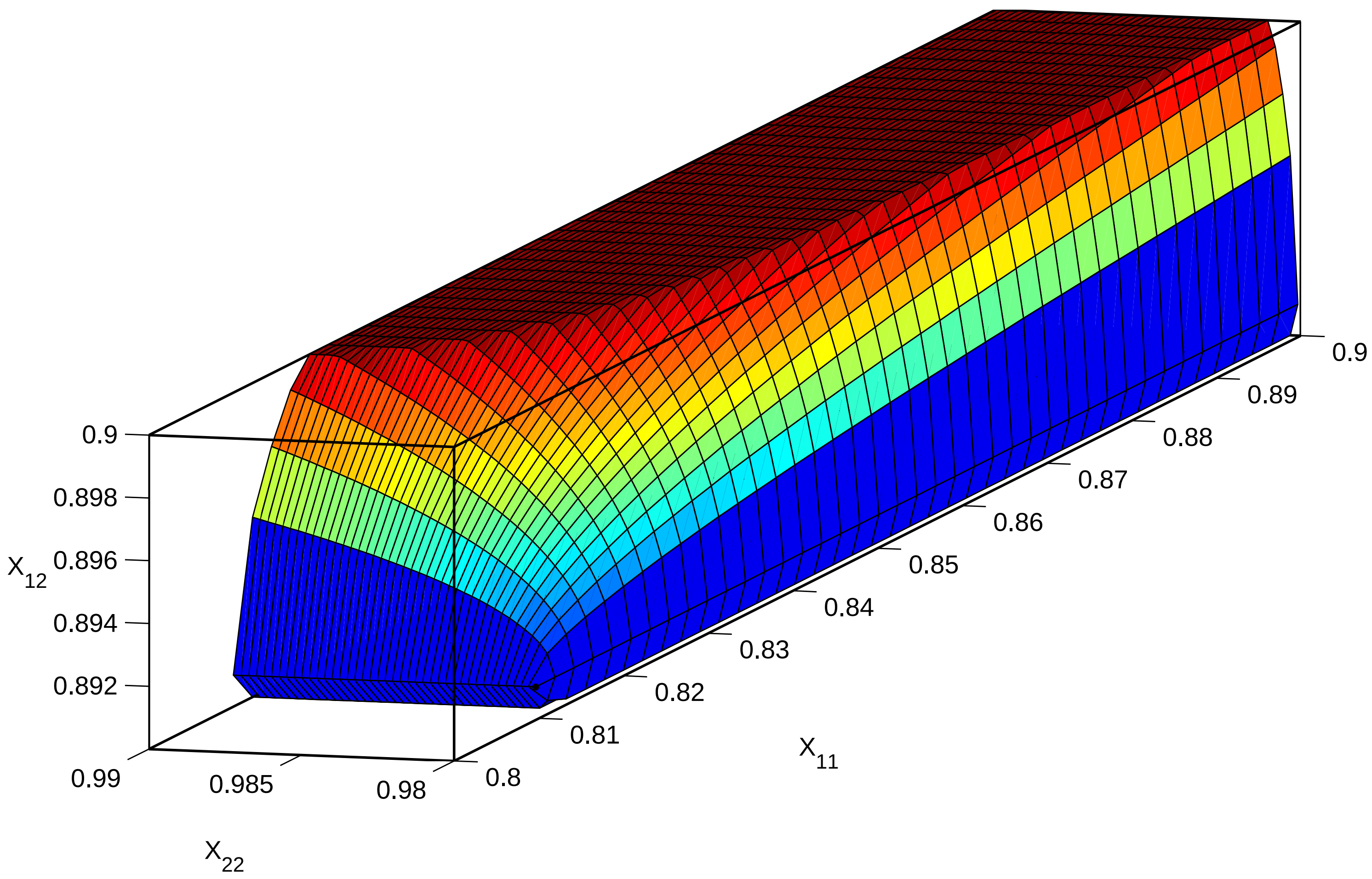


Figure 4: RLT versus SDP+RLT regions, $0 \leq x \leq e$, $x_1 = .9$, $x_2 = .99$.

3 Computational Results I: Box-constrained QP

Consider 15 box-constrained QP problems with $n = 30$, from Vandenbussche and Nemhauser (2003). Density of Q_0 varies from 60% to 100%. Compare bounds from Vandenbussche and Nemhauser polyhedral relaxation PS, BARON, RLT, SDP, and SDP+RLT. (Results for BARON are at root after tightening - courtesy of Dieter Vandenbusshe. SDP includes upper bound on diagonal components X_{ii}.)

Table 1: Comparison of bounds for indefinite box-constrained QP Problems

Problem instance	Optimal value	Bound value RLT	Bound value BARON	Bound value PS	Bound value SDP	SDP+ RLT	Gap to optimal value RLT	Gap to optimal value BARON	Gap to optimal value PS	Gap to optimal value SDP	Gap SDP+ RLT
30-60-1	706.00	1454.75	1430.20	1405.25	768.12	714.67	106.06%	102.58%	99.04%	8.80%	1.23%
30-60-2	1377.17	1699.50	1668.51	1637.00	1426.94	1377.17	23.41%	21.15%	18.87%	3.61%	0.00%
30-60-3	1293.50	2047.00	2006.83	1966.00	1370.13	1298.21	58.25%	55.15%	51.99%	5.92%	0.36%
30-70-1	654.00	1569.00	1547.43	1525.50	746.43	674.00	139.91%	136.61%	133.26%	14.13%	3.06%
30-70-2	1313.00	1940.25	1888.67	1836.25	1375.07	1313.00	47.77%	43.84%	39.85%	4.73%	0.00%
30-70-3	1657.40	2302.75	2251.55	2199.50	1719.77	1657.55	38.94%	35.85%	32.71%	3.76%	0.01%
30-80-1	952.73	2107.50	2072.29	2036.50	1050.76	965.25	121.21%	117.51%	113.75%	10.29%	1.31%
30-80-2	1597.00	2178.25	2158.29	2138.00	1622.81	1597.00	36.40%	35.15%	33.88%	1.62%	0.00%
30-80-3	1809.78	2403.50	2376.47	2349.00	1836.79	1809.78	32.81%	31.31%	29.79%	1.49%	0.00%
30-90-1	1296.50	2423.50	2385.44	2346.75	1348.48	1296.50	86.93%	83.99%	81.01%	4.01%	0.00%
30-90-2	1466.84	2667.00	2623.11	2578.50	1527.87	1466.84	81.82%	78.83%	75.79%	4.16%	0.00%
30-90-3	1494.00	2538.25	2499.69	2460.50	1516.81	1494.00	69.90%	67.32%	64.69%	1.53%	0.00%
30-100-1	1227.13	2602.00	2541.99	2481.00	1285.74	1227.13	112.04%	107.15%	102.18%	4.78%	0.00%
30-100-2	1260.50	2729.25	2699.12	2668.50	1365.32	1261.08	116.52%	114.13%	111.70%	8.32%	0.05%
30-100-3	1511.05	2751.75	2704.14	2655.75	1611.11	1513.08	82.11%	78.96%	75.76%	6.62%	0.13%
Average							76.94%	73.97%	70.95%	5.58%	0.41%

4 Computational Results II: Circle Packing

Consider the problem of maximizing the radius of n non–overlapping circles packed into the unit square in $\Re^2$. Via a simple, well-known transformation this is equivalent to the "point packing" problem

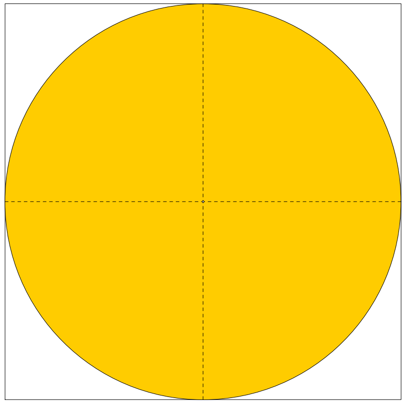
$$\begin{array}{ll} \max & \theta \\ \text{s.t.} & (x_i - x_j)^2 + (y_i - y_j)^2 \ge \theta, \quad 1 \le i < j \le n \\ & 0 \le x \le e, \quad 0 \le y \le e. \end{array}$$

Note that:

1. The variable θ represents the minimum squared distance separating n points in the unit square. The corresponding radius for n circles that can be packed into the unit square is $\sqrt{\theta}/[2(1+\sqrt{\theta})]$.
2. The problem formulation involves no terms of the form $x_i y_j$. As a result, the RLT and SDP bounds can both be based on matrices X and Y relaxing xx^T and yy^T, respectively.
3. Let $n_x = \lceil n/2 \rceil$, $n_y = \lceil n_x/2 \rceil$. By symmetry could assume $.5 \le x_i \le 1$, $i = 1, \ldots, n_x$ and $.5 \le y_i \le 1$, $i = 1, \ldots, n_y$.

N = 1

1 circle in the unit square

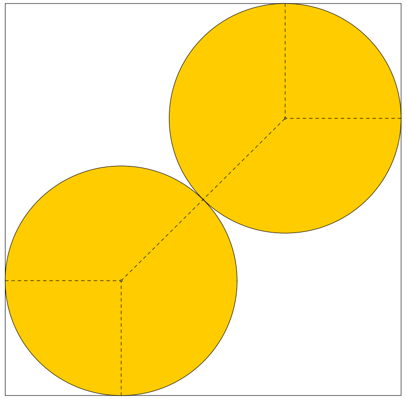


radius = 0.500000000000 density = 0.785398163397
contacts = 4

© E.Specht 10-SEP-1999

N = 2

2 circles in the unit square

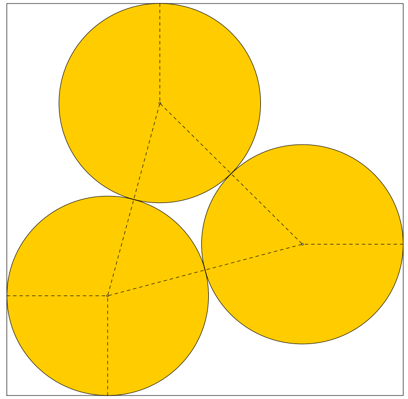


radius = 0.292893218813 density = 0.539012084453
distance = 1.414213562373 contacts = 5

© E.Specht 10-SEP-1999

N = 3

3 circles in the unit square

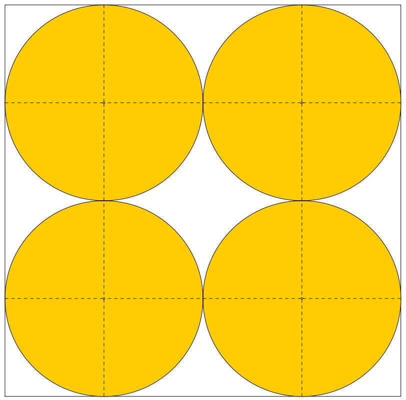


radius = 0.254333095030 density = 0.609644808741
distance = 1.035276180410 contacts = 7

© E.Specht 10-SEP-1999

N = 4

4 circles in the unit square

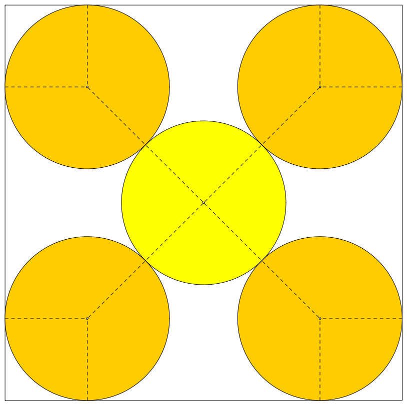


radius = 0.250000000000 density = 0.785398163397
distance = 1.000000000000 contacts = 12

© E.Specht 10-SEP-1999

N = 5

5 circles in the unit square

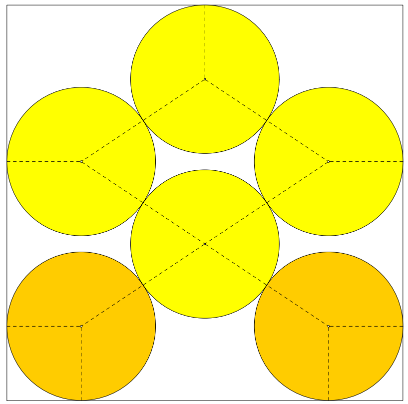


radius = 0.207106781187 density = 0.673765105566
distance = 0.707106781187 contacts = 12

© E.Specht 10-SEP-1999

N = 6

6 circles in the unit square

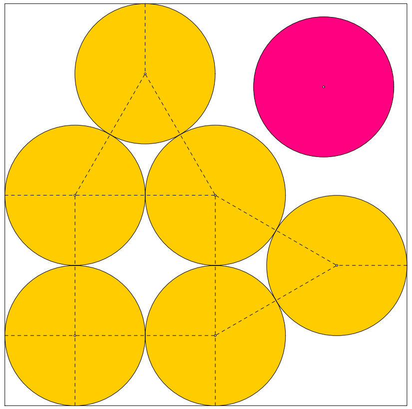


radius = 0.187680601147 density = 0.663956909464
distance = 0.600925212577 contacts = 13

© E.Specht 10-SEP-1999

$N = 7$

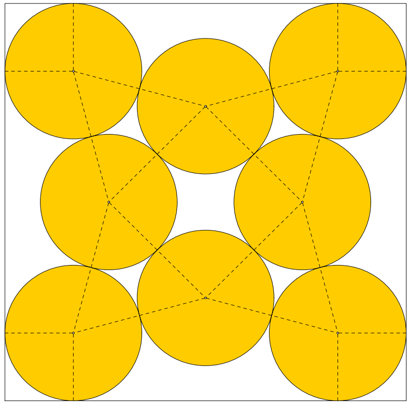
7 circles in the unit square



radius = 0.174457630187 density = 0.669310826841
distance = 0.535898384862 contacts = 14

$N = 8$

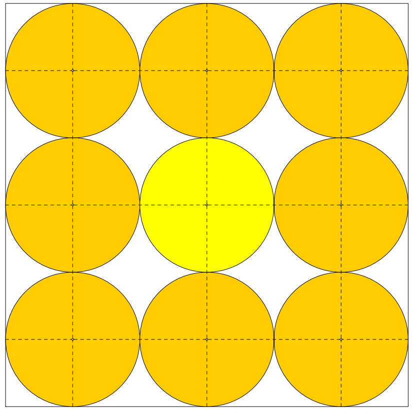
8 circles in the unit square



radius = 0.170540688701 density = 0.730963825254
distance = 0.517638090205 contacts = 20

$N = 9$

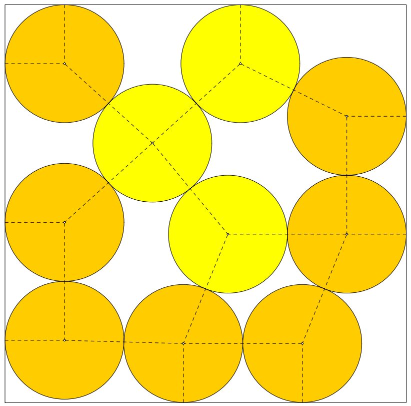
9 circles in the unit square



radius = 0.166666666667 density = 0.785398163397
distance = 0.500000000000 contacts = 24

$N = 10$

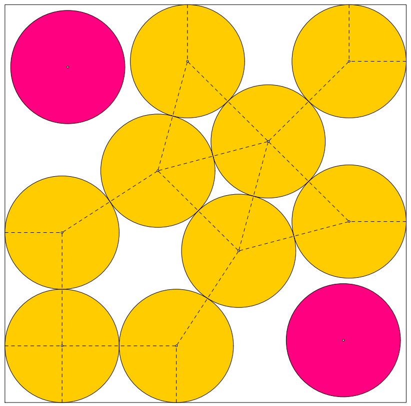
10 circles in the unit square



radius = 0.148204322565 density = 0.690035785264
distance = 0.421279543984 contacts = 21

$N = 11$

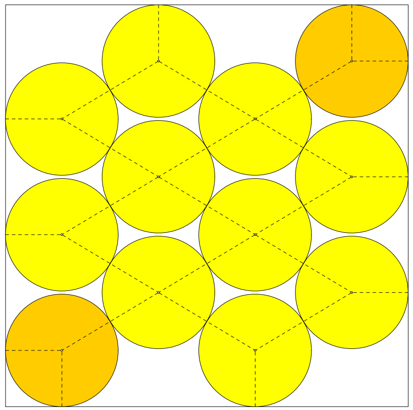
11 circles in the unit square



radius = 0.142399237696 density = 0.700741577756
distance = 0.398207310237 contacts = 20

$N = 12$

12 circles in the unit square



radius = 0.139958844038 density = 0.738468223884
distance = 0.388730126323 contacts = 25

Conjecture. Consider the RLT and SDP relaxations of the point packing problem for $n \geq 2$, where the SDP relaxation includes the upper bounds on X_{ii} and Y_{ii}. Then:

1. The optimal value for the RLT relaxation is 2.
2. The optimal value for the SDP relaxation is $1 + \frac{1}{n-1}$ and adding the RLT constraints does not change this value.
3. For $n \geq 5$ the optimal value for the RLT relaxation using symmetry is $\frac{1}{2}$.
4. For $n \geq 5$ the optimal value for the SDP relaxation using symmetry is

$$.25 + \frac{1}{4\lfloor (n-1)/4 \rfloor},$$

   equal to $.25 + \frac{1}{n-1}$ if $n - 1$ is divisible by 4.

Additional RLT constraints based on order

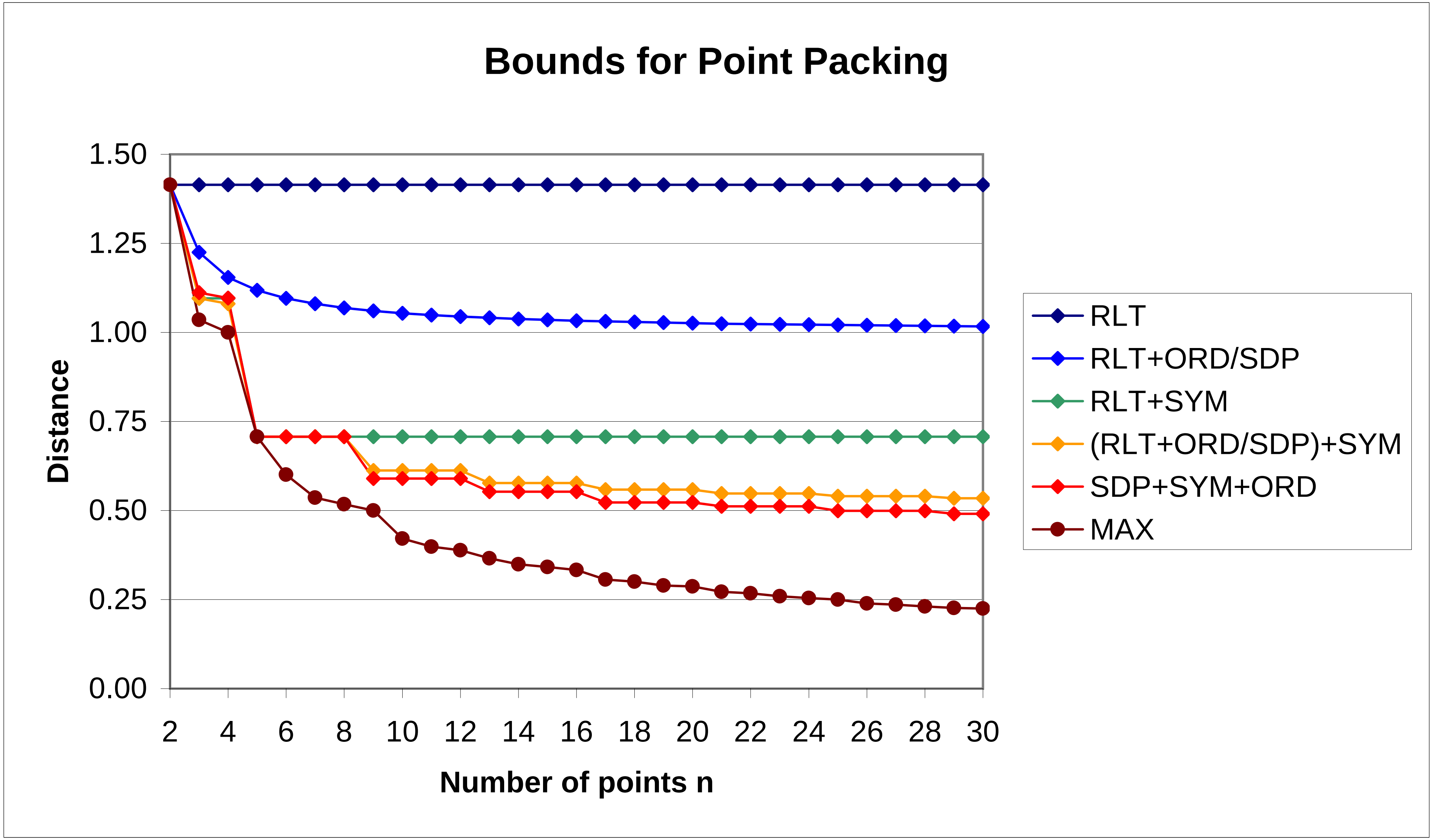
Note that one could assume w.l.o.g. that $x_1 \geq x_2 \geq \ldots \geq x_n$. Adding these constraints alone has no effect on the SDP or RLT relaxations. However, one can generate new RLT constraints by taking products of these constraints with each other and/or the original bound constraints. To limit the number of additional constraints, we consider the inequalities

$$x_i \geq x_{i+1} \quad i = 1, \ldots, n-1$$

and the constraints that result from products with the upper and lower bounds on x_i and x_{i+1}. This gives a total of $5(n-1)$ additional constraints. (If the tightened bounds based on symmetry are also used then the first n_y components of x are treated as a separate block from the remaining $n - n_y$ components.)

Conjecture. Consider the RLT and SDP relaxations of the point packing problem for $n \geq 2$, where the SDP relaxation includes the upper bounds on X_{ii} and Y_{ii}. Then:

1. The optimal value for the RLT relaxation with the additional order constraints is $1 + \frac{1}{n-1}$.
2. For $n \geq 5$ the optimal value for the RLT relaxation using symmetry and the additional order constraints is
$$.25 + \frac{1}{4\lfloor (n-1)/4 \rfloor}.$$
3. For $n \geq 9$ the optimal value for the SDP relaxation using symmetry and the additional order constraints is strictly less than that of the RLT relaxation using symmetry and the additional order constraints.


Bounds for Point Packing
Distance
1.50
1.25
1.00
0.75
0.50
0.25
0.00
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
Number of points n
RLT
RLT+ORD/SDP
RLT+SYM
(RLT+ORD/SDP)+SYM
SDP+SYM+ORD
MAX

5 What Next?

1. Lower bound on volume reduction (X_{ii}, X_{jj}, X_{ij}) for RLT+SDP compared to RLT.
2. Alternative treatment of symmetry/order in point packing problem.
3. Dynamic generation of constraints for RLT+SDP.
4. Additional Euclidean distance problems (protein folding local refinement).