

# Model Reduction of Periodic Descriptor Systems Using Balanced Truncation

Peter Benner, Mohammad-Sahadet Hossain, and Tatjana Stykel

**Abstract** Linear periodic descriptor systems represent a broad class of time evolutionary processes in micro-electronics and circuit simulation. In this paper, we consider discrete-time linear periodic descriptor systems and study the concepts of periodic reachability and observability Gramians. We also discuss a lifted representation of periodic descriptor systems and propose a balanced truncation model reduction method for such systems. The behaviour of the suggested model reduction technique is illustrated using a numerical example.

## 1 Introduction

Linear discrete-time periodic descriptor systems have received a lot of attention over the last twenty years. They are suitable models for several natural as well as man-made phenomena, and have applications in modeling of periodic time-varying filters and networks [15, 20], multirate sampled-data systems [15, 17], circuit simulation [2, 7, 9, 17], micro-electronics [18, 19], aerospace realm [32, 33], control of industrial processes and communication systems [1, 19].

A linear discrete-time periodic descriptor system with time-varying dimensions has the form

$$E_k x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k, \quad k \in \mathbb{Z}, \quad (1)$$

where  $E_k \in \mathbb{R}^{\mu_{k+1} \times n_{k+1}}$ ,  $A_k \in \mathbb{R}^{\mu_{k+1} \times n_k}$ ,  $B_k \in \mathbb{R}^{\mu_{k+1} \times m_k}$ ,  $C_k \in \mathbb{R}^{p_k \times n_k}$  are periodic with a period  $K \geq 1$  and  $\sum_{k=0}^{K-1} \mu_k = \sum_{k=0}^{K-1} n_k = n$ . The matrices  $E_k$  are allowed to be

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Peter Benner, Mohammad-Sahadet Hossain  
Fakultät für Mathematik, TU Chemnitz, D-09107 Chemnitz, Germany,  
e-mail: {benner@mathematik.tu-chemnitz.de, mohh@hrz.tu-chemnitz.de}

Tatjana Stykel  
Institut für Mathematik, MA 4-5, TU Berlin, Straße des 17.Juni 136, D-10623 Berlin, Germany,  
e-mail: stykel@math.tu-berlin.de

singular for all  $k$ . For system (1), a reduced-order model of dimension  $r$  would be a system of the form

$$\tilde{E}_k \tilde{x}_{k+1} = \tilde{A}_k \tilde{x}_k + \tilde{B}_k u_k, \quad \tilde{y}_k = \tilde{C}_k \tilde{x}_k, \quad k \in \mathbb{Z}, \quad (2)$$

where  $\tilde{E}_k \in \mathbb{R}^{\gamma_{k+1} \times r_{k+1}}$ ,  $\tilde{A}_k \in \mathbb{R}^{\gamma_{k+1} \times r_k}$ ,  $\tilde{B}_k \in \mathbb{R}^{\gamma_{k+1} \times m_k}$ ,  $\tilde{C}_k \in \mathbb{R}^{p_k \times r_k}$  are  $K$ -periodic matrices,  $\sum_{k=0}^{K-1} \gamma_k = \sum_{k=0}^{K-1} r_k = r$  and  $r \ll n$ . Apart from having a much smaller state-space dimension, it is also important that the reduced-order model preserves physical properties of the original system such as regularity, stability and passivity, and that the approximation error is small.

In this paper, we briefly review some basic concepts of discrete-time periodic descriptor systems (Section 2). In Section 3, we study the periodic reachability and observability Gramians from [5] using a lifted representation. A balanced truncation model reduction method for periodic descriptor systems is presented in Section 4. Section 5 contains a numerical example that illustrates the properties of the suggested model reduction technique.

## 2 Periodic Descriptor Systems

Lifted representations of discrete-time periodic systems play an important role in extending many theoretical results and numerical algorithms for time-invariant systems to the periodic setting [3, 6, 31]. We consider here the cyclic lifted representation which was introduced first for standard periodic systems in [16]. The *cyclic lifted representation* of the periodic descriptor system (1) is given by

$$\mathcal{E} \mathcal{X}_{k+1} = \mathcal{A} \mathcal{X}_k + \mathcal{B} u_k, \quad \mathcal{Y}_k = \mathcal{C} \mathcal{X}_k, \quad (3)$$

where

$$\mathcal{E} = \text{diag}(E_0, E_1, \dots, E_{K-1}), \quad \mathcal{B} = \text{diag}(B_0, B_1, \dots, B_{K-1}),$$

$$\mathcal{A} = \begin{bmatrix} 0 & \cdots & 0 & A_0 \\ A_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & A_{K-1} & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & \cdots & 0 & C_0 \\ C_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & C_{K-1} & 0 \end{bmatrix}. \quad (4)$$

A set of periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  is called *regular* if the pencil  $z\mathcal{E} - \mathcal{A}$  is regular, i.e.,  $\det(z\mathcal{E} - \mathcal{A}) \neq 0$ . In this case, we can define a transfer function of the lifted system (3) as  $\mathcal{H}(z) = \mathcal{C}(z\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$ .

The regular set of periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  can be transformed into a periodic Kronecker canonical form [5, 30]. For  $k = 0, 1, \dots, K-1$ , there exist nonsingular matrices  $W_k \in \mathbb{R}^{\mu_{k+1} \times \mu_{k+1}}$  and  $Z_k \in \mathbb{R}^{n_k \times n_k}$  such that

$$W_k E_k Z_{k+1} = \begin{bmatrix} I_{n_k^f} & 0 \\ 0 & E_k^b \end{bmatrix}, \quad W_k A_k Z_k = \begin{bmatrix} A_k^f & 0 \\ 0 & I_{n_k^\infty} \end{bmatrix}, \quad (5)$$

where  $Z_K = Z_0$ ,  $A_{k+K-1}^f A_{k+K-2}^f \cdots A_k^f = J_k$  is an  $n_k^f \times n_k^f$  matrix corresponding to the finite eigenvalues,  $E_k^b E_{k+1}^b \cdots E_{k+K-1}^b = N_k$  is an  $n_k^\infty \times n_k^\infty$  nilpotent matrix corresponding to the eigenvalue at infinity,  $n_k = n_k^f + n_k^\infty$  and  $\mu_k = n_{k+1}^f + n_k^\infty$ . The index  $\nu$  of the periodic descriptor system (1) is defined as  $\nu = \max(\nu_0, \nu_1, \dots, \nu_{K-1})$ , where  $\nu_k$  is the nilpotency index of  $N_k$ . Note that the finite eigenvalues of  $\{E_k, A_k\}_{k=0}^{K-1}$  coincide with the finite eigenvalues of the lifted pencil  $z\mathcal{E} - \mathcal{A}$ .

For  $k = 0, 1, \dots, K-1$ , the matrices

$$P_r(k) = Z_k \begin{bmatrix} I_{n_k^f} & 0 \\ 0 & 0 \end{bmatrix} Z_k^{-1}, \quad P_l(k) = W_k^{-1} \begin{bmatrix} I_{n_k^f} & 0 \\ 0 & 0 \end{bmatrix} W_k$$

are the *spectral projectors* onto the  $k$ -th right and left deflating subspaces of the periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  corresponding to the finite eigenvalues, and  $Q_r(k) = I - P_r(k)$  and  $Q_l(k) = I - P_l(k)$  are the complementary projectors. Let for every  $k = 0, 1, \dots, K-1$ , the vector  $Z_k^{-1} x_k = [(x_k^f)^T, (x_k^b)^T]^T$  and the matrices

$$W_k B_k = \begin{bmatrix} B_k^f \\ B_k^b \end{bmatrix}, \quad C_k Z_k = [C_k^f, C_k^b]$$

be partitioned in blocks conformally to the periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  in (1). Under this transformation, the system (1) can be decoupled into forward and backward periodic subsystems

$$x_{k+1}^f = A_k^f x_k^f + B_k^f u_k, \quad y_k^f = C_k^f x_k^f, \quad (6)$$

$$E_x^b x_{k+1}^b = x_k^b + B_k^b u_k, \quad y_k^b = C_k^b x_k^b, \quad (7)$$

respectively, with  $y_k = y_k^f + y_k^b$ ,  $k = 0, 1, \dots, K-1$ . The state transition matrix for the forward subsystem (6) is given by  $\Phi_f(i, j) = A_{i-1}^f A_{i-2}^f \cdots A_j^f$  for  $i > j$  and  $\Phi_f(i, i) = I_{n_i^f}$ . For the backward subsystem (7), the state transition matrix is defined as  $\Phi_b(i, j) = E_i^b E_{i+1}^b \cdots E_{j-1}^b$  for  $i < j$  and  $\Phi_b(i, i) = I_{n_i^\infty}$ . Using these matrices we can now define the forward and backward fundamental matrices of the periodic descriptor system (1) as

$$\Psi_{i,j} = \begin{cases} Z_i \begin{bmatrix} \Phi_f(i, j+1) & 0 \\ 0 & 0 \end{bmatrix} W_j, & i > j, \\ Z_i \begin{bmatrix} 0 & 0 \\ 0 & -\Phi_b(i, j) \end{bmatrix} W_j, & i \leq j. \end{cases} \quad (8)$$

These fundamental matrices play an important role in the definition of the reachability and observability Gramians of the periodic descriptor system (1) that we will consider in the next section.

### 3 Periodic Gramians and Matrix Equations

The dynamics of the periodic descriptor system (1) are often addressed by the eigenstructure of the periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$ .

**Definition 1.** The periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  are said to be *periodic stable* (*pd-stable*) if all the finite eigenvalues of  $\{E_k, A_k\}_{k=0}^{K-1}$  lie inside the unit circle.

In balanced truncation model reduction, Gramians play a fundamental role [14, 23, 26, 29]. For the periodic descriptor system (1), the reachability and observability Gramians have been first introduced in [5].

**Definition 2.** Suppose that the periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  are pd-stable. For  $k = 0, 1, \dots, K-1$ , we define the *causal* and *noncausal reachability Gramians*  $G_k^{cr}$  and  $G_k^{ncr}$  of system (1) as

$$G_k^{cr} = \sum_{j=-\infty}^{k-1} \Psi_{k,j} B_j B_j^T \Psi_{k,j}^T, \quad G_k^{ncr} = \sum_{j=k}^{k+\nu K-1} \Psi_{k,j} B_j B_j^T \Psi_{k,j}^T.$$

The *complete reachability Gramian*  $G_k^r$  is the sum of the causal and noncausal Gramians, i.e.,  $G_k^r = G_k^{cr} + G_k^{ncr}$  for  $k = 0, 1, \dots, K-1$ .

**Definition 3.** For the pd-stable matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  and  $k = 0, 1, \dots, K-1$ , the *causal* and *noncausal observability Gramians*  $G_k^{co}$  and  $G_k^{nco}$  of system (1) are defined as

$$G_k^{co} = \sum_{j=k}^{\infty} \Psi_{j,k-1}^T C_j^T C_j \Psi_{j,k-1}, \quad G_k^{nco} = \sum_{j=k-\nu K}^{k-1} \Psi_{j,k-1}^T C_j^T C_j \Psi_{j,k-1}.$$

The *complete observability Gramian*  $G_k^o$  is the sum of the causal and noncausal Gramians, i.e.,  $G_k^o = G_k^{co} + G_k^{nco}$  for  $k = 0, 1, \dots, K-1$ .

Note that the causal and noncausal Gramians correspond to the forward and backward subsystems (6) and (7), respectively.

#### 3.1 Periodic Projected Lyapunov Equations

It has been shown in [24] that the Gramians of discrete-time descriptor systems satisfy projected generalized discrete-time Lyapunov equations with special right-hand sides. A similar result also holds for periodic descriptor systems.

**Proposition 1 ([5]).** Consider a periodic discrete-time descriptor system (1), where the periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  are pd-stable.

1. For  $k = 0, 1, \dots, K-1$ , the causal reachability and observability Gramians  $\{G_k^{cr}\}_{k=0}^{K-1}$  and  $\{G_k^{co}\}_{k=0}^{K-1}$  are the unique symmetric, positive semidefinite solutions of the projected generalized discrete-time periodic Lyapunov equations (PGDPLEs)

$$\begin{aligned} A_k G_k^{cr} A_k^T - E_k G_{k+1}^{cr} E_k^T &= -P_l(k) B_k B_k^T P_l(k)^T, \\ G_k^{cr} &= P_r(k) G_k^{cr} P_r(k)^T, \end{aligned} \quad (9)$$

and

$$\begin{aligned} A_k^T G_{k+1}^{co} A_k - E_{k-1}^T G_k^{co} E_{k-1} &= -P_r(k)^T C_k^T C_k P_r(k), \\ G_k^{co} &= P_l(k-1)^T G_k^{co} P_l(k-1), \end{aligned} \quad (10)$$

respectively, where  $G_K^{cr} = G_0^{cr}$ ,  $G_K^{co} = G_0^{co}$ ,  $E_{-1} = E_{K-1}$ , and  $P_l(-1) = P_l(K-1)$ .

2. For  $k = 0, 1, \dots, K-1$ , the noncausal reachability and observability Gramians  $\{G_k^{ncr}\}_{k=0}^{K-1}$  and  $\{G_k^{nco}\}_{k=0}^{K-1}$  are the unique symmetric, positive semidefinite solutions of the PGDPLEs

$$\begin{aligned} A_k G_k^{ncr} A_k^T - E_k G_{k+1}^{ncr} E_k^T &= Q_l(k) B_k B_k^T Q_l(k)^T, \\ G_k^{ncr} &= Q_r(k) G_k^{ncr} Q_r(k)^T, \end{aligned} \quad (11)$$

and

$$\begin{aligned} A_k^T G_{k+1}^{nco} A_k - E_{k-1}^T G_k^{nco} E_{k-1} &= Q_r(k)^T C_k^T C_k Q_r(k), \\ G_k^{nco} &= Q_l(k-1)^T G_k^{nco} Q_l(k-1), \end{aligned} \quad (12)$$

respectively, where  $G_K^{ncr} = G_0^{ncr}$ ,  $G_K^{nco} = G_0^{nco}$  and  $Q_l(-1) = Q_l(K-1)$ .

Numerical solution of the PGDPLEs has been considered in [5]. The method proposed there extends the periodic Schur method [4, 27, 28] and the generalized Schur-Hammarling method [22] developed for periodic standard and projected generalized Lyapunov equations, respectively. This method is based on an initial reduction of the periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  to the generalized periodic Schur form [10, 30] and solving the resulting generalized periodic Sylvester and Lyapunov equations of (quasi)-triangular structure using the recursive block algorithms from [8]. Due to the computational complexity, the periodic generalized Schur-Hammarling method is restricted to problems of small and medium size. In [11, 27, 25], iterative methods based on Smith iterations [21] have been developed for periodic standard Lyapunov equations and also for projected generalized Lyapunov equations. These methods can also be extended to periodic projected Lyapunov equations by using the lifted representation of these equations. Therefore, we will discuss these lifted representations in the following subsection.

### 3.2 Lifted Representation of Periodic Lyapunov Equations

It is known that the Gramians of standard periodic systems satisfy the lifted form of the periodic Lyapunov equations and the solutions of these equations are block diagonal matrices [11, 27].

The following theorem describes the block structures of the solutions of periodic Lyapunov equations in lifted form and their relations to the corresponding solutions of the PGDPLEs in Proposition 1.

**Theorem 1.** *Consider the periodic discrete-time descriptor system (1) and its lifted representation (3), where the periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  are pd-stable. The causal and noncausal reachability Gramians  $\mathcal{G}^{cr}$  and  $\mathcal{G}^{ncr}$  satisfy the lifted projected Lyapunov equations*

$$\begin{aligned} \mathcal{A}\mathcal{G}^{cr}\mathcal{A}^T - \mathcal{E}\mathcal{G}^{cr}\mathcal{E}^T &= -\mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T, & \mathcal{G}^{cr} &= \mathcal{P}_r\mathcal{G}^{cr}\mathcal{P}_r^T, \\ \mathcal{A}\mathcal{G}^{ncr}\mathcal{A}^T - \mathcal{E}\mathcal{G}^{ncr}\mathcal{E}^T &= -\mathcal{Q}_l\mathcal{B}\mathcal{B}^T\mathcal{Q}_l^T, & \mathcal{G}^{ncr} &= \mathcal{Q}_r\mathcal{G}^{ncr}\mathcal{Q}_r^T, \end{aligned} \quad (13)$$

respectively, where  $\mathcal{E}$ ,  $\mathcal{A}$ ,  $\mathcal{B}$  are as in (4) and

$$\begin{aligned} \mathcal{G}^{cr} &= \text{diag}(G_1^{cr}, \dots, G_{K-1}^{cr}, G_0^{cr}), & \mathcal{G}^{ncr} &= \text{diag}(G_1^{ncr}, \dots, G_{K-1}^{ncr}, G_0^{ncr}), \\ \mathcal{P}_l &= \text{diag}(P_l(0), P_l(1), \dots, P_l(K-1)), & \mathcal{Q}_l &= I - \mathcal{P}_l, \\ \mathcal{P}_r &= \text{diag}(P_r(0), P_r(1), \dots, P_r(K-1)), & \mathcal{Q}_r &= I - \mathcal{P}_r. \end{aligned}$$

*Proof.* We will only sketch the proof due to space limitations. It follows from [5, 12] by writing out the first equation in (13) in block-structured form. A straightforward calculation shows that the resulting system of equations is equivalent to the periodic projected Lyapunov equations of Proposition 1. Since the periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  are pd-stable, the matrix pencil  $z\mathcal{E} - \mathcal{A}$  is regular and all its eigenvalues lie inside the unit circle. Thus, the projected discrete-time Lyapunov equation in (13) has a unique solution [22]. The proof for  $\mathcal{G}^{ncr}$  is similar.  $\square$

For the observability Gramians, the situation becomes a bit more complex. In that case we make a backward time-shift of the original system [27, 29]. The reason is that we do not want to destroy the block diagonal structure of the lifted solutions and we would like to use the lifted solution to find a balanced realization of the original system.

**Theorem 2.** *Consider the periodic discrete-time descriptor system and its lifted representation as in Theorem 1. The causal and noncausal observability Gramians  $\mathcal{G}^{co}$  and  $\mathcal{G}^{nco}$  satisfy the lifted projected Lyapunov equations*

$$\begin{aligned} \sigma\mathcal{E}\mathcal{G}^{co}\sigma\mathcal{E}^T - \sigma\mathcal{A}\mathcal{G}^{co}\sigma\mathcal{A}^T &= \mathcal{P}_r^T\sigma\mathcal{C}^T\sigma\mathcal{C}\mathcal{P}_r, & \mathcal{G}^{co} &= \mathcal{P}_l^T\mathcal{G}^{co}\mathcal{P}_l, \\ \sigma\mathcal{E}\mathcal{G}^{nco}\sigma\mathcal{E}^T - \sigma\mathcal{A}\mathcal{G}^{nco}\sigma\mathcal{A}^T &= \mathcal{Q}_r^T\sigma\mathcal{C}^T\sigma\mathcal{C}\mathcal{Q}_r, & \mathcal{G}^{nco} &= \mathcal{Q}_l^T\mathcal{G}^{nco}\mathcal{Q}_l, \end{aligned} \quad (14)$$

respectively, where  $\sigma\mathcal{E} = \text{diag}(E_{K-1}, E_0, \dots, E_{K-2})$  is the backward time-shift of  $\mathcal{E}$ ,  $\sigma\mathcal{C} = \text{diag}(C_0, C_1, \dots, C_{K-1})$  and

$$\sigma\mathcal{A} = \begin{bmatrix} 0 & \cdots & 0 & A_{K-1} \\ A_0 & & & 0 \\ & \ddots & & \vdots \\ 0 & & A_{K-2} & 0 \end{bmatrix},$$

$$\mathcal{G}^{co} = \text{diag}(G_0^{co}, G_1^{co}, \dots, G_{K-1}^{co}), \quad \mathcal{G}^{nco} = \text{diag}(G_0^{nco}, G_1^{co}, \dots, G_{K-1}^{nco}).$$

*Proof.* The proof is analogous to the previous proof of Theorem 2.  $\square$

Note that the lifted solutions  $\mathcal{G}^{co}$  and  $\mathcal{G}^{nco}$  also have block diagonal structure, but now the diagonal blocks appear in a different order.

### 3.3 Hankel Singular Values

Let the periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  be pd-stable. Then the causal and non-causal matrices  $M_k^c = G_k^{cr} E_{k-1}^T G_k^{co} E_{k-1}$  and  $M_k^{nc} = G_k^{ncr} A_k^T G_{k+1}^{nco} A_k$ ,  $k=0, 1, \dots, K-1$ , have real and nonnegative eigenvalues. These eigenvalues are used to define the causal and noncausal Hankel singular values of system (1).

**Definition 4.** Let the set of periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  be pd-stable. For  $k = 0, 1, \dots, K-1$ , the square roots of the largest  $n_k^f$  eigenvalues of the matrix  $M_k^c$ , denoted by  $\sigma_{k,j}$ , are called the *causal Hankel singular values*, and the square roots of the largest  $n_k^\infty$  eigenvalues of  $M_k^{nc}$ , denoted by  $\theta_{k,j}$ , are called the *noncausal Hankel singular values* of the periodic descriptor system (1).

Since the causal and noncausal reachability and observability Gramians are symmetric and positive semidefinite, there exist the Cholesky factorizations

$$G_k^{cr} = R_k R_k^T, \quad G_k^{co} = L_k^T L_k, \quad G_k^{ncr} = \check{R}_k \check{R}_k^T, \quad G_k^{nco} = \check{L}_k^T \check{L}_k, \quad (15)$$

Simple calculations show that  $\sigma_{k,j} = \zeta_j(L_k E_{k-1} R_k)$  and  $\theta_{k,j} = \zeta_j(\check{L}_{k+1} A_k \check{R}_k)$ , where  $\zeta_j(\cdot)$  denotes the singular values of the corresponding matrices.

## 4 Balanced Truncation Model Reduction

In this section, we present a generalization of a balanced truncation model reduction method for periodic descriptor systems. For a balanced system, the reachability and observability Gramians are both equal to a diagonal matrix [14, 23]. Balanced realizations for periodic descriptor systems have also been considered in [5].

**Definition 5.** A realization  $(E_k, A_k, B_k, C_k)$  of a periodic descriptor system (1) is called *balanced* if

$$G_k^{cr} = G_k^{co} = \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix}, \quad G_k^{ncr} = G_{k+1}^{nco} = \begin{bmatrix} 0 & 0 \\ 0 & \Theta_k \end{bmatrix},$$

where  $\Sigma_k = \text{diag}(\sigma_{k,1}, \dots, \sigma_{k,n_k^f})$  and  $\Theta_k = \text{diag}(\theta_{k,1}, \dots, \theta_{k,n_k^\infty})$ ,  $k = 0, 1, \dots, K-1$ .

Consider the Cholesky factorizations (15) of the reachability and observability Gramians and let

$$L_k E_{k-1} R_k = U_k \Sigma_k V_k^T, \quad \check{L}_{k+1} A_k \check{R}_k = \check{U}_k \Theta_k \check{V}_k^T \quad (16)$$

be the singular value decompositions of the matrices  $L_k E_{k-1} R_k$  and  $\check{L}_{k+1} A_k \check{R}_k$  for  $k = 0, 1, \dots, K-1$ . Here  $U_k, V_k, \check{U}_k, \check{V}_k$  are orthogonal, and  $\Sigma_k$  and  $\Theta_k$  are diagonal. If a realization  $(E_k, A_k, B_k, C_k)$  with pd-stable matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  is minimal, i.e.,  $\Sigma_k$  and  $\Theta_k$  are nonsingular, then there exist nonsingular periodic matrices

$$S_k = [L_{k+1}^T U_{k+1} \Sigma_{k+1}^{-1/2}, \check{L}_{k+1}^T \check{U}_k \Theta_k^{-1/2}], \quad T_k = [R_k V_k \Sigma_k^{-1/2}, \check{R}_k \check{V}_k \Theta_k^{-1/2}],$$

such that the transformed realization  $(S_k^T E_k T_{k+1}, S_k^T A_k T_k, S_k^T B_k, C_k T_k)$  is balanced [5]. Note that as in the case of standard state-space systems, the balancing transformation matrices for periodic discrete-time descriptor system (1) are not unique.

Model reduction via balanced truncation is discussed very widely for standard discrete-time periodic systems [11, 29] and also for continuous-time descriptor systems [13, 23]. For a balanced system, truncation of states related to the small causal Hankel singular values does not change system properties essentially. Unfortunately, we can not do the same for the noncausal Hankel singular values. If we truncate the states that correspond to the small non-zero noncausal Hankel singular values, then the pencil for the reduced-order system may have finite eigenvalues outside the unit circle that will lead to additional errors in the system approximation.

Assume that the periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  are pd-stable. Consider the Cholesky factorizations in (15). Let

$$L_k E_{k-1} R_k = [U_{k,1}, U_{k,2}] \begin{bmatrix} \Sigma_{k,1} & \\ & \Sigma_{k,2} \end{bmatrix} [V_{k,1}, V_{k,2}]^T, \quad \check{L}_{k+1} A_k \check{R}_k = \check{U}_k \Theta_k \check{V}_k^T,$$

be singular value decompositions of  $L_k E_{k-1} R_k$  and  $\check{L}_{k+1} A_k \check{R}_k$ , where

$$\Sigma_{k,1} = \text{diag}(\sigma_{k,1}, \dots, \sigma_{k,r_k^f}), \quad \Sigma_{k,2} = \text{diag}(\sigma_{k,r_k^f+1}, \dots, \sigma_{k,n_k^f}),$$

with  $\sigma_{k,1} \geq \dots \geq \sigma_{k,r_k^f} > \sigma_{k,r_k^f+1} \geq \dots \geq \sigma_{k,n_k^f} > 0$ , and  $\Theta_k = \text{diag}(\theta_{k,1}, \dots, \theta_{k,r_k^\infty})$  is nonsingular for  $k = 0, 1, \dots, K-1$ . Then the reduced-order system (2) can be computed as

$$\tilde{E}_k = S_{k,r}^T E_k T_{k+1,r}, \quad \tilde{A}_k = S_{k,r}^T A_k T_{k,r}, \quad \tilde{B}_k = S_{k,r}^T B_k, \quad \tilde{C}_k = C_k T_{k,r}, \quad (17)$$

where

$$S_{k,r} = [L_{k+1}^T U_{k+1,1} \Sigma_{k+1,1}^{-1/2}, \check{L}_{k+1}^T \check{U}_k \Theta_k^{-1/2}] \in \mathbb{R}^{\mu_{k+1}, r_{k+1}},$$

$$T_{k,r} = [R_k V_{k,1} \Sigma_{k,1}^{-1/2}, \check{R}_k \check{V}_k \Theta_k^{-1/2}] \in \mathbb{R}^{n_k, r_k},$$

with  $r_k = r_k^f + r_k^\infty$ .

Table 1: Norms and residuals of the reachability Gramians

k	$\ G_k^{cr}\ _2$	$\rho_k^{cr}$	$\ G_k^{ncr}\ _2$	$\rho_k^{ncr}$
0	$5.8182 \times 10^2$	$6.1727 \times 10^{-12}$	$1.3946 \times 10^1$	$1.5444 \times 10^{-14}$
1	$8.2981 \times 10^4$	$8.2172 \times 10^{-12}$	$1.3660 \times 10^1$	$1.7508 \times 10^{-14}$
2	$7.1107 \times 10^3$	$3.0961 \times 10^{-12}$	$1.4308 \times 10^1$	$3.3847 \times 10^{-14}$

**Theorem 3.** Let  $\mathcal{H}(z)$  be the transfer function of the lifted descriptor system (3), (4) and let  $\tilde{\mathcal{H}}(z)$  be the transfer function of the reduced-order lifted system formed from the reduced-order periodic descriptor system (2), (17). Then we have the following  $\mathbb{H}_\infty$ -norm error bound:

$$\|\mathcal{H} - \tilde{\mathcal{H}}\|_{\mathbb{H}_\infty} = \sup_{\omega \in [0, 2\pi]} \|\mathcal{H}(e^{i\omega}) - \tilde{\mathcal{H}}(e^{i\omega})\|_2 \leq 2 \sum_{k=0}^{K-1} \text{trace}(\Sigma_{k,2}), \quad (18)$$

where  $\|\cdot\|_2$  denotes the spectral norm of matrices and  $\Sigma_{k,2}$  contains the truncated causal Hankel singular values.

This error bound can be obtained similarly to the standard state space case [12, 29].

## 5 A Numerical Example

We consider the periodic discrete-time descriptor system with  $n = 10$ ,  $m = 2$ ,  $p = 3$ , and period  $K = 3$  presented in [5, Example 1]. We repeat some of the calculations from [5] using standard algorithms applied to the projected Lyapunov equations (13) and (14) in order to verify the validity of the lifted representation, while for the future, we plan to use the lifted representation in order to develop algorithms for large-scale, sparse problems based on [25]. We also show results for the balanced truncation approach discussed in Section 4.

The periodic matrix pairs  $\{E_k, A_k\}_{k=0}^{K-1}$  are pd-stable with  $n_k^f = 8$  and  $n_k^\infty = 2$  for  $k = 0, 1, 2$ . The norms of the computed solutions of the periodic Lyapunov equations and the corresponding residuals, e.g.,

$$\rho_k^{cr} = \|A_k G_k^{cr} A_k^T - E_k G_{k+1}^{cr} E_k^T + P_l(k) B_k B_k^T P_l(k)^T\|_2,$$

are shown in Table 1 and Table 2.

The original lifted system has order  $n = 30$ . Figure 1(a) shows the causal Hankel singular values for the different systems correspond to  $k = 0, 1, 2$ . We see that they decay fast, and, hence system (1) can be well approximated by a model of reduced order. We have 24 causal Hankel singular values for the original lifted system and the remaining 6 are noncausal Hankel singular values which are positive. We approximate system (1) to a tolerance  $10^{-2}$  by truncating the states corresponding to the smallest 7 causal Hankel singular values.

Table 2: Norms and residuals of the observability Gramians

k	$\ G_k^{co}\ _2$	$\rho_k^{co}$	$\ G_k^{nco}\ _2$	$\rho_k^{nco}$
0	$9.7353 \times 10^1$	$2.7678 \times 10^{-13}$	$1.6866 \times 10^0$	$1.3372 \times 10^{-15}$
1	$1.1373 \times 10^3$	$7.7003 \times 10^{-14}$	$1.7406 \times 10^0$	$2.1113 \times 10^{-15}$
2	$9.6984 \times 10^0$	$1.7859 \times 10^{-14}$	$1.6866 \times 10^0$	$1.1626 \times 10^{-15}$

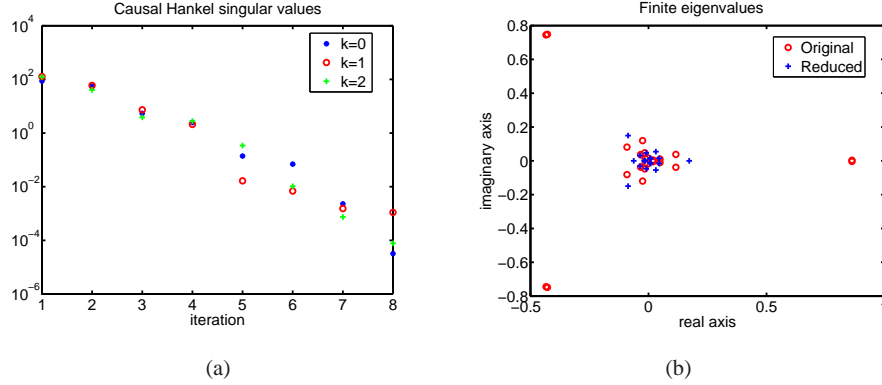


Fig. 1: (a) Causal Hankel singular values of different subsystems, (b) finite eigenvalues of the original and the reduced-order lifted systems.

Figure 1(b) shows the finite eigenvalues of the original and reduced-order lifted systems. We observe that stability is preserved for the reduced-order system. In Figure 2(a), we present the norms of the frequency responses  $\mathcal{H}(e^{i\omega})$  and  $\tilde{\mathcal{H}}(e^{i\omega})$  of the original and reduced-order lifted systems for a frequency range  $[0, 2\pi]$ . We observe good matching of the system norms.

In Figure 2(b) we display the absolute error  $\|\mathcal{H}(e^{i\omega}) - \tilde{\mathcal{H}}(e^{i\omega})\|_2$  and the error bound (18). One can see that the absolute error is smaller than the error bound.

## 6 Conclusions

In this paper, we have considered lifted representations of the reachability and observability Gramians as well as Hankel singular values of periodic descriptor systems. For such systems, a balanced truncation model reduction method has been presented. The proposed method delivers a reduced-order model that preserves the regularity and stability properties of the original system. A computable global error bound for the approximate system is also available. In the future, we plan to use the lifted representation in order to develop algorithms for large-scale, sparse problems based on the Smith iteration for large-scale projected Lyapunov equations so that problems of larger sizes than the toy example in Section 5 can be treated.

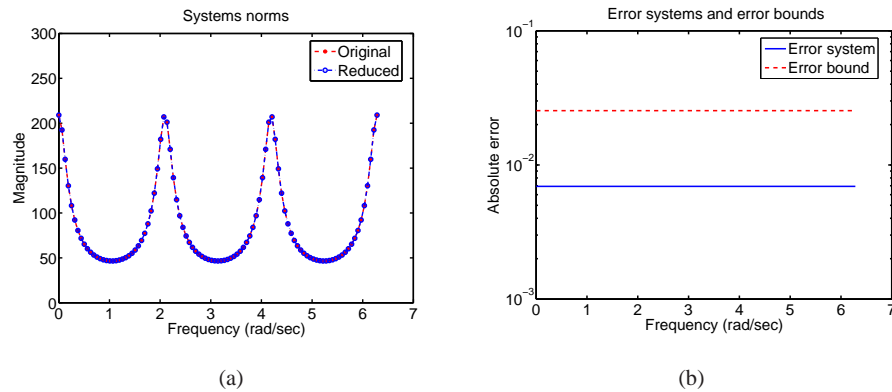


Fig. 2: (a) The frequency responses of the original and the reduced-order lifted systems; (b) absolute error and error bound.

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