On the Graph Bisection Cut Polytope

Michael Armbruster, Christoph Helmberg
Chemnitz University of Technology
Contact: helmberg@mathematik.tu-chemnitz.de

Marzena Fügenschuh, Alexander Martin
Darmstadt University of Technology
Contact: martin@mathematik.tu-darmstadt.de

October 8, 2007

Abstract

Given a graph $G = (V, E)$ with node weights $\varphi_v \in \mathbb{N} \cup \{0\}$, $v \in V$, and some number $F \in \mathbb{N} \cup \{0\}$, the convex hull of the incidence vectors of all cuts $\delta(S)$, $S \subseteq V$ with $\varphi(S) \leq F$ and $\varphi(V \setminus S) \leq F$ is called the *bisection cut polytope*. We study the facial structure of this polytope which shows up in many graph partitioning problems with applications in VLSI-design or frequency assignment. We give necessary and in some cases sufficient conditions for the knapsack tree inequalities introduced in [9] to be facet-defining. We extend these inequalities to a richer class by exploiting that each cut intersects each cycle in an even number of edges. Finally, we present a new class of inequalities that are based on non-connected substructures yielding non-linear right-hand sides. We show that the supporting hyperplanes of the convex envelope of this non-linear function correspond to the faces of the so-called *cluster weight polytope*, for which we give a complete description under certain conditions.

1 Introduction

Let $G = (V, E)$ be an undirected graph with $V = \{1, \ldots, n\}$ and $E \subseteq \{\{i, j\} : i, j \in V, i < j\}$. For given vertex weights $\varphi_v \in \mathbb{N} \cup \{0\}$ for all $v \in V$ and edge costs $w_{\{i,j\}} \in \mathbb{R}$ for all $\{i, j\} \in E$, a partition of the vertex set V into two disjoint clusters S and $V \setminus S$ with sizes $\varphi(S) \leq F$ and $\varphi(V \setminus S) \leq F$ for fixed $F \in \left[\frac{1}{2}\varphi(V), \varphi(V)\right] \cap (\mathbb{N} \cup \{0\})$ is called a *bisection*. Finding a bisection such that the total cost of edges in the cut $\delta(S) := \{\{i, j\} \in E : i \in S, j \in V \setminus S\}$ is minimal is the *minimum bisection problem* (MB). Its decision form is known to be NP-complete [11].

In this paper we investigate the *bisection cut polytope* P_{B} associated with MB. To define P_{B} note that a cut $\delta(S)$ can be described by its incidence vector $\chi^{\delta(S)}$ with respect to the edge set E. Then

$$P_{\mathrm{B}} := \operatorname{conv}\{\, y \in \mathbb{R}^{|E|} : \, y = \chi^{\delta(S)}, \, S \subseteq V, \, \varphi(S) \leq F, \, \varphi(V \setminus S) \leq F \,\}.$$

MB as well as P_{B} are related to other problems and polytopes in the literature. Obviously, the bisection cut polytope is contained in the *cut polytope* [3, 7]

$$P_{\mathrm{C}} := \mathrm{conv}\left\{ y \in \mathbb{R}^{|E|} : y = \chi^{\delta(S)}, S \subseteq V \right\} \ . \tag{1}$$

If $F = \varphi(V)$ then MB is equivalent to the maximum cut problem (using the negative cost function) and $P_{\mathrm{B}} = P_{\mathrm{C}}$. For $F = \lceil \frac{1}{2}\varphi(V) \rceil$ MB is equivalent to the *equipartition problem* [6] and the bisection cut polytope equals the *equipartition polytope* [4, 5, 15]

$$P_{\mathrm{E}} := \mathrm{conv}\{ y \in \mathbb{R}^{|E|} : y = \chi^{\delta(S)}, S \subseteq V, |\varphi(S) - \varphi(V \setminus S)| \leq 1 \}$$

Furthermore, MB is a special case of the minimum node capacitated graph partitioning problem (MNCGP) [9] where two or more clusters are available for the partition of the node set and each cluster has a limited capacity. The objective in MNCGP is the same as in MB, i.e., to minimize the total cost of edges having endpoints in distinct clusters. Finally, we mention the *knapsack polytope* [19]

$$P_{\mathrm{K}} := \mathrm{conv}\left\{ x \in \{0,1\}^{|V|} : \sum_{v \in V} \varphi_v x_v \leq F \right\} \ . \tag{2}$$

P_{K} plays a fundamental role in the inequalities which we derive for the bisection cut polytope.

Graph partitioning problems in general have numerous applications, for instance in numerics [13], VLSI-design [17], compiler-design [16] and frequency assignment [8].

The main contributions of this paper are threefold. First, in [9] the so-called knapsack tree inequalities have been introduced. These inequalities relate the knapsack conditions on the nodes with the edge variables defining the cuts and turn out to be computationally very effective. However, no theoretical justification has been found so far for this behavior. In this paper, we give necessary conditions for the knapsack tree inequality to be facet-defining, which turn out to be also sufficient in certain cases. Second, we can generalize the knapsack tree inequalities in the case of bisections by exploiting the well-known fact that any cut intersects a cycle an even number of times. This new class of inequalities, called *bisection knapsack walk inequalities*, subsume the knapsack tree inequalities and yield computationally more flexibility in finding strong inequalities. The third class of inequalities, called *capacity reduced bisection knapsack walk inequalities*, extends both classes of inequalities to non-connected substructures. The idea is to exploit the weights of the nodes that are not end-nodes of walks to reduce the capacity of the corresponding knapsack inequality yielding this way stronger right-hand sides for the knapsack tree and bisection knapsack walk inequalities. These stronger conditions result in non-linear right-hand sides. We consider the convex envelope of this non-linear function and show that the supporting hyperplanes are in one-to-one correspondence to the faces of a certain polytope, called *cluster weight polytope*. For the case of a star without capacity restriction we are able to give a complete description of the cluster weight polytope yielding in this case the tightest strengthening possible for the capacity reduced bisection knapsack walk inequalities.

The outline of the paper is as follows. In Section 2 we introduce an integer programming formulation for MB building on the formulation of MNCGP given in [9]. Section 3 treats the known knapsack tree inequalities valid for both MB and MNCGP while Section 5 introduces the new bisection knapsack walk inequalities which are only valid for MB and which subsume the knapsack tree inequalities. Section 4 shows a strengthening only applicable to knapsack tree inequalities. Furthermore, we state necessary and sufficient conditions for knapsack tree inequalities to define facets of P_{B}. Finally, Section 6 introduces a strengthening of the bisection knapsack walk inequalities. For this purpose we investigate the facial structure of the cluster weight polytope on stars.

We frequently denote an edge $\{i, j\}$ by ij. Let A and B be discrete sets such that $A \subseteq B$. The incidence vector of A with respect to B is a vector $\chi^A \in \{0,1\}^{|B|}$ with $\chi^A_a = \begin{cases} 1 \text{ if } a \in A \\ 0 \text{ if } a \in B \setminus A \end{cases}$. For a vector $x \in \mathbb{R}^{|B|}$ we define $x(A) = \sum_{a \in A} x_a$. $0^{|A|}$ is the zero vector of dimension $|A|$ and e_a is the unit vector of dimension $|A|$, which is indexed by the elements of A and has entry 1 in coordinate $a \in A$. For a graph $G = (V, E)$ the edge set of the subgraph induced by $\bar{V} \subseteq V$ will be denoted by $E\left(\bar{V}\right)$ and the node set of the subgraph induced by $\bar{E} \subseteq E$ by $V\left(\bar{E}\right)$. The convex hull of a set $A \subseteq \mathbb{R}^n$ will be denoted by $\operatorname{conv}\{A\}$.

2 An integer programming formulation of MB

The integer programming formulation for MB given below is based on the formulation for MNCGP presented in [9]. We introduce variables z_i^k for each node $i \in V$ and each cluster $k = 1, 2$ and edge variables y_{ij} for each edge $ij \in E$. z_i^k is set to 1 if node i is in cluster k and 0 otherwise. Variable y_{ij} is set to 1 if edge ij is in the cut, i.e., i and j are not in the same cluster. Then MB can be written as

$$
\text{(MB)} \qquad
\begin{array}{lll}
\min & \sum\limits_{e \in E} w_e y_e & \\
\text{s.t.} & z_i^1 + z_i^2 = 1 & \forall i \in V \\
& \sum\limits_{i \in V} \varphi_i z_i^k \leq F & k = 1, 2 \\
& y_{ij} \geq z_i^1 - z_j^1 & \forall ij \in E \\
& y_{ij} \geq z_j^1 - z_i^1 & \forall ij \in E \\
& y_{ij} \leq 2 - z_i^1 - z_j^1 & \forall ij \in E \\
& y_{ij} \leq 2 - z_i^2 - z_j^2 & \forall ij \in E \\
& y_{ij} \in \{0,1\} & \forall ij \in E \\
& z_i^k \in \{0,1\} & \forall i \in V, k = 1, 2 \ .
\end{array}
$$

The first constraints assure that each node i is packed into exactly one cluster k. The second constraints enforce the capacity restriction on each cluster k. The next four constraints transmit for each edge $ij \in E$ the values of variables z_i^1 and z_j^1 to the edge variable y_{ij} in the sense that $y_{ij} = 1$ if and only if $z_i^1 \neq z_j^1$ and $y_{ij} = 0$ otherwise. The last two constraints are the binary restrictions on the variables.
Noting that the variables z_i^k do not appear in the objective function we can consider model

$$
\begin{array}{ll}
\min & \sum\limits_{e \in E} w_e y_e \\
\text{s.t.} & y \in Y_{\mathrm{MB}},
\end{array}
$$

where $Y_{\mathrm{MB}} \subseteq \mathbb{R}^{|E|}$ is the projection onto the y-space of the feasible region of model (MB). It can be worked out that $P_{\mathrm{B}} = \mathrm{conv}(Y_{\mathrm{MB}})$.

3 Known valid inequalities for MNCGP and MB

A large variety of valid inequalities for the polytope associated to MNCGP is known and, since MB is a special case of MNCGP, all those inequalities are also valid for P_{B}: cycle inequalities of the cut polytope [3], tree inequalities [4], star inequalities [4], cycle inequalities of capacitated graph partitioning [5], cycle with tails inequalities [9], suspended tree inequalities [15], path block cycle inequalities [15], cycle with ear inequalities [9], strengthened cycle with ear inequalities [9], knapsack tree inequalities [9] and strengthened knapsack tree inequalities [9].

In the remainder of the paper we specialize and improve the knapsack tree inequality for MB. First we recall its definition for MNCGP from [9].

Definition 1 (Knapsack tree inequality [9])**.** *Let $\sum_{v\in V} a_v x_v \le a_0$ be a valid inequality for the knapsack polytope P_{K} with $a_v \ge 0$ for all $v \in V$. For a fixed node $r \in V$ and a subtree (T, E_T) of G rooted at r we define the* knapsack tree inequality

$$\sum_{v\in T} a_v \left(1 - \sum_{e\in P_{rv}} y_e\right) \le a_0 \tag{3}$$

where for each $v \in T$ the edge set of the path joining node v to root r in (T, E_T) is denoted by P_{rv}.

If (T, E_T) is a star rooted at r, i.e., $E_T = \{\{r,t\} : t \in T, t \neq r\}$, then we call the inequality (3) knapsack star inequality.

In general, there is an exponential number of these knapsack tree inequalities, since for each combination of a valid knapsack inequality with a choice of a rooted tree there is one knapsack tree inequality.

Proposition 2. [9] *The knapsack tree inequality (3) is valid for the polytope P_{B}.*

The above statement follows from the fact that MB is a special case of MNCGP.

It will be useful to write the inequality (3) in the form

$$\sum_{e\in E_T} \left(\sum_{v: e\in P_{rv}} a_v \right) y_e \ge \sum_{v\in T} a_v - a_0 \ . \tag{4}$$

The term on the right-hand side may be interpreted as the excess if all vertices $v \in T$ are packed into the cluster containing the root node r while we are only allowed to pack a total weight of a_0. The left-hand side has to compensate for this, i.e., it has to force some edges into the cut so that not all vertices are placed into the same cluster as the current root. We use this reformulation to apply a folklore approach to strengthen coefficients in general binary programs and obtain

$$\sum_{e\in E_T} \min\left\{ \sum_{v: e\in P_{rv}} a_v, \sum_{v\in T} a_v - a_0 \right\} y_e \ge \sum_{v\in T} a_v - a_0 \ . \tag{5}$$

We call this inequality *truncated knapsack tree inequality.*

Remark 3. Note that this strengthening was already proposed in Proposition 3.12 in [9] applied to the knapsack tree inequality for MNCGP. For MNCGP those authors also proposed a second case of strengthening, namely (in our notation) to reduce α_e to a_0. However, for MB we always have $\alpha_0 \leq a_0$ due to the following reason. In the bisection case if $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in P_{\mathrm{K}} \cap \{0,1\}^n$ then also $\tilde{x} = (1 - \bar{x}_1, \ldots, 1 - \bar{x}_n)^T$ lies in $P_{\mathrm{K}} \cap \{0,1\}^n$. This follows from the fact that the total node weight of each of the clusters $\{v \in V : \bar{x}_v = 1\}$ and $\{v \in V : \bar{x}_v = 0\}$ cannot exceed F. If $\sum_{i \in V} a_v \bar{x}_v \leq a_0$, then also $\sum_{v \in V} a_v \tilde{x}_v = \sum_{v \in V} a_v (1 - \bar{x}_v) \leq a_0$. Summing up these two inequalities yields $\sum_{v \in V} a_v \leq 2a_0$ and thus $\alpha_0 = \sum_{v \in V} a_v - a_0 \leq a_0$.

4 Minimum root strengthening of knapsack tree inequalities

Given a knapsack inequality $\sum_{v \in V} a_v x_v \leq a_0$ with $a_v \geq 0$, $v \in V$, let a corresponding knapsack tree inequality be defined on a tree (T, E_T) rooted at r. If we replace r by another node from T the paths change. The corresponding change of the coefficients of the inequality will be exploited in the strengthening of the truncated knapsack tree inequality presented in this section. Our strengthening aims at reducing the coefficients of the left-hand side while keeping the value of the right-hand side. We are going to show that the strongest or in some cases even facet-defining inequality is achieved if r enforces a sort of balance with respect to the cumulated node weights on the paths to r. To emphasize that the coefficients in (5) depend on the root node r we introduce the notation

$$\alpha_0 := \sum_{v \in T} a_v - a_0, \qquad \alpha_e^r := \min\{\sum_{v : e \in P_{rv}} a_v,\, \alpha_0\},\, e \in E_T, \tag{6}$$

and consider (5) in the form

$$\sum_{e \in E_T} \alpha_e^r y_e \geq \alpha_0. \tag{7}$$

Note that if we change the root of (T, E_T) the right-hand side of (7) remains the same, since by this operation we do not eliminate nodes of (T, E_T).

At first we derive some relations based on the definition of the coefficients α_e^r, $r \in T$, $e \in E_T$, which we will exploit in the proofs of the results presented in this section. The following lemma states that along a path from the root to some node the coefficients cannot increase.

Lemma 4. *Let (T, E_T) be a tree in G rooted at node r and let e and f be two edges on a path to r such that e is closer to r than f with respect to the number of edges. Then*

$$\alpha_e^r \geq \alpha_f^r. \tag{8}$$

Proof. It suffices to consider incident edges e and f. Setting $e := ij$ and $f := jk$ we obtain

$$\sum_{v : e \in P_{rv}} a_v = \sum_{v : f \in P_{rv}} a_v + a_j + \sum_{\bar{e} \in \bar{E}} \left(\sum_{v : \bar{e} \in P_{rv}} a_v \right) \geq \sum_{v : f \in P_{rv}} a_v,$$

where $\bar{E}$ contains edges incident to j except e and f. Hence if $\alpha_f^r = \alpha_0$, then also $\alpha_e^r = \alpha_0$, otherwise $\alpha_f^r \leq \min\{\sum_{v : e \in P_{rv}} a_v,\, \alpha_0\} = \alpha_e^r$. □

In the next lemma we investigate the change of coefficients if the root is moved from a node r to an adjacent node s.

Lemma 5. *Let (T, E_T) be a tree in G and $r, s \in T$ two adjacent nodes with $\bar{e} = rs$. We have*

(a) $\alpha_e^r = \alpha_e^s$ *for all* $e \in E_T$ *such that* $e \neq rs$,

(b) $\alpha_{\bar{e}}^r = \min\{a_s + \sum_{e\in\delta(\{s\})\setminus\{\bar{e}\}} \alpha_e^s, \alpha_0\}$,

(c) *if* $\alpha_{\bar{e}}^r \leq \alpha_{\bar{e}}^s$ *then* $\alpha_e^r \leq \alpha_e^{\bar{v}}$ *for all* $\bar{v} \in V_{\bar{e}}^r := \{v \in T : \bar{e} \in P_{rv}\}$ *and all* $e \in E_T$.

Proof. (a) For $e \neq rs$ we have $\{v : e \in P_{rv}\} = \{v : e \in P_{sv}\}$ and thus $\sum_{v:e\in P_{rv}} a_v = \sum_{v:e\in P_{sv}} a_v$.
(b) Using the notation from (6) we have:

$$\sum_{v:\bar{e}\in P_{rv}} a_v = a_s + \sum_{e\in\delta(\{s\})\setminus\{\bar{e}\}} \left(\sum_{v:e\in P_{sv}} a_v \right).$$

(c) Consider a $\bar{v} \in T$ with $\bar{e}$ on the path $\Pi = P_{r\bar{v}}$ with $V_\Pi = \{v_1, \dots, v_p\}$, $p \geq 2$, where $v_1 = r$, $v_2 = s$, $v_p = \bar{v}$ and v_k, v_{k+1}, $1 \leq k \leq p-1$ are adjacent in T. Applying (a) recursively to nodes v_i, v_{i+1}, $i = 1, \dots, p-1$ we obtain

$$\alpha_e^r = \alpha_e^{\bar{v}} \qquad \forall\, e \in E_T \setminus E_\Pi. \tag{9}$$

As $\bar{e}$ is outside the path from s to $\bar{v}$, the same argument for root s and the assumption yield

$$\alpha_{\bar{e}}^{\bar{v}} = \alpha_{\bar{e}}^s \geq \alpha_{\bar{e}}^r. \tag{10}$$

By Lemma 4, coefficients cannot increase along paths from the root, so

$$\begin{array}{ccccccccccc} & & \alpha_{\bar{v}v_{p-1}}^{\bar{v}} & \geq & \alpha_{v_{p-1}v_{p-2}}^{\bar{v}} & \geq \dots \geq & \alpha_{v_3v_2}^{\bar{v}} & \geq & \alpha_{v_2r}^{\bar{v}} & = & \alpha_{\bar{e}}^{\bar{v}}, \\ \alpha_{\bar{e}}^r & = & \alpha_{rv_2}^r & \geq & \alpha_{v_2v_3}^r & \geq \dots \geq & \alpha_{v_{p-2}v_{p-1}}^r & \geq & \alpha_{v_{p-1}\bar{v}}^r. & & \end{array} \tag{11}$$

Thus, putting (9)-(11) together the claim is proved. □

This allows to characterize exactly the set of roots giving the best coefficients.

Lemma 6. *Let (T, E_T) be a tree in G. The set of* minimal roots

$$\mathcal{R} := \{r \in T : \alpha_e^r \leq \alpha_e^v \text{ for all } v \in V_T \text{ and } e \in E_T\}$$

is nonempty and induces a connected subtree in T. A node $r \in T$ satisfies $r \in \mathcal{R}$ if and only if $\alpha_e^r \leq \alpha_e^s$ for all $e = rs \in E_T$.

Proof. To see that $\mathcal{R}$ is nonempty, orient the edges $e = uv \in T$ with $\alpha_e^u < \alpha_e^v$ towards u, do not orient edges with $\alpha_e^u = \alpha_e^v$. As T contains no cycle, there must be a node $r \in T$ so that all incident edges are either not oriented or point towards r, i.e., $\alpha_e^r \leq \alpha_e^s$ for all $e = rs \in E_T$. By Lemma 5(c) and using the notation defined there, this r is in $\mathcal{R}$, because $V = \bigcup_{rs\in E_T} V_{rs}^r \cup \{r\}$.

Next, we show connectedness for $r, s \in \mathcal{R}$. These satisfy $\alpha_e^r = \alpha_e^s$ for all $e \in E_T$. Assume there is some inner vertex $\bar{v} \in T$ on the path between r and s. Apply Lemma 5(a) on the edges outside the paths $P_{r\bar{v}}$ and $P_{s\bar{v}}$ to see that $\alpha_e^r = \alpha_e^{\bar{v}}$ for all $e \in E_T$ and so $\bar{v} \in \mathcal{R}$.

The characterization of the elements of $\mathcal{R}$ is obtained via Lemma 5(c) directly. □

For each choice of a roots out of R we obtain the same coefficient for each edge. Thus, the strongest truncated knapsack tree inequality is independent of the choice of $r \in R$.

Theorem 7. *Let (T, E_T) be a tree in G and $\mathcal{R}$ as defined in Lemma 6. The strongest truncated knapsack tree inequality, with respect to the knapsack inequality $\sum_{v \in V} a_v x_v \leq a_0$, $a_v \geq 0$, $v \in V$, defined on (T, E_T) is obtained for a root $r \in \mathcal{R}$, i.e., if $r \in \mathcal{R}$, then*

$$\sum_{e \in E_T} \alpha_e^s y_e \geq \sum_{e \in E_T} \alpha_e^r y_e \geq \alpha_0 \tag{12}$$

holds for all $s \in T$ and all $y \in P_{\mathrm{B}}$. In particular,

$$\sum_{e \in E_T} \alpha_e^r y_e = \sum_{e \in E_T} \alpha_e^s y_e \tag{13}$$

holds for all $r, s \in \mathcal{R}$ and all $y \in P_{\mathrm{B}}$.

Proof. Directly by Lemma 6. □

In the sequel the elements of the set $\mathcal{R}$ will be called *minimal roots* of a given tree (T, E_T). In order to obtain the strongest truncated knapsack tree inequality it is sufficient, by Theorem 7, to identify any minimal root. Given a tree (T, E_T) rooted at some node r one can find a minimal root along the lines of the proof of Lemma 6 by proceeding iteratively as follows. Select a node $s \in T$ adjacent to r such that $\alpha_{rs}^r = \max\{\alpha_{rv}^r : rv \in E_T\}$. If $\alpha_{rs}^r > \alpha_{rs}^s$, then also $\sum_{e \in E_T} \alpha_e^r > \sum_{e \in E_T} \alpha_e^s$ by Lemma 5 (c). Hence r can be discarded and s is marked as next root of (T, E_T). Otherwise $\sum_{e \in E_T} \alpha_e^r y_e \geq \alpha_0$ is the strongest truncated knapsack tree inequality with respect to all possible choices of roots in (T, E_T).

In the remainder of this section we show that the assumption on r to be a minimal root is not only a necessary condition for a truncated knapsack tree inequality to be facet-defining for the polytope P_{B}, which follows from Theorem 7, but in some cases also sufficient.

For this purpose we assume that $G = (T, E_T)$ is a tree and $\varphi_v = 1$ for all $v \in T$. Then the knapsack polytope P_{K} is defined by the inequality $\sum_{v \in T} x_v \leq F$ and the corresponding knapsack tree inequality (3) defined on (T, E_T) takes the form

$$\sum_{v \in T} (1 - \sum_{e \in P_{rv}} y_e) \leq F.$$

Applying the strengthening (5) and notation (6) we obtain $\alpha_0 = |T| - F$ and $\alpha_e^r = \min\{|V_e^r|, |T| - F\}$ for all $e \in E$, where V_e^r is the set of nodes, whose path to $r \in T$ contains the edge e, see e.g. Figure 3. To emphasize the special case that we treat in the sequel we set $\kappa_e^r := \alpha_e^r$, $\bar{F} := \alpha_0$ and consider the inequality $\sum_{e \in E_T} \kappa_e^r y_e \geq \bar{F}$ or $(\kappa^r)^T y \geq \bar{F}$ for short. For ease of exposition we call κ_e^r the *knapsack weight* of $e \in E$ with respect to the root r of (T, E_T). If $\kappa_e^r = \bar{F}$ and $\bar{F} < |V_e^r|$ we say that e has the *reduced knapsack weight.* Furthermore, we introduce the term *branch-less path*, which is a path in (T, E_T), whose inner nodes are all of degree 2. We consider a path consisting of an edge and both its end-nodes as a trivial case of a branch-less path. We call an edge a *leaf* if one of its endpoints is of degree one.

Theorem 8. *Assume that $G = (T, E_T)$ is a tree rooted at a node $r \in T$, $\varphi_v = 1$ for all $v \in T$ and $\frac{|T|}{2} + 1 \leq F < |T|$. The truncated knapsack tree inequality $(\kappa^r)^T y \geq \bar{F}$ is facet-defining for P_{B} if and only if one of the following conditions is satisfied:*

(a) *r is a minimal root and (T, E_T) satisfies the following* branch-less path condition*: each branch-less path with F nodes has one end-edge that is a leaf in (T, E_T),*

(b) *$F = |T| - 1$.*

Remark 9. Note that:

(1) Given a graph $G = (V, E)$, P_{B} is full-dimensional under assumptions that $\varphi_v = 1$ for all $v \in V$ and $F \geq \frac{|V|}{2} + 1$, see [9]. If $F = \frac{|V|}{2}$ the bisection cut polytope is not full dimensional, hence this case needs a special treatment which can be found in [10].

(2) In case $F = |V|$ the knapsack inequality $\sum_{v \in V} x_v \leq F$ is redundant for P_{K} and thus the corresponding truncated knapsack tree inequalities are redundant for P_{B}. Therefore we assume that $F < |V|$, in particular, $\bar{F} > 0$.

Due to the complexity of the proof of Theorem 8 we complete it in several steps. First we outline the general idea of the sufficiency part. Let $\mathcal{F}$ be a face of P_{B} induced by $(\kappa^r)^T y \geq \bar{F}$ and $\mathcal{F}_b$ be the facet of P_{B} defined by the inequality $b^T y \geq b_0$ such that $\mathcal{F} \subseteq \mathcal{F}_b$. To show that $(\kappa^r)^T y \geq \bar{F}$ is a facet-defining inequality for P_{B}, we prove that $\mathcal{F} = \mathcal{F}_b$, i.e., there exists $\gamma \in \mathbb{R} \setminus \{0\}$ such that

$$\begin{aligned} b_e &= \gamma \kappa^r_e, \quad \forall\, e \in E \\ b_0 &= \gamma \bar{F}. \end{aligned} \tag{14}$$

hold.

We introduce now further definitions and lemmas required to prove the above relations. Given a partition of the node set T we denote by V_r the cluster containing r, see e.g. Figure 3. We say that two edges $e, f \in E$ are *related*, if there exists a path to the root containing both e and f. An edge e is related to itself. For an edge e we set $B_e = \{f \in E : f \text{ is related to } e\}$ and call the graph $(\bigcup_{f \in B_e} f, B_e)$ a *branch* (induced by e); it is a subtree of (T, E_T) and is said to be incident on r if e and r are incident. If any two edges e and f are incident and related and such that e is closer to the root than f (with respect to the number of edges), then f is a *child* of e. We denote the set of children of e by

$$S_e = \{f \in E_T : f \text{ is a child of } e\}.$$

With this, the recursive construction rule of Lemma 5 (b) specialized to κ^r_e reads

$$\kappa^r_e = \min\{1 + \sum_{\bar{e} \in S_e} \kappa^r_{\bar{e}}, \bar{F}\}. \tag{15}$$

We say that a bisection cut $\delta(V_r)$ is *tight* for $(\kappa^r)^T y \geq \bar{F}$ if it satisfies $(\kappa^r)^T \chi^{\delta(V_r)} = \bar{F}$. As we will show soon, $|V_r| = F$ holds if $\delta(V_r)$ is tight for $(\kappa^r)^T y \geq \bar{F}$ and all $e \in \delta(V_r)$ satisfy $\kappa^r_e = |V^r_e| \leq \bar{F}$, i.e., all edges in the cut do not have reduced knapsack weights. In this case we will call the cut $\delta(V_r)$ *double-tight* for $(\kappa^r)^T y \geq \bar{F}$.

Next, we derive some properties of bisection cuts tight for $(\kappa^r)^T y \geq \bar{F}$.

Lemma 10. *No two edges in a bisection cut tight for $(\kappa^r)^T y \geq \bar{F}$ are related.*

Proof. Assume, for contradiction, that $\delta(V_r)$ is a bisection cut tight for $(\kappa^r)^T y \geq \bar{F}$ containing two related edges $e = v_{j-1}v_j$ and $f = v_{k-1}v_k$ on some path $P^r_{v_k} = \{v_1v_2, v_2v_3, \ldots, v_{j-1}v_j, \ldots, v_{k-1}v_k\}$ with $r = v_1$ and $1 < j < k$. W.l.o.g., we may assume that $v_{j-1} \in V_r$ and no further edge $v_{i-1}v_i$ is in the cut for $j < i < k$. Then $\{v_j, \ldots, v_{k-1}\} \subset T \setminus V_r$ and $v_k \in V_r$. The cut induced by $\bar{V}_r = V_r \cup \{v_j\} \setminus \{v_k\}$ is again a bisection cut with $\delta(\bar{V}_r) \subseteq (\delta(V_r) \setminus \{e, f\}) \cup S_e \cup S_f$; we know $e \notin \delta(\bar{V}_r)$ (this will not hold for f if $f \in S_e$). Note that e and f must be unreduced because $\delta(V_r)$ is tight, $(\kappa^r)^T \chi^{\delta(V_r)} = \bar{F}$. Thus, by (15),

$$(\kappa^r)^T \chi^{\delta(V_r)} - (\kappa^r)^T \chi^{\delta(\bar{V}_r)} \geq \kappa^r_e + \kappa^r_f - \sum_{\bar{e} \in S_e \cup S_f} \kappa^r_{\bar{e}} \geq 1,$$

and that yields $(\kappa^r)^T \chi^{\delta(\bar{V}_r)} + 1 \leq (\kappa^r)^T \chi^{\delta(V_r)} = \bar{F}$. This contradicts $(\kappa^r)^T \chi^{\delta(\bar{V}_r)} \geq \bar{F}$, which holds because $\delta(\bar{V}_r)$ is a bisection cut and $(\kappa^r)^T y \geq \bar{F}$ is feasible for P_B. □

Lemma 11. *Let (T, E_T) be rooted at $r \in T$. A bisection cut $\delta(V_r)$ is double-tight for $(\kappa^r)^T y \geq \bar{F}$ if and only if $|V_r| = F$ and $(V_r, E(V_r))$ is connected.*

Proof. Assume first that $\delta(V_r)$ is double-tight for $(\kappa^r)^T y \geq \bar{F}$. By Lemma 10 any two edges in $\delta(V_r)$ are not related. This implies that V_r induces a connected subgraph of (T, E_T). Hence $T \setminus V_r = \bigcup_{e \in \delta(V_r)} V^r_e$ and $V^r_e \cap V^r_f = \emptyset$ for any $e, f \in \delta(V_r)$. Furthermore, $\kappa^r_e = |V^r_e|$ holds for each $e \in \delta(V_r)$ and we obtain $|T \setminus V_r| = \sum_{e \in \delta(V_r)} \kappa^r_e = \bar{F}$, i.e., $|V_r| = F$.
Now consider a bisection $(V_r, T \setminus V_r)$ such that $(V_r, E(V_r))$ is connected and $|V_r| = F$ (i.e., $|T \setminus V_r| = \bar{F}$). We show first that $\delta(V_r)$ contains only edges, whose knapsack weights are not reduced. Assume for contradiction that $\delta(V_r)$ contains an edge f with a reduced knapsack weight. Since $\kappa^r_f = \bar{F}$, this is the only edge in $\delta(V_r)$, otherwise $\delta(V_r)$ is not tight for $(\kappa^r)^T y \geq \bar{F}$. Hence $\delta(V_r) = \{f\}$ and since f has a reduced knapsack weight, $|T \setminus V_r| = |V^r_f| > \bar{F}$ holds contradicting the assumption that $|V_r| = F$. To show that $\delta(V_r)$ is tight for $(\kappa^r)^T y \geq \bar{F}$, we use the assumption that $(V_r, E(V_r))$ is connected. We have

$$\sum_{e \in \delta(V_r)} \kappa^r_e = \sum_{e \in \delta(V_r)} |V^r_e| = |\bigcup_{e \in \delta(V_r)} V^r_e| = \bar{F}.$$

Hence $\delta(V_r)$ is double-tight for $(\kappa^r)^T y \geq \bar{F}$. □

Next, we provide some results following from the assumption that (T, E_T) is rooted at a minimal root. As we will show in the following lemmas, this assures the existence of bisection cuts tight for $(\kappa^r)^T y \geq \bar{F}$, which we will consider in the proof of further lemmas preceding the proof of Theorem 8.

Lemma 12. *Let $B = (V_B, E_B) \subseteq (T, E_T)$ be a branch incident on root $r \in T$. If r is a minimal root of (T, E_T), then $|V_B \setminus \{r\}| \leq F$.*

Proof. Let $B = (V_B, E_B)$ be a branch incident on r and assume that $|V_B \setminus \{r\}| > F$. We are going to show that in this case r cannot be a minimal root. Let s be the node in V_B adjacent to r, and let $f = rs \in E_B$, see Figures 1 and 2. Note that $V^r_f \dot\cup V^s_f = T$. Since $|V^r_f| = |V_B \setminus \{r\}| > F > \bar{F}$, we have $\kappa^r_f = \bar{F}$, on the one hand. On the other hand, $\kappa^s_f = \min\{|T \setminus V^r_f|, \bar{F}\} < \bar{F}$. Hence $\kappa^s_f < \kappa^r_f$ and by Lemma 6, r is not a minimal root. □

Lemma 13. *Assume (T, E_T), rooted at a minimal root r, has an edge $e \in E_T$ such that $\kappa^r_e = \bar{F}$. The cut $\delta(V^r_e) = \{e\}$ is a bisection cut tight for $(\kappa^r)^T y \geq \bar{F}$.*

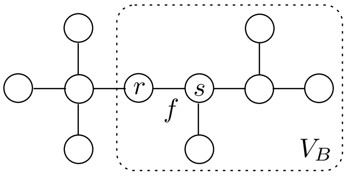

Figure 1: Node set V_B.

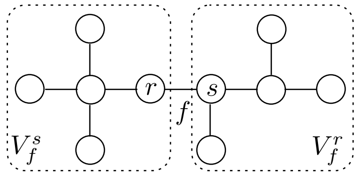

Figure 2: Node sets V_f^r and V_f^s.

Proof. Note that in the considered case we have $V_r = T \setminus V_e^r$. We show that the cut $\delta(V_r)$, which is obviously tight for $(\kappa^r)^T y \geq \bar{F}$, is also a bisection cut. Assume that $\delta(V_r)$ is not a bisection cut. Then either $|V_r| < \bar{F}$ or $|T \setminus V_r| < \bar{F}$. In the first case, let s be a node incident to e such that the path $\Pi_{rs} = (V_{rs}, E_{rs})$ joining r and s contains e, see Figure 3. For all $f \in E_T \setminus E_{rs}$ holds $\kappa_f^r = \kappa_f^s$ due to Lemma 5 (a). For $f \in E_{rs}$ we obtain by Lemma 4 that $\kappa_f^r \geq \kappa_e^r = \bar{F} > |V_r| \geq |V_f^s| = \kappa_f^s$. By Lemma 6 this contradicts the assumption that r is a minimal root. One can show in a similar way that $|T \setminus V_r| < \bar{F}$ not possible, either. □

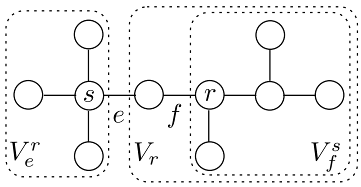

Figure 3: Node sets V_r, V_e^r and V_f^r.

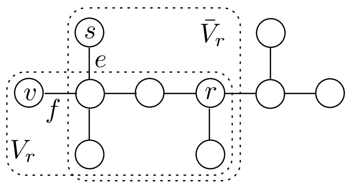

Figure 4: Node sets V_r and $\bar{V}_r$, Lemma 15 (b).

The following result exhibits the central structural property required for cross connecting double-tight cuts of which one contains edge e and the next contains its children S_e.

Lemma 14. *Given (T, E_T) with minimal root r. For each unreduced $e \in E_T$ that is not a leaf there exists a double-tight cut $\delta(V_r)$ that contains e, S_f and S_g for two further unreduced edges f and g with $\{e, f, g\}$ unrelated if and only if (T, E_T) satisfies the branch-less path condition.*

Proof. First suppose that in (T, E_T) the branch-less path condition does not hold, so there is a branch-less path on nodes $\bar{V} = \{v_1, \ldots, v_F\}$ with $\{v_i, v_{i+1}\} \in E_T$ for $1 \leq i < F$ and neither $\{v_1, v_2\}$ nor $\{v_{F-1}, v_F\}$ are leaves. By the minimality of r and Lemma 12 we must have $r \in \bar{V}$. One of the edges $\{v_1, v_2\}$ and $\{v_{F-1}, v_F\}$ must be unreduced (otherwise one of them would cut the graph into two parts, one containing more than F and one at least $\bar{F}+1$ nodes). W.l.o.g., let $e = \{v_1, v_2\}$ be unreduced and satisfy $\kappa_e^r \leq \kappa_{v_{F-1}v_F}^r$. For this e there exist no further two unrelated edges f and g satisfying the requirements, because at least one of them must be located on the path (by Lemma 11 we need $|V_r| = F$) and must thus be related either to e or to the other edge.
Now suppose that (T, E_T) satisfies the branch-less path conditions. Let e be an unreduced edge that is not a leaf. Edge e is contained in some branch incident on r, call this branch $B = (V_B, E_B)$. By lemmas 11 and 12 we may construct a bisection $(V_r, T \setminus V_r)$ with $e \in \delta(V_r)$ and $\delta(V_r)$ double-tight by extending $V_B \setminus V_e^r$ to V_r via successively adding adjacent points from $V \setminus V_B$ so that $|V_r| = F$ and the subgraph $\bar{T} = (V_r, E_r)$ induced by V_r is a tree. Note that by Lemma 12 root r can only be a leaf of $\bar{T}$ if it is incident to e. First suppose $\bar{T}$ has two leaves so that the corresponding nodes of degree 1 are not incident to e, then take these for

f and g. Otherwise $\bar{T}$ forms a path $\{v_1, \dots, v_F\}$ with v_i adjacent to v_{i+1} in $\bar{T}$ for $1 \le i < F$ with e incident to one of its endpoints, say $v_1 \in e$. By the branch-less path condition at least one of the nodes v_i, $1 \le i < F-1$, has degree at least three in T (otherwise one can find a branch-less path with F nodes not having one end edge a leaf), pick one and call it $\bar{v}$. Incident to this $\bar{v}$ there is an edge $f \in \delta(V_r)$ with $f \neq e$. Now observe that, by Lemma 11, $\delta(V_r \cup f \setminus \{v_F\})$ is a double-tight cut that contains e, S_f and S_g for $g = \{v_{F-2}, v_{F-1}\}$ and e, f, and g are unrelated. □

Lemma 15. *Given (T, E_T) satisfying the branch-less path condition and a minimal root $r \in T$. Let $\mathcal{F}, \mathcal{F}_b$ be faces of P_{B} defined by $(\kappa^r)^T y \ge \bar{F}$ and $b^T y \ge b_0$, respectively, such that $\mathcal{F} \subseteq \mathcal{F}_b$. For each branch $B = (V_B, E_B)$ in (T, E_T) incident on r there is a $\gamma_B \ge 0$ such that*

(*a*) *$b_e = b_0$ holds for any edge $e \in E_T$ with $\kappa^r_e = \bar{F}$,*

(*b*) *$b_e = b_f =: \gamma_B$ holds for any two leaves $e, f \in E_B$ of T,*

(*c*) *$b_e = \gamma_B \kappa^r_e$ holds for unreduced $e \in E_B$,*

(*d*) *$\gamma_B = b_0/\bar{F}$ and $b_e = \gamma_B \kappa^r_e$ holds for $e \in E_B$.*

Proof. (*a*) Let e be an edge in E_T with $\kappa^r_e = \bar{F}$. By Lemma 13 the cut $\delta(V^r_e) = \{e\}$ is a bisection cut tight for $(\kappa^r)^T y \ge \bar{F}$. Since $\chi^{\{e\}}$ is in $\mathcal{F}_b$, we obtain $b_e = b_0$.

(*b*) Let e and f be leaves in E_B and denote by s, v their respective end-nodes of degree 1. $|V_B \setminus \{s\}| \le F$ follows from Lemma 12. Hence there exists a bisection $(V_r, T \setminus V_r)$ with V_r connected such that $V_B \setminus \{s\} \subseteq V_r$, $s \notin V_r$ and $|V_r| = F$. Then, by Lemma 11, $\bar{V}_r = V_r \cup \{s\} \setminus \{v\}$ also yields a double-tight bisection cut, so $b_0 = \sum_{\bar{e} \in \delta(V_r)} b_{\bar{e}} = \sum_{\bar{e} \in \delta(\bar{V}_r)} b_{\bar{e}}$ and therefore $b_e = b_f =: \gamma_B$.

(*c*) We use induction by distance of the edges in E_B to the deepest leaves in their induced branches. By (b) the claim holds for the leaves. Let e be an unreduced edge in E_B that is not a leaf. By Lemma 14 there exists a double-tight cut $\delta(V_r)$ and unreduced edges f, g unrelated to e and to each other so that $e \in \delta(V_r)$, $S_f \subset \delta(V_r)$ and $S_g \subset \delta(V_r)$. Pick some $h \in S_e$. Lemma 4 implies that h is unreduced and hence (15) combined with the induction hypothesis yields

$$b_h - \sum_{\bar{e} \in S_h} b_{\bar{e}} = \gamma_B(\kappa^r_h - \sum_{\bar{e} \in S_h} \kappa^r_{\bar{e}}) = \gamma_B. \tag{16}$$

Denote by v_e (v_f, v_g, v_h) that node of e (f, g, h) whose distance to r is greater. Then by Lemma 11

$$\bar{V}_f = V_r \cup \{v_e\} \setminus \{v_f\},\ \bar{V}_g = V_r \cup \{v_e\} \setminus \{v_g\} \text{ and } \bar{V}_h = V_r \cup \{v_e, v_h\} \setminus \{v_f, v_g\},$$

see Figure 5, also yield double-tight cuts with edge sets (for appropriately chosen $D \subset E_T$)

$$\begin{aligned}
\delta(V_r) &= \{e\} \,\dot\cup\, S_f \,\dot\cup\, S_g \,\dot\cup\, D,\\
\delta(\bar{V}_f) &= S_e \,\dot\cup\, \{f\} \,\dot\cup\, S_g \,\dot\cup\, D,\\
\delta(\bar{V}_g) &= S_e \,\dot\cup\, S_f \,\dot\cup \{g\} \,\dot\cup\, D,\\
\delta(\bar{V}_h) &= (S_e \setminus \{h\}) \,\dot\cup\, S_h \,\dot\cup\, \{f\} \,\dot\cup\, \{g\} \,\dot\cup\, D.
\end{aligned}$$

Exploiting

$$b_0 = \sum_{\bar{e} \in \delta(V_r)} b_{\bar{e}} = \sum_{\bar{e} \in \delta(\bar{V}_f)} b_{\bar{e}} = \sum_{\bar{e} \in \delta(\bar{V}_g)} b_{\bar{e}} = \sum_{\bar{e} \in \delta(\bar{V}_h)} b_{\bar{e}}$$

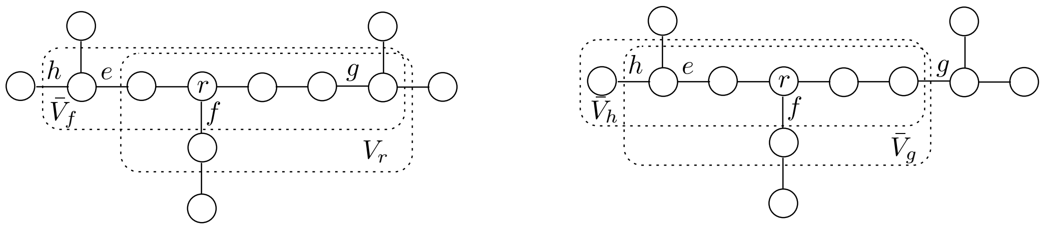

Figure 5: Node sets V_r, $\bar{V}_f$, $\bar{V}_g$ and $\bar{V}_h$.

one obtains after a few rearrangements together with (16)

$$b_e - \sum_{\bar{e} \in S_e} b_{\bar{e}} = b_f - \sum_{\bar{e} \in S_f} b_{\bar{e}} = b_g - \sum_{\bar{e} \in S_g} b_{\bar{e}} = b_h - \sum_{\bar{e} \in S_h} b_{\bar{e}} = \gamma_B.$$

By induction, $b_{\bar{e}} = \gamma_B \kappa^r_{\bar{e}}$ for all $\bar{e} \in S_e$, so $b_e = \gamma_B(1 + \sum_{\bar{e} \in S_e} \kappa^r_{\bar{e}}) = \gamma_B \kappa^r_e$.

(d) We first show $\gamma_{\bar{B}} = \gamma_B$ for all branches $\bar{B}$ incident on r with respect to a special branch B incident on r. Observe that there is always a branch $B = (V_B, E_B)$ incident on r with $|V_B \setminus \{r\}| < F$ (because there exist at least two branches by minimality of r and because $|T| \leq 2F$), so let B be this branch. Let $\bar{B} = (\bar{V}, \bar{E})$ be some other branch incident on r. Denote by $\bar{S} \subset \bar{E}$ the set of unreduced edges in $\bar{E}$ that have no unreduced predecessors, i.e.,

$$\bar{S} = \{\bar{e} \in \bar{E} : |V^r_{\bar{e}}| = \kappa^r_{\bar{e}} \text{ and if } \bar{e} \in S_{e'} \text{ for some } e' \in \bar{E} \text{ then } |V^r_{e'}| > |\bar{F}|\}.$$

Next, the set $\hat{V} = V_B \cup \bar{V} \setminus \bigcup_{\bar{e} \in \bar{S}} V^r_{\bar{e}}$ induces a tree $\hat{T} = (\hat{V}, \hat{E})$ with $|\hat{V}| \leq F$ ($|V_B| \leq F$ by construction and removing any leaf $v \in \hat{V} \setminus V_B$ corresponds to cutting a reduced edge f in $\bar{B}$ so that $|\hat{V} \setminus \{v\}| \leq |T| - |V^r_f| < |T| - \bar{F} = F$). Thus, we may extend $\hat{V}$ by nodes from $V \setminus (V_B \cup \bar{V})$ to a set V_r with $|V_r| = F$ so that $\delta(V_r)$ is a double-tight cut with $\bar{S} \subseteq \delta(V_r)$ and $E_B \cap \delta(V_r) = \emptyset$. Take a leaf $e \in E_B$, an edge $f \in \bar{S}$ and denote by v_e and v_f those of the two nodes of e and f whose distance to r is greater. Then $V_e = V_r \cup \{v_f\} \setminus \{v_e\}$ yields another double-tight cut $\delta(V_e) = \delta(V_r) \cup \{e\} \cup S_f \setminus \{f\}$, therefore $\gamma_B = b_e = b_f - \sum_{\bar{e} \in S_f} b_{\bar{e}} = \gamma_{\bar{B}}$.

As $\gamma_{\bar{B}} = \gamma_B$ for all branches $\bar{B}$ and B, any double-tight cut $\delta(V_r)$ yields with (c)

$$b_0 = \sum_{e \in \delta(V_r)} b_e = \gamma_B \sum_{e \in \delta(V_r)} \kappa^r_e = \gamma_B \bar{F}.$$

Thus, $\gamma_B = \frac{b_0}{\bar{F}}$. □

Proof of Theorem 8. (a) Because of (b) we require $\bar{F} \geq 2$ within the proof of (a). For sufficiency assume that the branch-less path condition is satisfied. Given a face $\mathcal{F}_b$ of P_{B} defined by $b^T y \geq b_0$ that contains all roots of $(\kappa^r)^T y \geq \bar{F}$, Lemma 15 shows that $b^T y \geq b_0$ is a nonnegative multiple of $(\kappa^r)^T y \geq \bar{F}$, so the latter is facet inducing.

Necessity of the minimal root condition is a consequence of Theorem 7, so consider the case that r is a minimal root but the branch-less path condition does not hold. In this case (T, E_T) contains a path on nodes $\bar{V} = \{v_1, \ldots, v_F\}$ with $\{v_i, v_{i+1}\} \in E_T$ for $1 \leq i < F$ and neither $\{v_1, v_2\}$ nor $\{v_{F-1}, v_F\}$ are leaves. By the minimality of r and Lemma 12 there must be some $k \in \{1, \ldots, F\}$ with $r = v_k$. Split (T, E_T) at r into two edge disjoint subtrees (V_1, E_1) and

(V_2, E_2) with $V_1 \cap V_2 = \{r\}$, $E_T = E_1 \cup E_2$ and set

$$\begin{aligned} S_1 &:= \delta(\{v_1\}) \setminus \{v_1 v_2\}, \\ S_2 &:= \delta(\{v_F\}) \setminus \{v_{F-1} v_F\}, \\ P_1 &:= \{v_i v_{i+1} : 1 \le i < k\}, \\ P_2 &:= \{v_i v_{i+1} : k \le i < F\}, \\ D_i &:= E_i \setminus P_i \qquad \text{for } i \in \{1,2\}, \\ n_i &:= \sum_{e \in S_i} |V_e^r| \qquad \text{for } i \in \{1,2\}. \end{aligned}$$

Note that $1 \le n_i < \bar{F}$ for $i \in \{1,2\}$, because otherwise the cut S_i would induce a partition requiring at least $n_i + F + 1 > \bar{F} + F = |T|$ nodes. Furthermore, $n_1 + n_2 = \bar{F}$ because $S_1 \cup S_2$ induces a double-tight cut by Lemma 11. We show that all roots of $(\kappa^r)^T y \ge \bar{F}$ also satisfy the equation

$$n_2 \sum_{e \in D_1} \kappa_e^r y_e + n_1 \sum_{e \in P_1} (\bar{F} - \kappa_e^r) y_e - n_2 \sum_{e \in P_2} (\bar{F} - \kappa_e^r) y_e - n_1 \sum_{e \in D_2} \kappa_e^r y_e = 0. \tag{17}$$

Indeed, let V_r be a node set inducing a bisection cut tight for $(\kappa^r)^T y \ge \bar{F}$ and consider the number of times the path can be cut in view of Lemma 10:

$|\delta(V_r) \cap (P_1 \cup P_2)| = 0$: This implies $\delta(V_r) = S_1 \cup S_2$ by Lemma 11 and (17) holds.

$|\delta(V_r) \cap (P_1 \cup P_2)| = 1$: By symmetry it suffices to consider the case $\delta(V_r) \cap P_1 = \{f\}$. By Lemma 10 we have $\delta(V_r) \cap D_1 = \emptyset$ and as the cut is tight, $\sum_{e \in D_2} \kappa_e^r y_e = \bar{F} - \kappa_f^r$, so (17) holds.

$|\delta(V_r) \cap (P_1 \cup P_2)| \ge 2$: By Lemma 10 it cannot be greater than two, but even two is impossible for a tight cut. Indeed, Lemma 10 ensures $\delta(V_r) \cap (P_1) = \{e_1\}$, $\delta(V_r) \cap (P_2) = \{e_2\}$ and $\delta(V_r) \cap (D_1 \cup D_2) = \emptyset$. Since $n_i < \bar{F}$ we obtain $\kappa_{e_i}^r \ge n_i + 1$ for $i \in \{1,2\}$, so $(\kappa^r)^T \chi^{\delta(V_r)} = \kappa_{e_1}^r + \kappa_{e_2}^r \ge n_1 + n_2 + 2 > \bar{F}$ yields the desired contradiction.

As (17) is not a scalar multiple of $(\kappa^r)^T y \ge \bar{F}$, the latter cannot be a facet if $\bar{F} \ge 2$ and the branch-less path condition is not fulfilled.
(b) If $\bar{F} = 1$ then $\kappa_e^r = 1$ for all $e \in E_T$ and each e alone forms a bisection cut that is tight for $(\kappa^r)^T y \ge \bar{F}$. As these are $|E_T|$ affinely independent roots, the inequality is facet inducing. □

Remark 16. The statements of Theorem 8 cannot be easily carried forward for graphs denser than trees. So far, we observed that if there are cycles in the graph, then not all trees in this graph yield a facet defining truncated knapsack tree inequality, even if conditions (a) in Theorem 8 are satisfied. Some additional assumptions must be figured out. For instance, it can be shown that if $F = |T| - 1$ (condition (b) in Theorem 8) and the graph G contains at least one cycle, then $(\kappa^r)^T y \ge \bar{F}$ does not define a facet of P_{B}.

5 The bisection knapsack walk inequalities for MB

In this section we exploit the special structure of MB in order to derive an improved version of the knapsack tree inequality. Note that in the MNCGP case with $K > 2$ a walk

$\{e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}\}$ with $y_{e_1} = y_{e_2} = 1$ does not imply any relation between nodes v_1 and v_3 while in the MB case where $K = 2$ it follows from $y_{e_1} = y_{e_2} = 1$ that v_1 and v_3 belong to the same cluster.

More generally, whenever there is a walk (for ease of exposition we assume throughout that a walk traverses an edge at most once; for the general case, see [1]) between two nodes of the graph with an even number of edges in the cut we know in the case of MB that the two end nodes of the walk have to be in the same cluster. We may therefore replace the indicator term $1 - \sum_{e \in P_{rv}} y_e$ of (3) by

$$1 - \sum_{e \in P_{rv} \setminus H_v} y_e - \sum_{e \in H_v} (1 - y_e) \tag{18}$$

where $H_v \subseteq P_{rv}$ with even cardinality. So if $y \in \{0,1\}^{|E|}$ is a valid solution of MB and P_{rv} is a walk from r to v in G with $H_v = \{e \in P_{rv} : y_e = 1\}$ and $|H_v|$ even, then expression (18) is equal to one, indicating that r and v belong to the same cluster. If, however, $H_v \neq \{e \in P_{rv} : y_e = 1\}$ the value of (18) is less than or equal to zero.

Lemma 17. *Given a root node $r \in V$, walks $P_{rv} \subseteq E$ and even subsets $H_v \subseteq P_{rv}$ for all $v \in V$. Let $y = \chi^{\delta(S)}$ for some $S \subseteq V$ with $r \in S$ and put $z = \chi^S$. Then for all $v \in V$*

$$1 - \sum_{e \in P_{rv} \setminus H_v} y_e - \sum_{e \in H_v} (1 - y_e) \leq z_v \ . \tag{19}$$

Proof. For $v \in S$ we have $z_v = 1$ and inequality (19) is satisfied, because $y_e \geq 0$ and $1 - y_e \geq 0$ for all $e \in E$. If $v \notin S_1$, the set $C = \{e \in P_{rv} : y_e = 1\}$ must be of odd cardinality (otherwise r and v would be together in S_1). Since H_v is of even cardinality and both C and H_v are subsets of P_{rv}, there exists an $e \in P_{rv}$ with $e \in C \setminus H_v$ or $e \in H_v \setminus C$. If $e \in C \setminus H_v$, then $y_e = 1$ and the left-hand side of (19) is smaller or equal to $1 - y_e = 0 = z_v$. If $e \in H_v \setminus C$, then $y_e = 0$ and the left-hand side of (19) is smaller or equal to $1 - (1 - y_e) = 0 = z_v$. □

Now we are ready to sum up all the evaluation terms.

Definition 18 (Bisection knapsack walk inequality)**.** *Let $\sum_{v \in V} a_v x_v \leq a_0$ be a valid inequality for the knapsack polytope P_{K} with $a_v \geq 0$ for all $v \in V$. For a subset $V' \subseteq V$, a fixed root node $r \in V'$, walks $P_{rv} \subseteq E$, and sets $H_v \subseteq P_{rv}$ with $|H_v|$ even, the* bisection knapsack walk inequality *reads*

$$\sum_{v \in V'} a_v \left(1 - \sum_{e \in P_{rv} \setminus H_v} y_e - \sum_{e \in H_v} (1 - y_e) \right) \leq a_0 \ . \tag{20}$$

Lemma 17 directly implies

Proposition 19. *The bisection knapsack walk inequality (20) is valid for the polytope P_{B}.*

Note that knapsack tree inequalities are a special case of the bisection knapsack walk inequalities where the walks P_{rv} form a tree, all nodes on these walks are contained in V' and all $H_v = \emptyset$. Again, we may rewrite the bisection knapsack walk inequality so as to pronounce its strength in forcing cut variables to increase:

$$\sum_{e \in E} \left(\sum_{v \in V' : e \in P_{rv}} a_v - \sum_{v \in V' : e \in H_v} 2a_v \right) y_e \geq \sum_{v \in V'} a_v - a_0 - \sum_{v \in V'} a_v |H_v| \ .$$

Remark 20. For $y = \chi^{\delta(S)}$ with $r \in S \subseteq V$, $z = \chi^S$, and $U_v \subseteq P_{rv}$ with $|U_v|$ odd $(v \in V \setminus \{r\})$ one can show $\sum_{e \in P_{rv} \setminus U_v} y_e + \sum_{e \in U_v} (1 - y_e) \geq z_v$ for $v \in V \setminus \{r\}$. In case of (MB), a valid knapsack inequality $\sum_{v \in V} a_v x_v \leq a_0$ implies validity of $\sum_{v \in V'} a_v z_v \geq a(V') - a_0$ for all $V' \subseteq V$. Thus the so-called *odd bisection knapsack walk inequality*

$$a_r + \sum_{v \in V' \setminus \{r\}} a_v \left(\sum_{e \in P_{rv} \setminus U_v} y_e + \sum_{e \in U_v} (1 - y_e) \right) \geq a(V') - a_0$$

is valid for P_{B}, too. Due to their close relation to the (even) bisection knapsack walk inequalities (20) we will not treat these inequalities further in this paper but refer the interested reader to [1].

Remark 21. The bisection knapsack walk inequalities are closely linked to the cycle inequalities [3] which are defined for cycles $C = (V_C, E_C)$ in G and subsets $U \subseteq E_C$ with $|U|$ odd by

$$\sum_{e \in E_C \setminus U} y_e - \sum_{e \in U} y_e \geq 1 - |U| \ .$$

Indeed, consider the case that in a bisection knapsack walk inequality a path P_{rv} with even subset H_v can be shortcut by the edge rv, put $E_C = P_{rv} \cup \{rv\}$ and $U = H_{rv} \cup \{rv\}$ and rewrite the corresponding cycle inequality in the form

$$1 - \sum_{e \in P_{rv} \setminus H_v} y_e - \sum_{e \in H_v} (1 - y_e) \leq 1 - y_{rv} \ . \tag{21}$$

If $1 - y_{rv}$ is interpreted as z_v (indicating whether v is in the same set as the root r) this is just (19). Thus, whenever all cycle inequalities are enforced and a direct edge rv exists between root r and v then $P_{rv} = \{rv\}$ is always the best possible choice. It is, however, not true in general that in the presence of all cycle inequalities a shorter path (with respect to the number of edges) dominates longer paths, see Example 22.

Example 22. Let G be the cycle on five nodes of Figure 6. The solution $y = (y_{12}, y_{23}, y_{34}, y_{45}, y_{15})^T = (0.5, 0.5, 0, 0, 0)^T$ fulfills all cycle inequalities because it is a convex combination of the two cuts $(0,0,0,0,0)^T$ and $(1,1,0,0,0)^T$. Now look at the bisection knapsack walk inequalities with $V' = \{1,3\}$ and $r = 1$. The shorter path P^s_{13} from root node 1 to node 3 uses the edge set $\{\{1,2\},\{2,3\}\}$ with $H^s_3 = \emptyset$ or $H^s_3 = \{\{1,2\},\{2,3\}\}$, the longer path P^l_{13} uses the edge set $\{\{3,4\},\{4,5\},\{1,5\}\}$ with $H^l_3 = \emptyset$, $H^l_3 = \{\{3,4\},\{4,5\}\}$, $H^l_3 = \{\{3,4\},\{1,5\}\}$ or $H^l_3 = \{\{4,5\},\{1,5\}\}$. For the shorter path of the two possible bisection knapsack walk inequalities the left-hand side value is $a_3 \cdot 0$ whereas the best possible bisection knapsack walk inequality on the longer path uses $H^l_3 = \emptyset$ and yields left-hand side value $a_3 \cdot 1$.

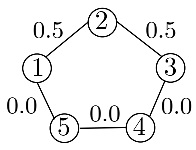

Figure 6: Graph for the counter example of Ex. 22

6 Capacity improved bisection knapsack walk inequalities and the lower envelope for stars

To motivate another strengthening for bisection knapsack walk inequalities consider the case of a disconnected graph with two components, one of them being a single edge $\{u, v\}$, the other connected one being $V' = V \setminus \{u, v\}$. Even though one cannot include the edge $\{u, v\}$ directly in a bisection knapsack walk inequality rooted at some $r \in V'$, one can at least improve the inequality if $y_{uv} = 1$. In this case u and v belong to different clusters and therefore the capacity F of both clusters can be reduced by $\min\{\varphi_u, \varphi_v\}$. Since F is the right-hand side of the inequality $\sum_{v \in V} \varphi_v x_v \leq F$ used to define the knapsack polytope P_{K}, this reduction may help to derive stronger bisection knapsack walk inequalities. For instance, one can look at a given valid inequality $\sum_{v \in V} a_v x_v \leq a_0$ for the original knapsack polytope with capacity F and in case $y_{uv} = 1$ we are allowed to reduce the right-hand side a_0 by $\min\{a_u, a_v\}$, thus also improving the bisection knapsack walk inequality.

To generalize this idea we define for $\bar{G} \subseteq G$ with $\bar{V} \subseteq V$, $\bar{E} \subseteq E(\bar{V})$ and $a \in \mathbb{R}_+^{|\bar{V}|}$ a function $\beta_{\bar{G}} : \{0,1\}^{|\bar{E}|} \to \mathbb{R} \cup \{\infty\}$ with

$$\beta_{\bar{G}}(y) = \inf\left\{a(S), a(\bar{V} \setminus S) : S \subseteq \bar{V}, \max\left\{a(S), a(\bar{V} \setminus S)\right\} \leq a_0, y = \chi^{\delta_{\bar{G}}(S)}\right\} . \tag{22}$$

Now we look at the lower convex envelope $\breve{\beta}_{\bar{G}} : \mathbb{R}^{|\bar{E}|} \to \mathbb{R} \cup \{\infty\}$ of $\beta_{\bar{G}}(y)$, i.e.,

$$\breve{\beta}_{\bar{G}}(x) = \sup\left\{\breve{\beta}(x) : \quad \breve{\beta} : \mathbb{R}^{|\bar{E}|} \to \mathbb{R},\ \breve{\beta} \text{ convex},\ \breve{\beta}(y) \leq \beta_{\bar{G}}(y),\ y \in \{0,1\}^{|\bar{E}|}\right\} . \tag{23}$$

Notice that $\breve{\beta}_{\bar{G}}$ is a piecewise linear function on its domain. We will see that given a bisection knapsack walk inequality (20) on some $V' \subseteq V$ and $\bar{V} \subseteq V \setminus V'$ subtracting any affine minorant $c_0 + \sum_{e \in \bar{E}} c_e y_e$ of $\breve{\beta}_{\bar{G}}$, i.e.,

$$c_0 + \sum_{e \in \bar{E}} c_e y_e \leq \breve{\beta}_{\bar{G}}(y), \tag{24}$$

on the right-hand side of (20) yields again a valid inequality for P_{B}. It yields an improvement with respect to a given y if the minorant is positive for this y. For convenience, the next proposition states this for several disjoint subsets $\bar{V}$.

Proposition 23. *Let $\sum_{v \in V} a_v x_v \leq a_0$ with $a_v \geq 0$ for all $v \in V$ be a valid inequality for the knapsack polytope P_{K}. Choose a non-empty $V' \subseteq V$ and subgraphs $\left(\bar{V}_l, \bar{E}_l\right) = \bar{G}_l \subset G$ with $\bar{V}_l \cap V' = \emptyset$, $\bar{E}_l \subseteq E(\bar{V}_l)$ for $l = 1, \ldots, L$ and pairwise disjoint sets $\bar{V}_l$. Find for each l an affine minorant $c_0^l + \sum_{e \in \bar{E}_l} c_e y_e$ for the convex envelope $\breve{\beta}_{\bar{G}_l}$ so that (24) holds for all y in P_{B}. Then the* capacity reduced bisection knapsack walk inequality

$$\sum_{v \in V'} a_v \left(1 - \sum_{e \in P_{rv} \setminus H_v} y_e - \sum_{e \in P_{rv} \cap H_v} (1 - y_e)\right) \leq a_0 - \sum_{l=1}^{L} \left(c_0^l + \sum_{e \in \bar{E}_l} c_e y_e\right) \tag{25}$$

is valid for P_{B}.

Proof. Let $y \in P_{\mathrm{B}}$ such that $y = \chi^{\delta(S)}$ with $S \subseteq V$, then $\varphi(S) \leq F$ and $\varphi(V \setminus S) \leq F$. W.l.o.g., let $r \in S$ and put $z = \chi^S$. Then for all $l = 1, \ldots, L$

$$c_0^l + \sum_{e \in \bar{E}_l} c_e y_e \leq \breve{\beta}_{\bar{G}_l}(y) \leq \beta_{\bar{G}_l}(y) = \min\left\{a(\bar{V}_l \cap S), a(\bar{V}_l \setminus S)\right\} \leq \sum_{v \in \bar{V}_l \cap S} a_v = \sum_{v \in \bar{V}_l \cap S} a_v z_v .$$

Furthermore, by Lemma 17 we have $1-\sum_{e\in P_{rv}\setminus H_v} y_e - \sum_{e\in H_v}(1-y_e) \le z_v$ for $v\in V'$. Thus

$$\sum_{v\in V'} a_v\left(1-\sum_{e\in P_{rv}\setminus H_v} y_e - \sum_{e\in H_v}(1-y_e)\right) + \sum_{l=1}^{L}\sum_{e\in \bar{E}_l} c_e y_e$$
$$\le \sum_{v\in V'} a_v z_v + \sum_{l=1}^{L}\sum_{v\in \bar{V}_l} a_v z_v \le \sum_{v\in V} a_v z_v \le a_0 \,. \qquad \square$$

Example 24. For the graph G displayed in Figure 7 with $\varphi_v = 1$ for all $v \in V$ the polytope P_{B} has 74 facets (computed by *polymake* [12]). Among these are 14 trivial facets, only 2 pure

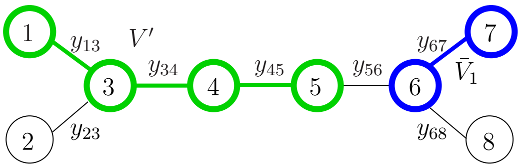

Figure 7: Graph considered in Example 24 (1). $F = 4$, $\varphi_i = 1$ for all $i \in V$, $\sum_{i\in V} x_i \le 4$.

bisection knapsack walk facets, 19 truncated bisection knapsack walk facets, 16 capacity reduced bisection knapsack walk facets (some truncated), 4 capacity reduced odd bisection knapsack walk facets and 19 facets for which we are not yet able to recognize a construction rule. Here we want to give a first simple example for a capacity reduced bisection knapsack walk inequality. Two more involved examples will follow at the end of this section. We use the knapsack inequality $\sum_{v\in V} x_v \le 4$ in all three examples, thus $a_v = 1$ for all $v \in V$:

(1) For $V' = \{1,3,4,5\}$, root node $r = 3$ and $H_v = \emptyset$ for all $v \in V'$ the bisection knapsack walk inequality is $1 + (1-y_{13}) + (1-y_{34}) + (1-y_{34}-y_{45}) \le 4$. We choose $\bar{G} = (\bar{V},\bar{E})$ with $\bar{V} = \{6,7\}$ and $\bar{E} = \{67\}$. We will see that the unique best minorizing function for $\check{\beta}_{\bar{G}}$ is y_{67}, thus the bisection knapsack walk inequality can be strengthened to $1 + (1-y_{13}) + (1-y_{34}) + (1-y_{34}-y_{45}) \le 4 - y_{67}$. Now rewrite this inequality to $y_{13} + 2y_{34} + y_{45} - y_{67} \ge 0$ and observe that, like in (5), the coefficient of y_{34} can be strengthened to 1 in order to find the facet $y_{13} + y_{34} + y_{45} - y_{67} \ge 0$ of P_{B}.

To find inequalities (24) to apply in Proposition 23 we take a closer look at the lower envelope defined in (23). In certain cases, e.g., for the case of $\bar{G} = (\bar{V},\bar{E})$ being a star with $a(\bar{V}) \le a_0$, we are able to give a full description of $\check{\beta}_{\bar{G}}$ by giving a complete description of the cluster weight polytope defined below. This will provide the tightest improvement possible in (25).

Definition 25. *Given a graph $G = (V,E)$ with node weights $a_v \ge 0$ for $v \in V$. For a set $S \subseteq V$ we define the following point in $\mathbb{R}^{|E|+1}$*

$$h^S_G = \begin{pmatrix} a(S) \\ \chi^{\delta(S)} \end{pmatrix} .$$

With respect to a given non-negative $a_0 \in \mathbb{R}$ we define

$$P_{\mathrm{CW}}(G) = \mathrm{conv}\{\, h^S_G : S \subseteq V,\ a(S) \le a_0,\ a(V\setminus S) \le a_0 \,\}$$

and call this set the cluster weight polytope*.*

As usual, we will drop G in h_G^S and $P_{\mathrm{CW}}(G)$ if the graph is clear from the context. The purpose of studying $P_{\mathrm{CW}}(\bar{G})$ is that its polyhedral description immediately yields the epigraph of $\breve{\beta}_{\bar{G}}$ via $\operatorname{epi}(\breve{\beta}_{\bar{G}}) = P_{\mathrm{CW}}(\bar{G}) + \{\lambda(1, 0^T)^T : \lambda \geq 0\}$. This is the content of the next proposition.

Proposition 26. *Given a subgraph $\bar{G} = (\bar{V}, \bar{E})$ of G with node weights $a_v \geq 0$ for $v \in V$, an inequality of the form $y_0 + \sum_{e\in\bar{E}} \gamma_e y_e \geq \gamma_0$ is valid for $P_{\mathrm{CW}}\big(\bar{G}\big)$ if and only if $\gamma_0 - \sum_{e\in\bar{E}} \gamma_e y_e$ is an affine minorant of $\breve{\beta}_{\bar{G}}$.*

Proof. $y_0 + \sum_{e\in\bar{E}} \gamma_e y_e \geq \gamma_0$ is valid for $P_{\mathrm{CW}}\big(\bar{G}\big)$ if and only if

$$y_0 + \sum_{e\in\bar{E}} \gamma_e y_e \geq \gamma_0 \text{ for } \begin{pmatrix} y_0 \\ y \end{pmatrix} \in \left\{ h_{\bar{G}}^S : S \subseteq \bar{V},\ a(S) \leq a_0,\ a(\bar{V} \setminus S) \leq a_0 \right\}$$

if and only if

$$y_0 \geq \gamma_0 - \sum_{e\in\bar{E}} \gamma_e y_e \text{ for } \left\{ \begin{pmatrix} \min\{a(S), a(\bar{V} \setminus S)\} \\ \chi^{\delta_{\bar{G}}(S)} \end{pmatrix} : S \subseteq \bar{V},\ \max\{a(S), a(\bar{V} \setminus S)\} \leq a_0 \right\}$$

if and only if $\beta_{\bar{G}}(y) \geq \gamma_0 - \sum_{e\in\bar{E}} \gamma_e y_e$ for $y \in \{0,1\}^{|\bar{E}|}$ (by (22), recall that $\inf \emptyset = \infty$ by definition) if and only if $\breve{\beta}_{\bar{G}}(y) \geq \gamma_0 - \sum_{e\in\bar{E}} \gamma_e y_e$ for $y \in \mathbb{R}^{|\bar{E}|}$ (see (23)). □

Hence, the "lower" facets of P_{CW} are in one to one correspondence to the linear components of $\breve{\beta}$. For a star $\bar{G} = \big(\bar{V}, \bar{E}\big)$ we are able to exhibit facets of $P_{\mathrm{CW}}\big(\bar{G}\big)$, which in certain problems enable us to strengthen bisection knapsack walk inequalities of P_{B} to facet-defining inequalities of P_{B} (see Example 40 at the end of this section).

Let us first look at a symmetry of P_{CW} for general graphs $G = (V, E)$, a property which we will later use frequently to cut down our efforts in the proofs.

Proposition 27. *P_{CW} is symmetric to the hyperplane $\{\, y \in \mathbb{R}^{|E|} : 2y_0 = a(V)\,\}$.*

Proof. Observe that for any point h^S used in the definition of P_{CW} the point $h^{V\setminus S}$ is contained in P_{CW}, too. Since $\chi^{\delta(S)} = \chi^{\delta(V\setminus S)}$, we have for all those pairs $\big(h^S, h^{V\setminus S}\big)$

$$\begin{pmatrix} \frac{1}{2}a(V) \\ \chi^{\delta(S)} \end{pmatrix} - h^S = h^{V\setminus S} - \begin{pmatrix} \frac{1}{2}a(V) \\ \chi^{\delta(S)} \end{pmatrix} .$$ □

Another useful result for a star $G = (V, E)$ is the following

Lemma 28. *Let $G = (V, E)$ be a star with center $r \in V$, $a_v \geq 0$ for all $v \in V$ and $a_{v'} = a(V \setminus \{v'\})$ for at least one $v' \in V \setminus \{r\}$. Then $a(S) = a(V \setminus S)$ for all $S \subseteq V$ with $v' \in S$ and $r \in V \setminus S$ if and only if $a_{v'} = a_r$ and $a_v = 0$ for all $v \in V \setminus \{v', r\}$.*

Proof. The sufficiency is obvious. We will show necessity: Suppose $a(S) = a(V \setminus S)$ for all $S \subseteq V$ with $v' \in S$ and $r \in V \setminus S$. Then, in particular, this is true for $V \setminus S = \{r\}$, i.e., $a_r = a(V \setminus \{r\}) = a_{v'} + a(V \setminus \{v', r\}) = a(V \setminus \{v'\}) + a(V \setminus \{v', r\}) = a_r + 2a(V \setminus \{v', r\})$. Thus, $a_v = 0$ for all $v \in V \setminus \{v', r\}$ and $a_{v'} = a_r$. □

In the remaining part of the section we will look into P_{CW} for stars $G = (V, E)$ with center node $r \in V$ and the constraint $\sum_{v\in V} a_v x_v \leq a_0$. At first we determine the dimension of the polytope.

Proposition 29. *For a star $G = (V, E)$ with center $r \in V$ and $a \geq 0$ with $a(V) \leq a_0$ the polytope P_{CW} has full dimension $|E| + 1$ for $a \neq 0^{|E|}$ and dimension $|E| = |V|$ for $a = 0^{|E|}$.*

Proof. Since G is a star and by assumption $a(V) \leq a_0$, the $1 + |E|$ points $h^{\emptyset}$ and $h^{\{v\}}$ for all $v \in V \setminus \{r\}$ are contained in P_{CW} and affinely independent. Thus the dimension of P_{CW} is at least $|E|$. If $a \neq 0^{|E|}$, then h^V is affinely independent from all points listed previously, thus P_{CW} is full-dimensional with dimension $|E| + 1$. For $a = 0^{|E|}$ all points lie on the hyperplane $y_0 = 0$. □

For $G = (V, E)$ a star with center $r \in V$, weights $a_v = 0$ for all $v \in V$ and $a_0 \geq 0$ it can easily be worked out that P_{CW} is completely described by the equality $y_0 = 0$ and the inequalities $0 \leq y_{rv} \leq 1$ for all $v \in V \setminus \{r\}$. So from now on we assume $a_v > 0$ for at least one $v \in V$. Let us first state trivial valid inequalities and facets of P_{CW}.

Proposition 30. *For a star $G = (V, E)$ with center $r \in V$, $a \geq 0$ with $a \neq 0^{|E|}$ and $a(V) \leq a_0$ the trivial inequalities*

$$0 \leq y_{rv} \leq 1, \quad \forall\, v \in V \setminus \{r\} \tag{26}$$

are facet-inducing for P_{CW} except for one particular case: if there is exactly one $v' \in V \setminus \{r\}$ with $a_{v'} = a_r = \frac{1}{2}a(V)$, then $y_{rv'} \leq 1$ does not induce a facet.

Proof. The validity of the inequalities $y_{rv'} \geq 0$ and $y_{rv'} \leq 1$ for all $v' \in V \setminus \{r\}$ follows from the definition of P_{CW}. In general, to prove that a valid inequality defines a facet of P_{CW} we have to find $\dim(P_{\text{CW}})$ affinely independent points of P_{CW} which fulfill it with equality. From Proposition 29 we know that $\dim(P_{\text{CW}}) = |V|$ if $a \neq 0^{|E|}$. For $y_{rv'} \geq 0$ we choose the $|V|$ points $h^{\emptyset}$, h^V and $h^{\{v\}}$ for all $v \in V \setminus \{r, v'\}$. For $y_{rv'} \leq 1$ the accumulation of affinely independent points on the inequality is a bit more involved: If $a_{v'} \neq a(V \setminus \{v'\})$ we can choose the $|V|$ points $h^{\{v'\}}$, $h^{V \setminus \{v'\}}$ and $h^{\{v', v\}}$ for all $v \in V \setminus \{r\}$ with $v \neq v'$. If $a_{v'} = a(V \setminus \{v'\})$ we look at two cases:

1. $a_r \neq a_{v'}$: Then there is a $\tilde{v} \in V \setminus \{r, v'\}$ with $a_{\tilde{v}} > 0$. Furthermore, since $a_{v'} = a(V \setminus \{v'\})$, we have $a_{v'} = \frac{1}{2}a(V)$. Together with $a_{\tilde{v}} > 0$ this implies $a(\{v', \tilde{v}\}) \neq a(V \setminus \{v', \tilde{v}\})$, i.e., $h^{\{v', \tilde{v}\}} \neq h^{V \setminus \{v', \tilde{v}\}}$. Thus we can choose the $|V|$ points $h^{\{v'\}}$, $h^{\{v', v\}}$ for all $v \in V \setminus \{r, v'\}$ and $h^{V \setminus \{v', \tilde{v}\}}$.

2. $a_r = a_{v'}$: The set of points contained in the definition of P_{CW} which fulfill $y_{rv'} = 1$ is $\{h^S, h^{V \setminus S} : S \subseteq V, v' \in S, r \in V \setminus S\}$. Lemma 28 implies for every pair $(h^S, h^{V \setminus S})$ in this set that $a(S) = a(V \setminus S)$. Since $a(S) + a(V \setminus S) = a(V)$, we get $a(S) = \frac{1}{2}a(V)$ for all S with $y = \chi^{\delta(S)}$ and $y_{rv'} = 1$. Thus all vertices of P_{CW} fulfilling $y_{rv'} = 1$ live in the hyperplane $\{y \in \mathbb{R}^{|E|+1} : y_0 = \frac{1}{2}a(V)\}$. Therefore, $y_{rv'} \leq 1$ cannot induce a facet of P_{CW}. □

In the following two propositions we look into non-trivial facets of P_{CW}. Proposition 31 deals with the case $a(V \setminus \{r\}) > a_r$ and Proposition 32 with the case $a(V \setminus \{r\}) \leq a_r$.

Proposition 31. *Given a star $G = (V, E)$ with center $r \in V$, $a \geq 0$ with $a \neq 0^{|E|}$, $a(V) \leq a_0$ and $a(V \setminus \{r\}) > a_r$. We call a triple $(V_p, \bar{v}, V_n)$ feasible if it fulfills $V = \{r, \bar{v}\} \dot\cup V_p \dot\cup V_n$ and $a(V_p) \leq \frac{1}{2}a(V) < a(V_p) + a_{\bar{v}}$. For all feasible triples $(V_p, \bar{v}, V_n)$ the inequalities*

$$y_0 + \sum_{v \in V_p} a_v y_{rv} + \left(a(V) - 2a(V_p) - a_{\bar{v}}\right) y_{r\bar{v}} - \sum_{v \in V_n} a_v y_{rv} \leq a(V) \tag{27}$$

$$y_0 - \sum_{v \in V_p} a_v y_{rv} - (a(V) - 2a(V_p) - a_{\bar{v}})\, y_{r\bar{v}} + \sum_{v \in V_n} a_v y_{rv} \geq 0 \tag{28}$$

are facet-inducing for P_{CW}.

Note, that it is possible that either V_p or V_n of feasible triples $(V_p, \bar{v}, V_n)$ might be empty, but for $a(V \setminus \{r\}) > a_r$ there always is the special element $\bar{v}$.

Proof of Proposition 31. To cut down our efforts in this proof and the ones to follow observe that for each feasible triple $(V_p, \bar{v}, V_n)$ the corresponding pair of inequalities (27) and (28) is symmetric to the hyperplane $\{\, y \in \mathbb{R}^{|E|} : 2y_0 = a(V)\,\}$. To see this, subtract the equation $2y_0 = a(V)$ from (27) to obtain (28). Thus, by Prop. 27, it suffices to show that (27) is valid and facet-defining. Furthermore, to show the validity of (27) it is sufficient to only look at the "upper" points defining P_{CW}, i.e., if w.l.o.g., $S \subseteq V$ such that $a(S) \geq a(V \setminus S)$, then we only need to check validity of (27) for $h^S = (a(S), \chi^{\delta(S)})^T$.
Consider an arbitrary $S \subseteq V$ such that $a(S) \geq a(V \setminus S)$. Let $V^1 = \{v \in V : rv \in \delta(S)\}$. We discern the following four cases:

1. $\bar{v} \in V^1 = S$: For $\begin{pmatrix} a(S) \\ \chi^{\delta(S)} \end{pmatrix}$ the left-hand side of (27) equals

$$\begin{aligned} a(V^1) + a(V_p \cap V^1) + a(V) - 2a(V_p) - a_{\bar{v}} - a(V_n \cap V^1) &= \\ 2a(V_p \cap V^1) + a(V) - 2a(V_p) &= \\ a(V) - 2a(V_p \setminus V^1) &\leq a(V) \end{aligned}$$

   where the first equality uses $a(V^1) = a(V_p \cap V^1) + a_{\bar{v}} + a(V_n \cap V^1)$ and the inequality is due to $a(V_p \setminus V^1) \geq 0$.

2. $\bar{v} \notin V^1 = S$: For $\begin{pmatrix} a(S) \\ \chi^{\delta(S)} \end{pmatrix}$ the left-hand side of (27) equals

$$a(V^1) + a(V_p \cap V^1) - a(V_n \cap V^1) = 2a(V_p \cap V^1) \leq 2a(V_p) \leq a(V)$$

   where the equality uses $a(V^1) = a(V_p \cap V^1) + a(V_n \cap V^1)$ and the last inequality is due to $a(V_p) \leq \frac{1}{2}a(V)$ by the definition of V_p.

3. $\bar{v} \in V^1 = V \setminus S$: For $\begin{pmatrix} a(S) \\ \chi^{\delta(S)} \end{pmatrix}$ the left-hand side of (27) equals

$$\begin{aligned} a(V) - a(V^1) + a(V_p \cap V^1) + a(V) - 2a(V_p) - a_{\bar{v}} - a(V_n \cap V^1) &= \\ 2a(V) - 2a(V_p) - 2a_{\bar{v}} - 2a(V_n \cap V^1) &< \\ 2a(V) - a(V) - 2a(V_n \cap V^1) &\leq a(V) \end{aligned}$$

   where the first equality uses $a(V^1) = a(V_p \cap V^1) + a_{\bar{v}} + a(V_n \cap V^1)$, the strict inequality is due to $a(V_p) + a_{\bar{v}} > \frac{1}{2}a(V)$ by the definition of V_p and $\bar{v}$ and the inequality holds since $a(V_n \cap V^1) \geq 0$.

4. $\bar{v} \notin V^1 = V \setminus S$: For $\begin{pmatrix} a(S) \\ \chi^{\delta(S)} \end{pmatrix}$ the left-hand side of (27) equals

$$a(V) - a(V^1) + a(V_p \cap V^1) - a(V_n \cap V^1) = a(V) - 2a(V_n \cap V^1) \leq a(V)$$

   where the first equality uses $a(V^1) = a(V_p \cap V^1) + a(V_n \cap V^1)$ and the inequality is due $a(V_n \cap V^1) \geq 0$.

In order to show that (27) is also facet-defining, let $V_p = \{v_1^p, \dots, v_{|V_p|}^p\}$ and $V_n = \{v_1^n, \dots, v_{|V_n|}^n\}$. Then the $|V|$ points

$$h^V,\ h^{V\setminus\{v_1^p\}}, \dots, h^{V\setminus\{v_1^p,\dots,v_{|V_p|}^p\}},\ h^{\{v_1^p,\dots,v_{|V_p|}^p,\bar{v}\}},\ h^{\{v_1^p,\dots,v_{|V_p|}^p,\bar{v},v_1^n\}}, \dots, h^{\{v_1^p,\dots,v_{|V_p|}^p,\bar{v},v_1^n,\dots,v_{|V_n|}^n\}}$$

fulfill the inequality (27) with equality and are affinely independent, thus (27) is a facet-inducing inequality. □

In the case of $a(V \setminus \{r\}) \le a_r$ the set V_n is empty, there is no $\bar{v}$ and the inequalities (27) and (28) take the following form.

Proposition 32. *For a star $G = (V, E)$ with root $r \in V$ and $a \ge 0$ with $a \ne 0^{|E|}$, $a(V) \le a_0$ and $a(V \setminus \{r\}) \le a_r$ the inequalities*

$$y_0 + \sum_{v \in V\setminus\{r\}} a_v y_{e_v} \le a(V) \tag{29}$$

$$y_0 - \sum_{v \in V\setminus\{r\}} a_v y_{e_v} \ge 0 \tag{30}$$

are facet-inducing for P_{CW}.

Proof. We start again by observing the symmetry of the inequalities (29) and (30) to the hyperplane $\{\, y \in \mathbb{R}^{|E|} : 2y_0 = a(V)\,\}$. To see this, subtract the equation $2y_0 = a(V)$ from inequality (29) to obtain inequality (30). Thus, by Prop. 27, it suffices to prove the validity and facet-induction of (29). Take an $S \subseteq V$ with $a(S) \ge a(V \setminus S)$. Then $h^S = \begin{pmatrix} a(S) \\ \chi^{\delta(S)} \end{pmatrix}$ is one of the points defining P_{CW}. We see that $V \setminus S = \{v \in V : rv \in \delta(S)\}$. Now plug h^S into the left-hand side of (29) to get $a(S) + a(V \setminus S) = a(V)$. The point $h^{V\setminus S} = \begin{pmatrix} a(V\setminus S) \\ \chi^{\delta(V\setminus S)} \end{pmatrix}$ can also not violate (29) since $a(V \setminus S) \le a(S)$, thus (29) is valid for P_{CW}.

In order to show that (29) is facet-inducing let $v_1, \dots, v_{|V|-1}$ be an arbitrary ordering of the nodes in $V \setminus \{r\}$. Since $a(S) + a(V \setminus S) = a(V)$ holds, the $\dim(P_{\mathrm{CW}}) = |V|$ points

$$h^V,\ h^{V\setminus\{v_1\}},\ \dots,\ h^{V\setminus\{v_1,\dots,v_{|V|-1}\}}$$

fulfill the inequality (29) with equality and are affinely independent. □

All possible facets of P_{CW} fall into one of the following three classes:

$$y_0 + \sum_{v \in V\setminus\{r\}} \gamma_v y_{rv} \le \gamma_0 \tag{31}$$

$$\sum_{v \in V\setminus\{r\}} \gamma_v y_{rv} \le \gamma_0 \tag{32}$$

$$-y_0 + \sum_{v \in V\setminus\{r\}} \gamma_v y_{rv} \le \gamma_0 \tag{33}$$

In the next two lemmas we will look closer into coefficients of facets of the form (31). The following three propositions state that we have found all facets of P_{CW} of the forms (31), (32) and (33), respectively. Finally, Theorem 38 summarizes the results. The section is accompanied by two small examples on how to apply the inequalities to derive capacity reduced bisection knapsack walk inequalities.

Lemma 33. *For an arbitrary facet of* P_{CW} *of the form (31) we have for all* $v \in V \setminus \{r\}$

$$-a_v \leq \gamma_v \leq a_v \ .$$

Proof. We give the proof for the case $\gamma_{\tilde{v}} > 0$ (the case $\gamma_{\tilde{v}} < 0$ can be proved by analogous arguments). The facet has a root $\left(\hat{y}_0, \hat{y}^T\right)^T$ with $\hat{y}_{r\tilde{v}} = 0$, because otherwise all roots $\hat{y}$ would lie on the equation $\hat{y}_{r\tilde{v}} = 1$, thus (31) could not induce a facet. Let $S \subseteq V$ be the corresponding subset satisfying $\hat{y} = \chi^{\delta(S)}$ and $\hat{y}_0 = a(S) \geq a(V \setminus S)$. To bound $\gamma_{\tilde{v}}$ we look at $\bar{y} = \hat{y} + e_{r\tilde{v}}$, i.e., the cut $\delta(S) \cup \{r\tilde{v}\}$. We discern three cases concerning the location of node $\tilde{v}$ and the size of the bigger cluster:

1. $\tilde{v} \in V \setminus S$: Because $a(S) \geq a(V \setminus S)$ we obtain $a(S \cup \{\tilde{v}\}) \geq a(V \setminus (S \cup \{\tilde{v}\}))$. Set $\bar{y}_0 = a(S \cup \{\tilde{v}\})$, i.e., $\left(\bar{y}_0, \bar{y}^T\right)^T = h^{S \cup \{\tilde{v}\}} \in P_{\text{CW}}$. In order for (31) to be feasible for $\left(\bar{y}_0, \bar{y}^T\right)^T$ we need $\gamma_0 \geq \bar{y}_0 + \sum_{v \in V\setminus\{r\}} \gamma_v \bar{y}_{rv}$. Since $\left(\hat{y}_0, \hat{y}^T\right)^T$ is a root of (31), we have $\gamma_0 = \hat{y}_0 + \sum_{v \in V\setminus\{r\}} \gamma_v \hat{y}_{rv}$. Thus, $\hat{y}_0 + \sum_{v \in V\setminus\{r\}} \gamma_v \hat{y}_{rv} \geq \bar{y}_0 + \sum_{v \in V\setminus\{r\}} \gamma_v \bar{y}_{rv}$, i.e., $\hat{y}_0 \geq \bar{y}_0 + \gamma_{\tilde{v}}$. Therefore, $\gamma_{\tilde{v}} \leq \hat{y}_0 - \bar{y}_0 = -a_{\tilde{v}}$. This contradicts our assumption $\gamma_{\tilde{v}} > 0$, thus the case $\tilde{v} \in V \setminus S$ is not possible.

2. $\tilde{v} \in S$ and $a(S \setminus \{\tilde{v}\}) \geq a((V \setminus S) \cup \{\tilde{v}\})$: Set $\bar{y}_0 = a(S \setminus \{\tilde{v}\})$, i.e., $\left(\bar{y}_0, \bar{y}^T\right)^T = h^{S \setminus \{\tilde{v}\}} \in P_{\text{CW}}$. As $\left(\bar{y}_0, \bar{y}^T\right)^T$ is feasible for (31) we derive, as in the previous case, $\hat{y}_0 \geq \bar{y}_0 + \gamma_{\tilde{v}}$, hence $\gamma_{\tilde{v}} \leq \hat{y}_0 - \bar{y}_0 = a(S) - a(S \setminus \{\tilde{v}\}) = a_{\tilde{v}}$.

3. $\tilde{v} \in S$ and $a(S \setminus \{\tilde{v}\}) < a((V \setminus S) \cup \{\tilde{v}\})$: This implies $a(S \setminus \{\tilde{v}\}) < \frac{1}{2}a(V)$. Set $\bar{y}_0 = a((V \setminus S) \cup \{\tilde{v}\})$, i.e., $\left(\bar{y}_0, \bar{y}^T\right)^T = h^{(V \setminus S) \cup \{\tilde{v}\}} \in P_{\text{CW}}$. From the feasibility of (31) we conclude $\hat{y}_0 \geq \bar{y}_0 + \gamma_{\tilde{v}}$. Therefore, $\gamma_{\tilde{v}} \leq \hat{y}_0 - \bar{y}_0 = a(S) - a((V \setminus S) \cup \{\tilde{v}\}) = a_{\tilde{v}} + 2a(S \setminus \{\tilde{v}\}) - a(V) < a_{\tilde{v}}$, where the last inequality uses $a(S \setminus \{\tilde{v}\}) - \frac{1}{2}a(V) < 0$.

□

Lemma 34. *For an arbitrary facet of* P_{CW} *of the form (31) we have* $\gamma_0 = a(V)$ *and* $\sum_{v \in V\setminus\{r\}} \gamma_v \leq a_r$.

Proof. In order for (31) to be valid for $h^V = \left(a(V), (\chi^{\delta(V)})^T\right)^T \in P_{\text{CW}}$ we get $\gamma_0 \geq a(V)$. We discern two cases regarding the weight of the root node r.

$a_r < a(V \setminus \{r\})$: (31) has to be valid for $\left(a(V \setminus \{r\}), (\chi^{\delta(V\setminus\{r\})})^T\right)^T = h^{V\setminus\{r\}} \in P_{\text{CW}}$, thus $\sum_{v \in V\setminus\{r\}} \gamma_v \leq \gamma_0 - a(V \setminus \{r\})$.

$a_r \geq a(V \setminus \{r\})$: (31) has to be valid for $\left(a_r, (\chi^{\delta(\{r\})})^T\right)^T = h^{\{r\}} \in P_{\text{CW}}$, thus $\sum_{v \in V\setminus\{r\}} \gamma_v \leq \gamma_0 - a_r \leq \gamma_0 - a(V \setminus \{r\})$.

Thus in any case we have

$$\sum_{v \in V\setminus\{r\}} (a_v + \gamma_v) \leq \gamma_0 \ . \tag{34}$$

Now use $a_v + \gamma_v \geq 0$ (by Lemma 33) and $y_{rv} \in [0, 1]$ for all $\left(y_0, y^T\right)^T \in P_{\text{CW}}$ to conclude that

$$\sum_{v \in V\setminus\{r\}} (a_v + \gamma_v) y_{rv} \leq \gamma_0 \tag{35}$$

is a valid inequality for P_{CW}. Additionally, $a_r = a(V) - a(V \setminus \{r\})$, thus it is sufficient to show that $\gamma_0 = a(V)$ if (31) induces a facet of P_{CW}, because then (34) implies $\sum_{v \in V \setminus \{r\}} \gamma_v \leq a_r$. So assume, for contradiction, that $\gamma_0 > a(V)$ with (31) facet defining. In search for roots of (31) let $S \subseteq V$ be such that $\tilde{y}_0 = a(S) \geq a(V \setminus S)$, put $\tilde{y} = \chi^{\delta(S)}$ and consider the following two cases.

$r \in S$: Then $\tilde{y}_0 + \sum_{v \in V \setminus \{r\}} \gamma_v \tilde{y}_{rv} = a(S) + \sum_{v \in V \setminus S} \gamma_v \leq a(V) < \gamma_0$, where the $\leq$-inequality is due to $\gamma_v \leq a_v$ by Lemma 33. Therefore, $\left(\tilde{y}_0, \tilde{y}^T\right)^T$ cannot lie on the facet.

$r \in V \setminus S$: We show that all such roots also satisfy (35) with equality and so (31) cannot not be a facet. Indeed, if it is a root, $\left(\tilde{y}_0, \tilde{y}^T\right)^T$ satisfies $\tilde{y}_0 + \sum_{v \in V \setminus \{r\}} \gamma_v \tilde{y}_{rv} = \gamma_0$. Since $\tilde{y}_0 = \sum_{v \in V \setminus \{r\}} a_v \tilde{y}_{rv}$, we obtain $\gamma_0 = \sum_{v \in V \setminus \{r\}} (a_v + \gamma_v) \tilde{y}_{rv}$.

Hence, any facet inducing inequality (31) has $\gamma_0 = a(V)$. □

Proposition 35. *For a star $G = (V, E)$ with root $r \in V$, $a \geq 0$ with $a \neq 0^{|E|}$ and $a(V) \leq a_0$ all facets of the form (31) for P_{CW} are defined by (27) if $a(V \setminus \{r\}) > a_r$ and by (29) if $a(V \setminus \{r\}) \leq a_r$.*

Proof. We have shown in Lemma 33 that each coefficient γ_v for all $v \in V \setminus \{r\}$ in all facets of P_{CW} of the form (31) fulfills

$$-a_v \leq \gamma_v \leq a_v \ . \tag{36}$$

Lemma 34 tells us that for each individual facet of P_{CW} of the form (31) the coefficients fulfill

$$\sum_{v \in V \setminus \{r\}} \gamma_v \leq a_r \tag{37}$$

and

$$\gamma_0 = a(V) \ . \tag{38}$$

For any given $y \in [0,1]^{|E|}$ we will now determine the best γ_0 and γ subject to the constraints (36), (37) and (38) so that the right hand side of $y_0 \leq \gamma_0 - \sum_{v \in V \setminus \{r\}} \gamma_v y_{rv}$ is as small as possible. If we can always exhibit an optimal solution γ_0^*, γ^* that corresponds to the coefficients of (27) if $a(V \setminus \{r\}) > a_r$ or of (29) if $a(V \setminus \{r\}) \leq a_r$, then the proof is complete. At first note that (38) directly fixes γ_0 to $a(V)$ which corresponds to the right-hand sides of (27) and (29). Now look at the problem

$$\begin{aligned} \min \quad & a(V) - \sum_{v \in V \setminus \{r\}} y_{rv} \gamma_v \\ \text{s.t.} \quad & \sum_{v \in V \setminus \{r\}} \gamma_v \leq a_r \\ & -a_v \leq \gamma_v \leq a_v \ \forall v \in V \setminus \{r\} \ . \end{aligned} \tag{39}$$

In case $a(V \setminus \{r\}) \leq a_r$, the optimal solution sets $\gamma_v = a_v$ for all $v \in V \setminus \{r\}$ and we have determined inequality (29). So assume $a(V \setminus \{r\}) > a_r$. Using the variable transformation $\tilde{\gamma}_v = \gamma_v + a_v$, problem (39) is equivalent to

$$\begin{aligned} \max \quad & \sum_{v \in V \setminus \{r\}} y_{rv} \tilde{\gamma}_v - \sum_{v \in V \setminus \{r\}} y_{rv} a_v \\ \text{s.t.} \quad & \sum_{v \in V \setminus \{r\}} \tilde{\gamma}_v \leq a(V) \\ & 0 \leq \tilde{\gamma}_v \leq 2a_v \ \forall v \in V \setminus \{r\} \ . \end{aligned} \tag{40}$$

Problem (40) is the canonical continuous bounded knapsack problem (see Sections 3.2 and 3.3.1 in [14]) with continuous variables γ_v, profits y_{rv}, weights 1 and upper bound $2a_v$ for all items $v \in V \setminus \{r\}$ and knapsack capacity $a(V)$. An optimal solution can be found by sorting the items v with respect to non-increasing profit-to-weight ratios $y_{rv}/1$, w.l.o.g., let this ordering be $1, 2, \ldots, |V| - 1$, and by using this ordering to pack the knapsack in the following way: $\tilde{\gamma}_v = 2a_v$ for all $v = 1, \ldots, \bar{v} - 1$ with $\bar{v}$ so that $2a(\{1, \ldots, \bar{v} - 1\}) \leq a(V)$ and $2a(\{1, \ldots, \bar{v} - 1\}) + 2a_{\bar{v}} > a(V)$, $\tilde{\gamma}_{\bar{v}} = a(V) - 2a(\{1, \ldots, \bar{v} - 1\})$, and $\tilde{\gamma}_v = 0$ for all $v = \bar{v} + 1, \ldots, |V| - 1$. The item $\bar{v}$ is called the critical item. Note that if one $\bar{v}$ can be chosen as the critical item, then so can all $v \neq \bar{v}$ with $y_{rv} = y_{r\bar{v}}$.
Backsubstitution of $\tilde{\gamma}_v = \gamma_v + a_v$ yields the optimal solution of problem (39): $\gamma_v = a_v$ for all $v = 1, \ldots, \bar{v} - 1$ with $a(\{1, \ldots, \bar{v} - 1\}) \leq \frac{1}{2}a(V)$ and $a(\{1, \ldots, \bar{v} - 1\}) + a_{\bar{v}} > \frac{1}{2}a(V)$, $\gamma_{\bar{v}} = a(V) - 2a(\{1, \ldots, \bar{v} - 1\}) - a_{\bar{v}}$, and $\gamma_v = -a_v$ for all $v = \bar{v} + 1, \ldots, |V| - 1$. Finally we observe that we have determined a feasible triple $(V_p = \{1, \ldots, \bar{v} - 1\}, \bar{v}, V_n = \{\bar{v} + 1, \ldots, |V| - 1\})$, i.e., we have found an inequality of (27). □

Proposition 36. *For a star $G = (V, E)$ with root $r \in V$, $a \geq 0$ with $a \neq 0^{|E|}$ and $a(V) \leq a_0$ all facets of the form (33) for P_{CW} are defined by (28) if $a(V \setminus \{r\}) > a_r$ and by (30) if $a(V \setminus \{r\}) \leq a_r$.*

Proof. Use the symmetry of P_{CW}, of pairs (31) and (33) with the same γ_v and γ_0, of pairs (27) and (28) and of pairs (29) and (30) to the hyperplane $\{ y \in \mathbb{R}^{|E|} : 2y_0 = a(V) \}$ and apply Proposition 35. □

Proposition 37. *For a star $G = (V, E)$ with root $r \in V$, $a \geq 0$ with $a \neq 0^{|E|}$ and $a(V) \leq a_0$ all facets of the form (32) for P_{CW} are defined by (26).*

Proof. It is trivial to show that facets of a polytope with coefficient zero for a fixed variable are also facets of the projection of this polytope if one projects out this variable. Since the hyperplanes defined by inequalities of the form (32) have coefficient zero for variable y_0, we have to look at the projection of P_{CW} onto the space $\mathbb{R}^{|E|}$ and have to show that this projection only has facets of the form (26). A point $\big(a(S), (\chi^{\delta(S)})^T\big)^T \in \mathbb{R}^{|E|+1}$ used to define P_{CW} is projected to $\chi^{\delta(S)} \in \mathbb{R}^{|E|}$, and since $a(V) \leq a_0$ the polytope P_{CW} contains the points $\big(a(S), (\chi^{\delta(S)})^T\big)^T \in \mathbb{R}^{|E|+1}$ for all $S \subseteq V$, thus its projection contains all possible points $\{0, 1\}^{|E|}$. Furthermore, the projection of any other point of P_{CW} can be written as the convex combination of points $\{0, 1\}^{|E|}$. Thus the projection of P_{CW} is exactly the $|E|$-dimensional hypercube. To finish the proof we note that the $|E|$-dimensional hypercube is completely described by the projection of the inequalities (26). □

Theorem 38. *For a star $G = (V, E)$ with root $r \in V$, $a \geq 0$ with $a \neq 0^{|E|}$ and $a(V) \leq a_0$ we have*

$$P_{\text{CW}} = \{ y \in \mathbb{R}^{|E|+1} : y \ \textit{fulfills} \ (26), (27) \ \textit{and} \ (28) \} =: Y, \ \ \textit{if} \ a(V \setminus \{r\}) > a_r, \ \textit{and}$$
$$P_{\text{CW}} = \{ y \in \mathbb{R}^{|E|+1} : y \ \textit{fulfills} \ (26), (29) \ \textit{and} \ (30) \} =: Y^r, \ \ \textit{if} \ a(V \setminus \{r\}) \leq a_r.$$

Proof. If $a(V \setminus \{r\}) > a_r$, propositions 30 and 31 show that $Y \supseteq P_{\text{CW}}$, and to show $Y \subseteq P_{\text{CW}}$ we can use propositions 35, 36 and 37. If $a(V \setminus \{r\}) \leq a_r$, propositions 30 and 32 show that $Y^r \supseteq P_{\text{CW}}$ and to prove $Y^r \subseteq P_{\text{CW}}$ we can use again propositions 35, 36 and 37. □

Remark 39. Note that in all assertions of this section we have assumed $a(V) \leq a_0$. This assumption guarantees that every $S \subseteq V$ contributes its point h^S to P_{CW}. If we reduce a_0 below $a(V)$ the facial structure of P_{CW} becomes much more complicated, because suddenly the whole complexity of the knapsack polytope P_{K} comes into play. So far a complete description of P_{CW} with $a(V) > a_0$ seems out of reach for non-trivial graphs, even if we assume $a_v = 1$ for all $v \in V$.

Example 40. We continue Example 24. For the choice of the subgraphs $\bar{G}_l$ compare Figure 8.

(2) The bisection knapsack walk inequality on $V' = \{1, 2, 3\}$ with root node $r = 3$ and $H_v = \emptyset$ for all $v \in V'$ is $1 + (1 - y_{13}) + (1 - y_{23}) \leq 4$. With $\bar{G}_1$ and $\bar{G}_2$ such that $\bar{V}_1 = \{4, 5\}$, $\bar{V}_2 = \{6, 7\}$, $\bar{E}_1 = \{45\}$ and $\bar{E}_2 = \{67\}$ the capacity reduced bisection knapsack walk inequality reads $1 + (1 - y_{13}) + (1 - y_{23}) \leq 4 - y_{45} - y_{67}$ and is a facet of P_{B}.

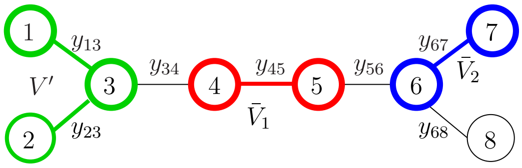

Figure 8: Graph for Example 40 (2). $F = 4$, $\varphi_i = 1$ for all $i \in V$, $\sum_{i \in V} x_i \leq 4$.

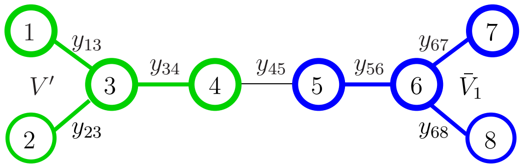

Figure 9: Graph for Example 40 (3). $F = 4$, $\varphi_i = 1$ for all $i \in V$, $\sum_{i \in V} x_i \leq 4$.

(3) For $V' = \{1, 2, 3, 4\}$, $r = 3$ and $H_v = \emptyset$ for all $v \in V'$ the bisection knapsack walk inequality is $1 + (1 - y_{13}) + (1 - y_{23}) + (1 - y_{34}) \leq 4$. Proposition 31 establishes that for $\bar{G}$ with $\bar{V} = \{5, 6, 7, 8\}$ and $\bar{E} = \{56, 67, 68\}$ one of the best minorizing functions for $\breve{\beta}_{\bar{G}}$ is $y_{56} + y_{67} - y_{68}$. Thus the resulting capacity reduced bisection knapsack walk inequality reads $1 + (1 - y_{13}) + (1 - y_{23}) + (1 - y_{34}) \leq 4 - y_{56} - y_{67} + y_{68}$. It is a facet of P_{B}.

7 Conclusion

We investigated the bisection cut polytope P_{B} associated with the minimum graph bisection problem MB. In particular, we exploited the knapsack condition ($f(S) \leq F$ and $f(V \setminus S) \leq F$, $S \subseteq V$) in the formulation of the problem, which makes it NP-hard. As one would expect, inequalities basing on this knapsack constraint define high dimensional faces of P_{B}. In the first part of the paper we showed that in case the underlying graph G is a tree the knapsack tree inequalities define facets of P_{B}. The situation becomes more complicated if one considers a denser graph. We suppose, that also in this case there are facet-defining knapsack tree inequalities for P_{B}. However, so far we have not been able to identify sufficient conditions which must be fulfilled by the tree supporting the inequality. Here there is certainly room for further research. In the second part of the paper we worked out a version of knapsack related inequalities – the bisection knapsack walk inequalities – which exploit the special bisection case. We took a closer look at their strengthening resulting in capacity reduced bisection knapsack walk inequalities. The right-hand sides of these inequalities may be reduced by exploiting weights of nodes that are not endpoints of walks within the respective inequality. The best possible reduction of the right-hand side is achieved by applying facets of the newly introduced cluster weight polytope P_{CW}. As one would expect, the facial structure of P_{CW} is not trivial due to its relation to the knapsack polytope. We gave the full description of P_{CW} for the case that the complement of the walk in G is a star, all whose nodes fit into the knapsack. Even though this simple case was already challenging we encourage to investigate P_{CW} on more complex graphs than a star and on graphs with an active capacity restriction on the node weight. The practical value of the strengthenings will be investigated in a follow up paper, that also features extensive results on a comparison of semidefinite versus pure linear branch and cut approaches for bisection problems on large sparse graphs.

References

[1] Armbruster, M. (2007): *Branch-and-Cut for a Semidefinite Relaxation of the Minimum Bisection Problem*, PhD-Thesis, Chemnitz University of Technology, Chemnitz 2007.

[2] Barahona, F. (1983): *The max-cut problem in graphs not contractible to K_5*, Operations Research letters, 2, pp. 107-111.

[3] Barahona, F., Mahjoub, A.R. (1986): *On the cut polytope*, Mathematical Programming, 36, pp. 157-173.

[4] Conforti, M., Rao, M.R., Sassano, A. (1990): *The equipartition polytope I*, Mathematical Programming, 49, pp. 49-70.

[5] Conforti, M., Rao, M.R., Sassano, A. (1990): *The equipartition polytope II*, Mathematical Programming, 49, pp. 71-90.

[6] de Souza, C.C. (1993): *The graph equipartition problem: Optimal solutions, extensions and applications*, PhD-Thesis, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.

[7] Deza, M., Laurent, M. (1997): *Geometry of Cuts and Metrics*, volume 15 of Algorithms and Combinatorics, Springer.

[8] Eisenblätter, A. (2001): *Frequency Assignment in GSM Networks*, PhD-Thesis, Technische Universität Berlin, Berlin 2001.

[9] Ferreira, C.E., Martin, A., de Souza, C.C., Weismantel, R., Wolsey, L.A. (1996): *Formulations and valid inequalities for the node capacitated graph partitioning problem*, Mathematical Programming, 74, pp. 247-266.

[10] Fügenschuh, M. (2007): *Relaxations and Solutions for the Minimum Graph Bisection Problem*, PhD-Thesis, Darmstadt University of Technology, Darmstadt 2007.

[11] Garey, M. R., Johnson D.S. (1979): *Computers and Intractability*, W.H. Freeman and Company, New York.

[12] Gawrilow, E., Joswig, M. (2000): *polymake: A Framework for Analyzing Convex Polytopes*, Kalai, G., Ziegler, M. (Eds.): Polytopes — Combinatorics and Computation, Birkhäuser, pp. 43-74. See also *http://www.math.tu-berlin.de/polymake*.

[13] Gilbert, J.R., Tarjan R.E. (1987): *The analysis of a nested dissection algorithm*, Numer. Math., 50, pp. 377-404.

[14] Martello, S., Toth, P. (1990): *Knapsack Problems – Algorithms and Computer Implementations*, John Wiley and Sons Ltd, Chichester.

[15] Laurent, M., de Souza, C.C. (1995): *Some new classes of facets for the equicut polytope*, Discrete Applied Mathematics, 62, pp. 167-191.

[16] Johnson E., Mehrotra, A., Nemhauser, G. (1993): *Min-cut clustering*, Mathematical Programming, 62, pp. 133-152.

[17] Lengauer, T. (1990): *Combinatorial Algorithms for Integrated Circuit Layout*, John Wiley and Sons Ltd, Chichester.

[18] Schrijver, A. (2003): *Combinatorial Optimization – Polyhedra and Efficiency*, Springer-Verlag, Berlin.

[19] Weismantel, R. (1997): *On the 0/1 Knapsack polytope*, Mathematical Programming, 77, pp. 49-68.