SOME RESULTS ON SURJECTIVITY OF AUGMENTED SEMI-ELLIPTIC DIFFERENTIAL OPERATORS

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ABSTRACT. We show that for a semi-elliptic polynomial P on $\mathbb{R}^2$ surjectivity of $P(D)$ on $\mathscr{D}'(\Omega)$ implies surjectivity of the augmented operator $P^+(D)$ on $\mathscr{D}'(\Omega\times\mathbb{R})$, where $P^+(x_1,x_2,x_3) := P(x_1,x_2)$. For arbitrary dimension n we give a sufficient geometrical condition on $\Omega\subset\mathbb{R}^n$ such that an analogous implication is true for semi-elliptic P. Moreover, we give an alternative proof of a result due to Vogt which says that for elliptic P the operator $P^+(D)$ is surjective if this is true for $P(D)$.

1. INTRODUCTION

Let $\Omega\subset\mathbb{R}^n$ be open and $P\in\mathbb{C}[X_1,\ldots,X_n]$ be a non-zero polynomial. Consider the corresponding differential operator $P(D)$, where as usual $D_j = -i\frac{\partial}{\partial x_j}$, acting on $\mathscr{D}'(\Omega)$. We denote by $P^+(D)$ the augmented operator, i.e. $P(D)$ acting "on the first n variables" on $\mathscr{D}'(\Omega\times\mathbb{R})$.

In [1, Problem 9.1] it is asked if it is true that $P^+(D)$ is surjective if $P(D)$ is surjective (not only on the space of ordinary distributions over Ω but more general for ultradistributions of Beurling type). This question is closely connected with the parameter dependence of solutions of the differential equation

$$P(D)u_\lambda = f_\lambda,$$

see [1]. It is shown in [1, Proposition 8.3] that the answer to the above question is in the affirmative, if and only if $\mathscr{N}_P(\Omega)$, the kernel of the operator, possesses the linear topological invariant $(P\Omega)$. It was shown by Vogt [3, Proposition 2.5] that $\mathscr{N}_P(\Omega)$ has $(P\Omega)$ if the polynomial P is elliptic (in this case the property $(P\Omega)$ equals the linear topological invariant (Ω)).

The paper is organized as follows. In section 2 we show that the above problem is equivalent to the question whether P-convexity for supports as well as for singular supports of Ω implies P^+-convexity for singular supports of $\Omega\times\mathbb{R}$. Moreover, we observe that due to the fact that P^+ carries a muted variable it is easier to evaluate a certain numerical quantity $\sigma_{P^+}(W)$ for subspaces W which arises in the theory of continuation of differentabilty due to Hörmander. Based on this observation we consider semi-elliptic polynomials P and characterize those subspaces W for which $\sigma_{P^+}(W)=0$ in section 3. This knowledge together with sufficient conditions for P-convexity given in section 4 enable us to present an alternative proof of the above mentioned result of Vogt in section 5, as well as a positive answer to the problem for semi-elliptic polynomials if $\Omega\subset\mathbb{R}^2$ or if Ω satisfies a certain additional "geometric" property in case of $n>2$.

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2. Preliminaries

As is well-known, for a non-zero polynomial $P \in \mathbb{C}[X_1, \ldots, X_n]$ the differential operator $P(D)$ is surjective on $\mathscr{D}'(\Omega)$ if and only if Ω is P-convex for supports as well as P-convex for singular supports, i.e. for each compact subset K of Ω there is another compact subset L of Ω such that for all $\phi \in \mathscr{D}(\Omega)$ one has supp $P(-D)\phi \subset L$ whenever supp $\phi \subset K$, resp. for all $\mu \in \mathscr{E}'(\Omega)$ one has sing supp $P(-D)\mu \subset L$ whenever sing supp $\mu \subset K$.

Therefore, the problem whether $P^+(D)$ is surjective on $\mathscr{D}'(\Omega \times \mathbb{R})$ if $P(D)$ is surjective on $\mathscr{D}'(\Omega)$ is equivalent to the problem if $\Omega \times \mathbb{R}$ is P^+-convex for supports as well as P^+-convex for singular supports if Ω is P-convex for supports and P-convex for singular supports. As we will see, P-convexity for supports is trivial.

Proposition 1. *Let $P \in \mathbb{C}[X_1, \ldots, X_n]$ and $\Omega \subseteq \mathbb{R}^n$ be open such that Ω is P-convex for supports. Then $\Omega \times \mathbb{R}$ is P^+-convex for supports.*

Proof. Let $K \subset \Omega$ and $K' \subset \mathbb{R}$ be compact. Ω being P-convex for supports there is a compact subset L of Ω such that for every $\phi \in \mathscr{D}(\Omega)$ satisfying supp $P(-D)\phi \subset K$ already supp $\phi \subset L$ holds. If $\phi \in \mathscr{D}(\Omega \times \mathbb{R})$ is of the form $\phi(x, s) = \phi_1(x)\phi_2(s)$ with $\phi_1 \in \mathscr{D}(\Omega)$ and $\phi_2 \in \mathscr{D}(\mathbb{R})$ obviously $P^+(-D)\phi = (P(-D)\phi_1)\phi_2$ so that supp $P^+(-D)\phi \subset K \times K'$ implies supp $\phi \subset L \times K'$. Since functions of the form $\phi = \phi_1\phi_2$ span a dense linear subspace in $\mathscr{D}(\Omega \times \mathbb{R})$ the proposition follows. □

An alternative proof of the above proposition can be given by using tensor products. That an analogous implication for P-convexity for singular supports is not true in general is shown in Example 9 below. Hence the original problem is equivalent to whether P-convexity for supports as well as P-convexity for singular supports of Ω imply P^+-convexity for singular supports of $\Omega \times \mathbb{R}$.

Recalling that Ω is P-convex for supports if and only if $P(D) : \mathscr{E}(\Omega) \to \mathscr{E}(\Omega)$ is surjective we obtain the following result as an immediate consequence.

Corollary 2. *Let $P \in \mathbb{C}[X_1, \ldots, X_n]$ and $\Omega \subseteq \mathbb{R}^n$ be open. If $P(D) : \mathscr{E}(\Omega) \to \mathscr{E}(\Omega)$ is surjective then $P^+(D) : \mathscr{E}(\Omega \times \mathbb{R}) \to \mathscr{E}(\Omega \times \mathbb{R})$ is surjective.*

In order to deal with P^+-convexity for singular supports, we will use the following notion introduced by Hörmander in connection with continuation of differentiability (cf. [2, Section 11.3, vol. II]). For a subspace V of $\mathbb{R}^n$

$$\sigma_P(V) = \inf_{t>1} \liminf_{\xi \to \infty} \tilde{P}_V(\xi, t)/\tilde{P}(\xi, t),$$

where $\tilde{P}_V(\xi, t) := \sup\{|P(\xi + \eta)|;\, \eta \in V, |\eta| \leq t\}$, $\tilde{P}(\xi, t) := \tilde{P}_{\mathbb{R}^n}(\xi, t)$. This quantity is intimately connected with the so called localizations at infinity of the polynomial P which in turn are related to the bounds for the wave front set and singular support of a regular fundamental solution of P. Roughly speaking, $\sigma_P(V) \neq 0$ implies that regularity of $P(D)u$ continues along the subspace V to regularity of u (cf. [2, Theorem 11.3.6, vol. II]).

The way we will use $\sigma_P(V)$ is given by the following result which is nothing but a reformulation of [2, Corollary 11.3.7, vol. II].

Corollary 3. *Let $\Omega_1 \subset \Omega_2$ be open and convex, and let P be a non-constant polynomial. Then the following are equivalent:*

- i) *If $u \in \mathscr{D}'(\Omega_2)$ satisfies $P(D)u \in C^\infty(\Omega_2)$ as well as sing supp $u \subset \Omega_2 \backslash \Omega_1$ then sing supp $u = \emptyset$.*
- ii) *Every hyperplane $H = \{x; \langle x, N\rangle = \alpha\}$ with $\sigma_P(span\{N\}) = 0$ which intersects Ω_2 already intersects Ω_1.*

Proof. That i) implies ii) is just a special case of [2, Corollary 11.3.7, vol. II]. Let $u \in \mathscr{D}'(\Omega_2)$ satisfy $P(D)u \in C^\infty(\Omega_2)$ as well as $u|_{\Omega_1} \in C^\infty(\Omega_1)$. By the convexity of Ω_2 we find $v \in C^\infty(\Omega_2)$ such that $P(D)v = P(D)u$. Therefore $w := u - v \in \mathscr{D}'(\Omega_2)$ satisfies $P(D)w = 0$ and $w|_{\Omega_1} \in C^\infty(\Omega_1)$. Now it follows from ii) and [2, Corollary 11.3.7, vol. II] that $w \in C^\infty(\Omega_2)$, thus $u \in C^\infty(\Omega_2)$. □

So, for us it will be important to know for which (one-dimensional) subspace W of $\mathbb{R}^{n+1}$ we have $\sigma_{P^+}(W) = 0$. The next lemma will be very helpful in this.

Lemma 4. *Let $P \in \mathbb{C}[X_1, \ldots, X_n]$ and let Π be the orthogonal projection of $\mathbb{R}^{n+1}$ onto the first n coordinates. For a subspace W of $\mathbb{R}^{n+1}$ we identify $W' := \Pi(W)$ with the corresponding subspace of R^n. Then the following hold.*

i)

$$\sigma_{P^+}(W' \times \{0\}) = \sigma_{P^+}(W' \times \mathbb{R}) = \inf_{t>1, \xi\in\mathbb{R}^n} \frac{\tilde{P}_{W'}(\xi,t)}{\tilde{P}(\xi,t)}.$$

ii) $\sigma_{P^+}(W) = 0$ *if and only if* $\inf_{t>1,\xi\in\mathbb{R}^n} \frac{\tilde{P}_{W'}(\xi,t)}{\tilde{P}(\xi,t)} = 0$.

Proof. We write $x = (x', x_{n+1})$ for $x \in W$ with $x' \in \mathbb{R}^n$ and $x_{n+1} \in \mathbb{R}$.
By definition we have for $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}$

$$\begin{aligned}
\tilde{P}^+_{W'\times\mathbb{R}}((\xi,\eta),t) &= \sup\{|P(\xi + x')|;\ (x', x_{n+1}) \in W' \times \mathbb{R}, |(x', x_{n+1})| \le t\} \\
&= \sup\{|P(\xi + x')|;\ x' \in W', |x'| \le t\} \\
&= \tilde{P}_{W'}(\xi,t) = \tilde{P}^+_{W'\times\{0\}}((\xi,\eta),t).
\end{aligned}$$

In particular, this implies

$$\tilde{P}^+((\xi,\eta),t) = \tilde{P}(\xi,t).$$

Hence

$$\begin{aligned}
\liminf_{(\xi,\eta)\to\infty} \frac{\tilde{P}^+_{W'\times\mathbb{R}}((\xi,\eta),t)}{\tilde{P}^+((\xi,\eta),t)} &= \sup_{r>0} \inf_{|(\xi,\eta)|>r} \frac{\tilde{P}^+_{W'\times\mathbb{R}}((\xi,\eta),t)}{\tilde{P}^+((\xi,\eta),t)} \\
&= \sup_{r>0} \inf_{|(\xi,\eta)|>r} \frac{\tilde{P}_{W'}(\xi,t)}{\tilde{P}(\xi,t)} \\
&= \inf_{\xi\in\mathbb{R}^n} \frac{\tilde{P}_{W'}(\xi,t)}{\tilde{P}(\xi,t)}
\end{aligned}$$

as well as

$$\liminf_{(\xi,\eta)\to\infty} \frac{\tilde{P}^+_{W'\times\{0\}}((\xi,\eta),t)}{\tilde{P}^+((\xi,\eta),t)} = \inf_{\xi\in\mathbb{R}^n} \frac{\tilde{P}_{W'}(\xi,t)}{\tilde{P}(\xi,t)}$$

which gives

$$\sigma_{P^+}(W' \times \mathbb{R}) = \inf_{t>1} \liminf_{(\xi,\eta)\to\infty} \frac{\tilde{P}^+_W((\xi,\eta),t)}{\tilde{P}^+((\xi,\eta),t)} = \inf_{t>1,\xi\in\mathbb{R}^n} \frac{\tilde{P}_{W'}(\xi,t)}{\tilde{P}(\xi,t)},$$

as well as

$$\sigma_{P^+}(W' \times \{0\}) = \inf_{t>1,\xi\in\mathbb{R}^n} \frac{\tilde{P}_{W'}(\xi,t)}{\tilde{P}(\xi,t)}.$$

Thus i) is proved.

In order to prove ii) assume first that W is contained in the kernel of Π, i.e. $W \subset \{0\} \times \mathbb{R}$. Then we have for $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}$

$$\tilde{P}^+_W((\xi,\eta),t) = \sup\{|P(\xi)|; (0, x_{n+1}) \in W, |x_{n+1}| \le t\} = |P(\xi)| = \tilde{P}_{W'}(\xi,t).$$

As in the proof of i) it then follows that

$$\sigma_{P^+}(W) = \inf_{t>1,\xi\in\mathbb{R}^n} \frac{\tilde{P}_{W'}(\xi,t)}{\tilde{P}(\xi,t)}.$$

Hence, without loss of generality, let $W \not\subseteq \{0\}\times\mathbb{R}$. Then, by setting $p_1 := \|\Pi_{|W}\|$ we get $p_1 > 0$ as well as

$$\begin{aligned}\tilde{P}^+_W((\xi,\eta),t) &= \sup\{|P(\xi+x')|;\ (x',x_{n+1})\in W, |(x',x_{n+1})|\leq t\}\\ &\leq \sup\{|P(\xi+x')|;\ x'\in W', |x'|\leq tp_1\}\\ &= \tilde{P}_{W'}(\xi,tp_1).\end{aligned}$$

Now we distinguish two cases. If $\Pi_{|W}: W\to W'$ is not injective we clearly have $\{(0,y);\ y\in\mathbb{R}\}\subset W$. Therefore, recalling that Π as an orthogonal projection satisfies $p_1 = \|\Pi_{|W}\|\leq\|\Pi\|\leq 1$

$$\sup\{|P(\xi+x')|; x'\in W', |x'|\leq tp_1\} = \sup\{|P(\xi+x')|; (x',x_{n+1})\in W, |(x',x_{n+1})|\leq t\}$$

because if $x_0'\in W'$ with $|x_0'|\leq tp_1$ is a point where the supremum on the left hand side is attained then $(x_0',0)\in W$ with $|(x_0',0)|\leq t$. Therefore

$$\tilde{P}_{W'}(\xi,tp_1) = \tilde{P}^+_W((\xi,\eta),t).$$

In case of $\Pi_{|W}: W\to W'$ being injective $(\Pi_{|W})^{-1}: W'\to W$ is well-defined and continuous and we get

$$\begin{aligned}\tilde{P}_{W'}(\xi,t\,\|(\Pi_{|W})^{-1}\|^{-1}) &= \sup\{|P(\xi+x')|;\ x'\in W', |x'|\leq t\,\|(\Pi_{|W})^{-1}\|^{-1}\}\\ &\leq \sup\{|P(\xi+x')|;\ (x',x_{n+1})\in W, |(x',x_{n+1})|\leq t\}\\ &= \tilde{P}^+_W((\xi,\eta),t).\end{aligned}$$

Hence, in both cases there are $p_1,p_2>0$ such that

$$\tilde{P}_{W'}(\xi,tp_2)\leq\tilde{P}^+_W((\xi,\eta),t)\leq\tilde{P}_{W'}(\xi,tp_1)$$

for all $\xi\in\mathbb{R}^n,\eta\in\mathbb{R},t\geq 1$. Altogether this yields

$$\inf_{\xi\in\mathbb{R}^n}\frac{\tilde{P}_{W'}(\xi,tp_2)}{\tilde{P}(\xi,t)}\leq\liminf_{(\xi,\eta)\to\infty}\frac{\tilde{P}^+_W((\xi,\eta),t)}{\tilde{P}^+((\xi,\eta),t)}\leq\inf_{\xi\in\mathbb{R}^n}\frac{\tilde{P}_{W'}(\xi,tp_1)}{\tilde{P}(\xi,t)},$$

so that

$$\inf_{t\geq 1,\xi\in\mathbb{R}^n}\frac{\tilde{P}_{W'}(\xi,tp_2)}{\tilde{P}(\xi,t)}\leq\sigma_{P^+}(W)\leq\inf_{t\geq 1,\xi\in\mathbb{R}^n}\frac{\tilde{P}_{W'}(\xi,tp_1)}{\tilde{P}(\xi,t)}. \tag{1}$$

Now, recall that on the finite dimensional vector space

$$\{Q_{|W'}; Q\in\mathbb{C}[X_1,\dots,X_n], degQ\leq degP\}$$

all norms are equivalent. Hence there are $C_j>0, j=1,2$, such that for every $Q\in\mathbb{C}[X_1,\dots,X_n]$ with $degQ\leq degP$ we have for $j=1,2$

$$1/C_j\sup_{x'\in W',|x'|\leq p_j}|Q(x')|\leq\sup_{x'\in W',|x'|\leq 1}|Q(x')|\leq C_j\sup_{x'\in W',|x'|\leq p_j}|Q(x')|.$$

Since for arbitrary $\xi\in\mathbb{R}^n$, and $t>1$ the degree of the polynomial $y\mapsto P(\xi+ty)$ equals that of P it follows that for $j=1,2$

$$1/C_j\frac{\tilde{P}_{W'}(\xi,tp_j)}{\tilde{P}(\xi,t)}\leq\frac{\tilde{P}_{W'}(\xi,t)}{\tilde{P}(\xi,t)}\leq C_j\frac{\tilde{P}_{W'}(\xi,tp_j)}{\tilde{P}(\xi,t)} \tag{2}$$

for all $\xi\in\mathbb{R}^n$ and $t>1$. Now ii) follows from the inequalities (1) and (2). □

3. Properties of semi-elliptic polynomials

In this section we will characterize the subspaces W of $\mathbb{R}^{n+1}$ which satisfy $\sigma_{P^+}(W) = 0$ for a semi-elliptic polynomial P on $\mathbb{R}^n$. For $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and $\alpha \in \mathbb{N}_0^n$ let $|\alpha : \mathbf{m}| := \sum_{j=1}^n \alpha_j/m_j$. If $P(\xi) = \sum_\alpha a_\alpha \xi^\alpha$ is a polynomial with $|\alpha : \mathbf{m}| \leq 1$ for every α with $a_\alpha \neq 0$, i.e.

$$P(\xi) = \sum_{|\alpha:\mathbf{m}|\leq 1} a_\alpha \xi^\alpha$$

set

$$P^0(\xi) := \sum_{|\alpha:\mathbf{m}|=1} a_\alpha \xi^\alpha.$$

If $P^0(\xi) \neq 0$ for every $\xi \in \mathbb{R}^n\backslash\{0\}$ then P is called semi-elliptic. Clearly, if P is of degree m and $m_j = m$ for every j then P^0 is nothing but the principal part P_m of P. Hence elliptic polynomials are semi-elliptic. Moreover, taking $m_1 = 1$ and $m_j = 2$ for $j > 1$ shows that the polynomial $P(\xi) = i\xi_1 + \xi_2^2 + \ldots + \xi_n^2$, i.e. the heat polynomial, is semi-elliptic.

In order to simplify the notation in the following proofs we write $f \lesssim g$ or $g \gtrsim f$ for two positive functions f, g if there is a positive constant C such that $f \leq Cg$.

The next lemma recalls some facts about semi-elliptic polynomials which can be found in [2, proof of Theorem 11.1.11, vol. II].

Lemma 5. *Let $P(\xi) = \sum_{|\alpha:\mathbf{m}|\leq 1} a_\alpha \xi^\alpha$ be a semi-elliptic polynomial, $P^0(\xi) = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha \xi^\alpha$. Then the following hold.*

- i) *For every $\xi \in \mathbb{R}^n$ we have $\sum_{j=1}^n |\xi_j|^{m_j} \lesssim |P^0(\xi)|$.*
- ii) *For α with $|\alpha : \mathbf{m}| \leq 1$ we have $|\xi^\alpha| \leq 1 + \sum_{j=1}^n |\xi_j|^{m_j}$.*

Recall that two polynomials P and Q on $\mathbb{R}^n$ are called equally strong if there is a positive constant C such that $1/C \leq \tilde{Q}(\xi, 1)/\tilde{P}(\xi, 1) < C$ for all $\xi \in \mathbb{R}^n$.

Proposition 6. *Let $P(\xi) = \sum_{|\alpha:\mathbf{m}|\leq 1} a_\alpha \xi^\alpha$ be a semi-elliptic polynomial of degree m, $P^0(\xi) = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha \xi^\alpha$. Then the following properties hold.*

- i) *The degree m of P equals $\max_{1\leq j\leq n} m_j$.*
- ii) *The principal part P_m is a part of P^0, i.e. there is a polynomial R of degree $\leq m-1$ such that $P^0 = P_m + R$ and $P(\xi) - P_m(\xi) - R(\xi) = \sum_{|\alpha:\mathbf{m}|<1} a_\alpha \xi^\alpha$.*
- iii) *$P_m(x) = 0$ for $x \in \mathbb{R}^n$ if and only if $x_j = 0$ for every j with $m_j = m$. In particular, $\{P_m = 0\}$ is a subspace of $\mathbb{R}^n$.*
- iv) *P^0 and P are equally strong.*

Proof. In case of $n = 1$ part i) is trivial so let $n > 1$. Not every monomial appearing in P^0 depends on ξ_1, for if this was true then $P^0(0, \xi_2, \ldots, \xi_n) = 0$ for every choice of $\xi_2, \ldots, \xi_n \in \mathbb{R}$ contradicting the semi-ellipticity of P. If $n > 2$ from these monomials independent of ξ_1, not every monomial depends of ξ_2 for this would yield $P^0(0, 0, \xi_3, \ldots, \xi_n) = 0$ for all $\xi_3, \ldots, \xi_n \in \mathbb{R}$ again contradicting the semi-ellipticity of P. Continuing in that way we finally find a monomial in P^0 which only depends on ξ_n. For the exponent α of this monomial we have, since it is part of P^0, that $1 = |\alpha : \mathbf{m}| = \alpha_n/m_n$. Because $|\alpha| \leq m$ this gives $m_n \leq m$. In the same way we get $m_j \leq m$ for every $j = 1, \ldots, n$.

Now, for every α with $|\alpha| = m$ and $a_\alpha \neq 0$ we have $1 \geq |\alpha : \mathbf{m}|$. If $m > m_j$ for some j with $\alpha_j \neq 0$ we get $1 \geq \sum \frac{\alpha_l}{m_l} > \sum \frac{\alpha_l}{m}$ contradicting $|\alpha| = m$. This shows $m = \max m_j$ and $m_j = m$ for every j such that there is α with $|\alpha| = m, a_\alpha \neq$

$0, \alpha_j \neq 0$ which implies i) and ii). Moreover, if α is the exponent of a monomial in P_m we have $m_j = m$ for every j with $\alpha_j \neq 0$. Therefore, $P_m(x) = 0$ if $x_j = 0$ for every j with $m_j = m$.

To prove necessity in iii), note that semi-ellipticity of P gives $\sum |\xi_j|^{m_j} \lesssim |P^0(\xi)|$ for all $\xi \in \mathbb{R}^n$ by Lemma 5 i). If $P_m(x) = 0$ it follows from the homogeneity of P_m and ii) that for l with $m_l = m$ and $t > 0$ sufficiently large

$$t^m |x_l|^m \leq \sum_{j=1}^n |tx_j|^{m_j} \lesssim |P^0(tx)| \lesssim t^{m-1}$$

which shows $x_l = 0$.

To prove iv) we set $S := P - P^0$. For $\xi \in \mathbb{R}^n$ we have

$$|S(\xi)|^2 \lesssim \sum_{|\alpha:\mathbf{m}|<1} |a_\alpha|^2 |\xi^\alpha|^2.$$

Without loss of generality, let $m_1 = m$ so that for $t > 0$ we have with Lemma 5 i)

$$\begin{aligned}
\tilde{P^0}(\xi,t)^2 &= \sup_{|\eta|<1} |P^0(\xi + t\eta)|^2 \gtrsim \sup_{|\eta|<1} (\sum_{j=1}^n |\xi_j + t\eta_j|^{m_j})^2 \\
&\gtrsim \sup_{|\eta|<1} (\sum_{j=1}^n |\xi_j + t\eta_j|^{2m_j}) \gtrsim \sup_{\sigma\in\{-1,1\}} (\sum_{j=2}^n \xi_j^{2m_j} + (\xi_1 + \sigma t)^{2m}) \\
&\gtrsim (\sum_{j=1}^n \xi^{2m_j} + t^{2m}).
\end{aligned}$$

From this and the fact that for α with $|\alpha : \mathbf{m}| < 1$ we have $\alpha_l < m_l \leq m$ for some l we get for $t \geq 1$

$$\begin{aligned}
\frac{|S(\xi)|^2}{\tilde{P^0}(\xi,t)^2} &\lesssim \sum_{|\alpha:\mathbf{m}|<1} |a_\alpha|^2 \prod_{j=1}^n \frac{\xi_j^{2\alpha_j}}{\sum_{k=1}^n \xi_k^{2m_k} + t^{2m}} \\
&\lesssim \sum_{|\alpha:\mathbf{m}|<1} |a_\alpha|^2 \frac{\xi_l^{2(m_l-1)}}{\xi_l^{2m_l} + t^{2m}} \\
&\lesssim \sum_{|\alpha:\mathbf{m}|<1} |a_\alpha|^2 (t^{2m})^{-1/m_l} \lesssim t^{-2}
\end{aligned}$$

where in the third inequality we used that $f : [0,\infty) \to \mathbb{R}, f(x) := x^{2m_l-2}/(x^{2m_l}+c)$ for $c > 0$ is bounded by Mc^{-1/m_l} for some constant M.

It follows that

$$\inf_{t>1} (\sup_{\xi\in\mathbb{R}^n} \frac{|S(\xi)|}{\tilde{P^0}(\xi,t)}) = 0$$

so that by [2, Theorem 10.4.6, vol. II] P^0 dominates S which by [2, Corollary 10.4.8, vol. II] implies the equivalence of P^0 and $P^0 + S = P$. □

Lemma 7. *Let $P(\xi) = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha \xi^\alpha$ be a semi-elliptic polynomial on $\mathbb{R}^n$ of degree m. Moreover, let W be a subspace of $\mathbb{R}^{n+1}$. Then we have $\sigma_{P^+}(W) = 0$ if and only if W' is a subspace of $\{P_m = 0\}$.*

Proof. By Proposition 6 iii) W' is a subspace of $\{P_m = 0\}$ if and only if for each $x \in W'$ we have $x_j = 0$ for every j with $m_j = m$.

Assume there is $x \in W'$ such that $x_l \neq 0$ for some l with $m_l = m$. Without loss of generality let $|x| = 1$. Then by Lemma 5 ii)

$$\begin{aligned}
\tilde{P}_{W'}(\xi,t)^2 &\geq \sup_{|\lambda|\leq t} |P(\xi+\lambda x)|^2 \\
&\gtrsim \sup_{|\lambda|\leq t} (\sum_{j=1}^n |\xi_j + \lambda x_j|^{m_j})^2 \\
&\gtrsim \sum_{j=1}^n ((\xi_j + t x_j)^{2m_j} + (\xi_j - t x_j)^{2m_j}) \\
&\gtrsim \sum_{j=1}^n \xi_j^{2m_j} + \sum_{j=1}^n t^{2m_j} x_j^{2m_j} \\
&\geq \sum_{j=1}^n \xi_j^{2m_j} + t^{2m} x_l^{2m}.
\end{aligned}$$

Since for α with $|\alpha : \mathbf{m}| \leq 1$ we have $|\xi^\alpha| \leq 1 + \sum_{j=1}^n |\xi_j|^{m_j}$ by Lemma 5 ii) we get for $r \geq 1$ using the equivalence of norms on $\mathbb{R}^2$

$$\begin{aligned}
\tilde{P}(\xi,t)^2 &= \sup_{|y|\leq t} |P(\xi+y)|^2 \lesssim 1 + \sup_{|y|\leq t} (\sum_{j=1}^n |\xi_j + y_j|^{m_j})^2 \\
&\lesssim 1 + \sum_{j=1}^n \xi_j^{2m_j} + n t^{2m} \leq \sum_{j=1}^n \xi_j^{2m_j} + (n+1) t^{2m}.
\end{aligned}$$

Observing that $x_l \leq 1$, these estimates give

$$\frac{\tilde{P}_{W'}(\xi,t)^2}{\tilde{P}(\xi,t)^2} \gtrsim \frac{\sum_{j=1}^n \xi_j^{2m_j} + t^{2m} x_l^{2m}}{\sum_{j=1}^n \xi_j^{2m_j} + (n+1) t^{2m}} \gtrsim \frac{x_l^{2m}}{n+1} > 0,$$

so that by Lemma 4 ii) we have $\sigma_{P^+}(W) > 0$.

On the other hand, if W' is a subspace of $\{x \in \mathbb{R}^n;\ x_j = 0\ \forall j \text{ with } m_j = m\}$ we get using Lemma 5 ii) and the equivalence of norms on $\mathbb{R}^2$

$$\begin{aligned}
\tilde{P}_{W'}(\xi,t)^2 &= \sup_{|x|\leq 1, x\in W'} |P(\xi+tx)|^2 \\
&\lesssim 1 + \sup_{|x|\leq 1, x\in W'} (\sum_{j=1}^n |\xi_j + t x_j|^{m_j})^2 \\
&\lesssim 1 + \sup_{|x|\leq 1, x\in W'} (\sum_{j=1}^n |\xi_j|^{m_j} + |t x_j|^{m_j})^2 \\
&\lesssim 1 + \sum_{j=1}^n \xi_j^{2m_j} + \sup_{|x|\leq 1, x\in W'} \sum_{j=1}^n t^{2m_j} |x_j|^{2m_j} \\
&\lesssim 1 + \sum_{j=1}^n \xi_j^{2m_j} + k t^{2(m-1)}.
\end{aligned}$$

Here k equals the number of m_js stictly less than m. Observe that W' is a subspace of $\{x \in \mathbb{R}^n;\ x_j = 0\ \forall j \text{ with } m_j = m\}$!

Since P is semi-elliptic we have $|P(\xi)| \gtrsim \sum_{j=1}^n |\xi_j|^{m_j}$ by Lemma 5 i). Without loss of generality we assume $m_1 = m$ and obtain

$$\begin{aligned}\tilde{P}(\xi,t)^2 &\gtrsim \sup_{|x|\leq t}(\sum_{j=1}^n |\xi_j + x_j|^{m_j})^2\\ &\gtrsim \sup_{\tau\in\{-1,1\}}((\xi_1+\tau t)^{2m} + \sum_{j=2}^n \xi_j^{2m_j})\\ &\gtrsim \sum_{j=1}^n \xi_j^{2m_j} + t^{2m}.\end{aligned}$$

With these estimates we conclude

$$\frac{\tilde{P}_{W'}(\xi,t)^2}{\tilde{P}(\xi,t)^2} \lesssim \frac{1+\sum_{j=1}^n \xi_j^{2m_j} + kt^{2m-2}}{\sum_{j=1}^n \xi_j^{2m_j} + t^{2m}},$$

so that $\sigma_{P^+}(W) = 0$ by Lemma 4 ii). □

Theorem 8. *Let $P(\xi) = \sum_{|\alpha:\mathbf{m}|\leq 1} a_\alpha \xi^\alpha$ be a semi-elliptic polynomial of degree m on $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^{n+1}$. Then we have $\sigma_{P^+}(W) = 0$ if and only if W' is a subspace of $\{P_m = 0\}$.*

Proof. By Proposition 6 the polynomials $P^0(\xi) = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha\xi^\alpha$ and P are equally strong, thus P^+ and $(P^0)^+$ are equally strong, too. By [2, Theorem 11.3.14, vol. II] we therefore have $\sigma_{P^+}(W) = 0$ if and only if $\sigma_{(P^0)^+}(W) = 0$ so that the lemma follows from the previous lemma and Proposition 6. □

The following example shows that contrary to Proposition 1 P-convexity for singular supports of Ω in general does not imply P^+-convexity for singular supports of $\Omega\times\mathbb{R}$. However, in this example the set Ω is not P-convex for supports hence it does not yield an answer to the general question.

Example 9. Consider $P(\xi_1,\xi_2) = i\xi_1 + \xi_2^2$, i.e. the heat polynomial in one space dimension. As illustrated at the beginning of this section, P is then semi-elliptic hence hypoelliptic by [2, Theorem 11.1.11]. Therefore

$$\Omega := \{x\in\mathbb{R}^2;\, x_1 > 0\}\cap\{x\in\mathbb{R}^2; x_1^2 + x_2^2 > 1\}$$

is P-convex for singular supports. Consider the affine subspace

$$V = \{(2,t,0);\, t\in\mathbb{R}\} = (2,0,0) + \mathit{span}\{(0,1,0)\}$$

of $\mathbb{R}^3$. The orthogonal space $W = \mathit{span}\{(1,0)\}\times\mathbb{R}$ of $\mathit{span}\{(0,1,0)\}$ clearly satisfies $W' \subset \{x\in\mathbb{R}^2;\, P_2(x) = 0\}$ so that by Theorem 8 we have $\sigma_{P^+}(W) = 0$.

Let $K := \{(2,t,0);\, t\in[-3,3]\}$. Then $K\subset V$ and the boundary of K relative V consists of the points $(2,-3,0)$ and $(2,3,0)$. Since

$$\mathit{dist}(K,(\Omega\times\mathbb{R})^c) = 1 < 2 = \mathit{dist}(\{(2,-3,0),(2,3,0)\},(\Omega\times\mathbb{R})^c)$$

it follows from [2, Corollary 11.3.2, vol. II] that $\Omega\times\mathbb{R}$ is not P^+-convex for singular supports.

On the other hand, $V'\subset\mathbb{R}^2$ is clearly a characteristic hyperplane for P. Since the boundary of K' relative V' consists of the points $(2,-3)$ and $(2,3)$ and

$$\mathit{dist}(K',\Omega^c) = 1 < 2 = \mathit{dist}(\{(2,-3),(2,3)\},\Omega^c)$$

it follows from [2, Theorem 10.8.1, vol. II] that Ω is not P-convex for supports.

Compare this example with Corollary 15.

4. Sufficient conditions for P-convexity

For $x, y \in \mathbb{R}^n$ we denote by $[x, y]$ the closed convex hull of $\{x, y\}$. Moreover, for $\Omega \subset \mathbb{R}^n$ open, $x \in \Omega$, $r \in \mathbb{R}^n \backslash \{0\}$, we define

$$\lambda(x, r) := \sup\{\lambda > 0; \forall\, 0 \leq \mu < \lambda : [x, x + \mu r] \subset \Omega\}.$$

In case of $\lambda(x, r) = \infty$ we simply write $[x, x + \lambda(x, r)r]$ instead of $\cup_{0<\lambda<\lambda(x,r)}[x, x + \lambda r]$. The next lemma gives a sufficient condition for P-convexity for supports.

Lemma 10. *Let Ω be an open subset of $\mathbb{R}^n$ and P a non-zero polynomial of degree m. Assume that for each compact subset K of Ω there is another compact subset L of Ω such that for every $x \in \Omega \backslash L$ one can find $r \in \{P_m = 0\}^\perp \backslash \{0\}$ satisfying*

$$[x_0, x_0 + \lambda(x_0, r)r] \cap K = \emptyset.$$

Then Ω is P-convex for supports.

Proof. Let $\phi \in \mathscr{D}(\Omega)$ and $K := \operatorname{supp} P(-D)\phi$. Choose L for K as stated in the hypothesis. For $x_0 \in \Omega \backslash L$ there is $r \in \{P_m = 0\}^\perp \backslash \{0\}$ such that

$$[x_0, x_0 + \lambda(x_0, r)r] \cap K = \emptyset.$$

From the compactness of $\operatorname{supp} \phi$ it follows that there is $\lambda \in (0, \lambda(x_0, r))$ such that $x_1 := x_0 + \lambda r \notin \operatorname{supp} \phi$. Therefore, $[x_0, x_1] \subset \Omega$ and we can find $\rho > 0$ such that $\Omega_1 := B(x_1, \rho) \subset \Omega \backslash \operatorname{supp} \phi$ and $\Omega_2 := [x_0, x_1] + B(0, \rho) \subset \Omega \backslash K$.

$\Omega_1 \subset \Omega_2$ are open and convex, and $\phi_{|\Omega_1} = 0$ as well as $P(-D)\phi_{|\Omega_2} = 0$. Let $H = \{x; \langle x, \xi \rangle = \alpha\}$ be a characteristic hyperplane for P, i.e. $\xi \neq 0$ satisfies $P_m(\xi) = 0$. If H intersects Ω_2 there are $\gamma \in [0, 1], b \in B(0, \rho)$ satisfying

$$\begin{aligned} \alpha &= \langle \gamma x_0 + (1 - \gamma)x_1 + b, \xi \rangle = \langle x_0 + (1 - \gamma)\lambda r + b, \xi \rangle \\ &= \langle x_0 + b, \xi \rangle = \langle x_1 - \lambda r + b, \xi \rangle = \langle x_1 + b, \xi \rangle \end{aligned}$$

where we used $\langle r, \xi \rangle = 0$. So H already intersects Ω_1. [2, Theorem 8.6.8, vol. I] now gives $\phi_{|\Omega_2} = 0$ so that $x_0 \notin \operatorname{supp} \phi$. Since $x_0 \in \Omega \backslash L$ was arbitrary it follows $\operatorname{supp} \phi \subset L$ proving the lemma. □

In order to formulate a similar condition for P-convexity for singular supports we introduce for a non-zero polynomial P the subspace

$$S_P := \bigcap(\{V \subset \mathbb{R}^n; V \text{ one-dimensional subspace}, \sigma_P(V) = 0\}^\perp).$$

The non-zero elements r of S_P are the directions which lie in every hyperplane $H = \{x; \langle x, \xi \rangle = \alpha\}$ with $\sigma_P(span\{\xi\}) = 0$. Hence, due to these directions an application of Corollary 3 instead of [2, Theorem 8.6.8, vol. I] makes it possible to prove the next lemma in a very similar way to the previous one. Indeed, the proof is mutatis mutandis the same. Nevertheless, we include it for the reader's convenience.

Lemma 11. *Let Ω be an open subset of $\mathbb{R}^n$ and P a non-zero polynomial. Assume that for each compact subset K of Ω there is another compact subset L of Ω such that for every $x \in \Omega \backslash L$ one can find $r \in S_P \backslash \{0\}$ with*

$$[x, x + \lambda(x, r)r] \cap K = \emptyset.$$

Then Ω is P-convex for singular supports.

Proof. Let $\mu \in \mathscr{E}'(\Omega)$ and $K := \operatorname{sing\,supp} P(-D)\mu$. Choose L for K as stated in the hypothesis. For $x_0 \in \Omega \backslash L$ there is $r \in S_P \backslash \{0\}$ such that

$$[x_0, x_0 + \lambda(x_0, r)r] \cap K = \emptyset.$$

From the compactness of $\operatorname{sing\,supp} \mu$ it follows that there is $\lambda \in (0, \lambda(x_0, r))$ such that $x_1 := x_0 + \lambda r \notin \operatorname{sing\,supp} \mu$. Therefore, $[x_0, x_1] \subset \Omega$ and we can find $\rho > 0$ such that $\Omega_1 := B(x_1, \rho) \subset \Omega \backslash \operatorname{sing\,supp} \mu$ and $\Omega_2 := [x_0, x_1] + B(0, \rho) \subset \Omega \backslash K$. We will show that $\mu_{|\Omega_2} \in C^\infty(\Omega_2)$ implying $x_0 \notin \operatorname{sing\,supp} \mu$. Since $x_0 \in \Omega \backslash L$ was chosen arbitrarily this implies $\operatorname{sing\,supp} \mu \subset L$ proving P-convexity for singular supports of Ω.

By definition of K we have $P(-D)\mu_{|\Omega_2} \in C^\infty(\Omega_2)$. Moreover, Ω_1 is convex and $\operatorname{sing\,supp} \mu_{|\Omega_2} \subset \Omega_2 \backslash \Omega_1$. To show that $\mu_{|\Omega_2} \in C^\infty(\Omega_2)$, let $H = \{x; \langle x, \xi \rangle = \alpha\}, \xi \neq 0$, be a hyperplane with $\sigma_P(\operatorname{span}\{\xi\}) = 0$. Since $r \in S_P$ we have $\langle r, \xi \rangle = 0$. If H intersects Ω_2 it follows exactly as in the proof of Lemma 10 that H already intersects Ω_1. Now Corollary 3 gives $\mu_{|\Omega_2} \in C^\infty(\Omega_2)$ thus proving the lemma. □

Having seen that $\{P_m = 0\}$ is a subspace for semi-elliptic P the next proposition will be useful to apply the above lemmas in the semi-elliptic case.

Proposition 12. *Let $\Omega \subset \mathbb{R}^n$ be open and $M \subset \mathbb{R}^n$ a subspace. The following condition i) implies ii):*

i) *For each $x \in \Omega$ there is $r \in M \backslash \{0\}$ such that $dist(x, \Omega^c) \geq dist(y, \Omega^c)$ for all $y \in [x, x + \lambda(x, r)r]$*
ii) *For each compact subset K of Ω there is another compact subset L of Ω such that for every $x \in \Omega \backslash L$ there is $r \in M \backslash \{0\}$ satisfying $[x, x+\lambda(x, r)r] \cap K = \emptyset$.*

Proof. For $m \in \mathbb{N}$ let $\Omega_m := \{x \in \Omega;\ |x| < m,\ \operatorname{dist}(x, \Omega^c) > 1/m\}$. For $K \subset \Omega$ compact choose m such that $K \subset \Omega_m$ and set $L := \overline{\Omega_m}$. For $x \in \Omega \backslash L$ let r be as in i).

If $|x| > m$ either $\{x+\lambda r; \lambda > 0\} \subset \mathbb{R}^n \backslash \overline{B(0, m)}$ or $\{x-\lambda r; \lambda > 0\} \subset \mathbb{R}^n \backslash \overline{B(0, m)}$ so that ii) follows with r or $-r$. If $|x| \leq m$ we have $1/m \geq \operatorname{dist}(x, \Omega^c) \geq \operatorname{dist}(y, \Omega^c)$ for every $y \in [x, x + \lambda(x, r)r]$ because of $x \in \Omega \backslash L$, hence $[x, x + \lambda(x, r)r] \cap K = \emptyset$.□

5. Main results

The next theorem is an immediate consequence of Theorem 8, Lemma 10, Lemma 11, Proposition 12, and Proposition 1.

Theorem 13. *Let $\Omega \subset \mathbb{R}^n$ be open and P a non-zero polynomial with principal part P_m. If for every $x \in \Omega$ there is $r \in \{P_m = 0\}^\perp \backslash \{0\}$ such $dist(x, \partial\Omega) \geq dist(y, \partial\Omega)$ for every $y \in \{x + \lambda r; \lambda \in (0, \lambda(x, r))\}$ then Ω is P-convex for supports.*

Moreover, if P is semi-elliptic then $\Omega \times \mathbb{R}$ is P^+-convex for singular supports, hence $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ as well as $P^+(D) : \mathscr{D}'(\Omega \times \mathbb{R}) \to \mathscr{D}'(\Omega \times \mathbb{R})$ are surjective.

A result of Vogt (cf. [3, Proposition 2.5]) says that the kernel of an elliptic differential operator always has the linear topological invariant (Ω). Since in this context (Ω) equals the property $(P\Omega)$ it follows from [1, Proposition 8.3] that for an elliptic polynomial P the augmented operator $P^+(D)$ is surjective on $\mathscr{D}'(\Omega \times \mathbb{R})$ if $P(D)$ is surjective on $\mathscr{D}'(\Omega)$. This interpretation of Vogt's result can be derived as a direct application of the above theorem.

Corollary 14. *Let $\Omega \subset \mathbb{R}^n$ be open and P an elliptic polynomial. Then $P^+(D)$ is surjective on $\mathscr{D}'(\Omega \times \mathbb{R})$.*

Proof. This follows immediately from Theorem 13, $\{P_m = 0\}^\perp = \mathbb{R}^n$, and Proposition 1. □

Corollary 15. *Let $\Omega \subset \mathbb{R}^2$ be open and P a semi-elliptic polynomial such that $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is surjective.*

Then $P^+(D) : \mathscr{D}'(\Omega \times \mathbb{R}) \to \mathscr{D}'(\Omega \times \mathbb{R})$ is surjective.

Proof. By Corollary 14 we can assume without loss of generality that P is not elliptic. Then by Proposition 6 $\{P_m = 0\}$ is a one-dimensional subspace of $\mathbb{R}^2$. Therefore a hyperplane H is characteristic if and only if $H = \{x + \lambda r; \lambda \in \mathbb{R}\}$ for some $x \in \mathbb{R}^2, r \in \mathbb{R}^2\backslash\{0\}$ with $r \in \{P_m = 0\}^\perp$.

Let $x_0 \in \Omega$ and $r \in \{P_m = 0\}^\perp\backslash\{0\}$. Then the hyperplane

$$H := \{x_0 + \lambda r; \lambda \in \mathbb{R}\}$$

is characteristic. Assuming that there are $\lambda^+ \in (0, \lambda(x_0, r))$ and $\lambda^- \in (0, \lambda(x_0, -r))$ such that $dist(x_0+\lambda^+ r, \Omega^c) > dist(x_0, \Omega^c)$ as well as $dist(x_0-\lambda^- r, \Omega^c) > dist(x_0, \Omega^c)$ it follows for the compact subset $K := [x_0 - \lambda^- r, x_0 + \lambda^+ r]$ of $\Omega \cap H$ that

$$\begin{aligned} dist(\partial_H K, \Omega^c) &= \min\{dist(x_0 + \lambda^+ r, \Omega^c), dist(x_0 - \lambda^- r, \Omega^c)\} > dist(x_0, \Omega^c) \\ &\geq dist(K, \Omega^c), \end{aligned}$$

where $\partial_H K$ denotes the boundary of K as a subset of H. On the other hand, since Ω is P-convex for supports by hypothesis, we have $dist(\partial_H K, \Omega^c) = dist(K, \Omega^c)$ by [2, Theorem 10.8.1, vol. II] giving a contradiction. Hence, $dist(y, \Omega^c) \leq dist(x_0, \Omega^c)$ for all $y \in [x_0, x_0 + \lambda(x_0, r)r]$ or all $y \in [x_0, x_0 - \lambda(x_0, -r)r]$.

It follows from Proposition 12 that for each compact subset K of Ω there is another compact subset L of Ω such that for every $x \in \Omega\backslash L$ there is $r \in \{P_m = 0\}^\perp\backslash\{0\}$ satisfying $[x, x + \lambda(x, r)r] \cap K = \emptyset$.

Now, since P is semi-elliptic we have $S_{P^+} = \{P_m = 0\}^\perp \times \{0\}$ by Theorem 8. Thus the above gives that for each compact subset K of $\Omega \times \mathbb{R}$ there is another compact subset L of $\Omega \times \mathbb{R}$ such that for every $x \in (\Omega \times \mathbb{R})\backslash L$ there is $r \in S_{P^+}\backslash\{0\}$ satisfying $[x, x+\lambda(x, r)r] \cap K = \emptyset$. Lemma 11 applied to $\Omega \times \mathbb{R}$ therefore yields the result. □

We do not know if an analogous conclusion for semi-elliptic operators is true for arbitrary dimension. In particular, the main problem remains open for the heat operator in arbitrary many variables.

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