On the behaviour of power series in the absence of Hadamard-Ostrowski gaps
Sur le comportement des séries entières en l'absence de lacunes de Hadamard-Ostrowski*

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Abstract

We show that the partial sums $(S_n f)_{n\in\mathbb{N}}$ of a power series f with radius of convergence one tend to ∞ in capacity on (arbitrarily large) compact subsets of the complement of the closed unit disk, if f does not have so-called Hadamard-Ostrowski gaps. Regarding a recent result of Gardiner, this covers a large class of functions f holomorphic in the unit disk.

Nous montrons que les sommes partielles $(S_n f)_{n\in\mathbb{N}}$ d'une série entière f de rayon de convergence 1 tendent vers ∞ en capacité sur les ensembles compacts (arbitrairement grands) du complémentaire du disque unité fermé, si f ne contient pas de lacunes de Hadamard-Ostrowski. Tenant compte d'un résultat récent de Gardiner, ceci couvre une grande classe de fonctions f holomorphes sur le disque unité.

Key words. Convergence in capacity, Hadamard-Ostrowski gaps.

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1 Introduction and main result

For a power series $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ with radius of convergence one, we denote the partial sums by

$$S_n f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \ .$$

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Then $(S_n f)(z_0)$ is unbounded at each point z_0 with $|z_0| > 1$. More precisely, we have

$$\limsup_{n\to\infty} |S_n f(z_0)|^{1/n} = |z_0| \tag{1}$$

and, for $R \geq 1$,

$$\limsup_{n\to\infty} \max_{|z|=R} |S_n f(z)|^{1/n} = R\,. \tag{2}$$

This does not prevent subsequences of $(S_n f)$ from being convergent, even on large subsets of the plane. Indeed, it is well-known (see e.g. the expository articles [6] and [8]) that there exist functions f holomorphic in the unit disk $\mathbb{D}$ having the property that the sequence $(S_n f)_{n\in\mathbb{N}}$ is universal outside $\mathbb{D}$ in the sense that for each compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ with connected complement and each continuous function $h : K \to \mathbb{C}$ which is holomorphic in the interior of K, there is a subsequence of $(S_n f)_{n\in\mathbb{N}}$ tending to h uniformly on K.

It turns out that for such power series the sequence of coefficients $(a_\nu)_{\nu\in\mathbb{N}_0}$ necessarily exhibits a strong kind of irregularity in the sense of having so-called Ostrowski gaps (see [5], [10], [12]). In contrast, if the sequence $(a_\nu)_{\nu\in\mathbb{N}_0}$ behaves regularly, as for example in the case of the geometric series $f(z) = \sum_{\nu=0}^{\infty} z^\nu$, the partial sums $(S_n f(z))_{n\in\mathbb{N}}$ tend to be attracted by ∞ for z outside the closed unit disk. Our aim is to show that, for a reasonable class of power series, this turns out to be true in a certain sense.

In the sequel, we use the term capacity for logarithmic capacity. For unexplained notions from potential theory see [13].

Definition 1.1 A sequence $(h_n)_{n\in\mathbb{N}}$ of Borel-measurable functions is said to *converge in capacity* to ∞ on a set $D \subset \mathbb{C}$, if for every $M > 0$ we have

$$\lim_{n\to\infty} \operatorname{cap}(\{z \in D\colon |h_n(z)| \leq M\}) = 0\,.$$

Furthermore, if D is open, (h_n) is said to *converge locally in capacity* on D to ∞, if the sequence converges in capacity to ∞ on every (non-polar) compact subset of D.

This definition is in accordance with the definition of convergence in capacity to finite limit functions, as considered e.g. in [9]. The notion of convergence in capacity is well-known in Padé approximation (see, e.g. [1, Section 6.6]). Since $\operatorname{cap}(K) \geq \sqrt{|K|/\pi}$, where $|K|$ denotes the area of a compact plane set K, convergence in capacity implies convergence in (plane Lebesgue) measure.

Definition 1.2 Let $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ have radius of convergence one. We say that f possesses *Hadamard-Ostrowski gaps* if there are sequences $(p_k)_{k\in\mathbb{N}}$ and $(q_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ with $p_1 < q_1 \leq p_2 < q_2 \leq \ldots$, such that

1. $\liminf\limits_{k\to\infty} \dfrac{q_k}{p_k} > 1$,
2. $\limsup\limits_{I\ni\nu\to\infty} |a_\nu|^{\frac{1}{\nu}} < 1$ for $I := \bigcup\limits_{k\in\mathbb{N}} \{p_k + 1, p_k + 2, \ldots, q_k - 1\}$.

Remark 1.3 A recent result of Gardiner [3, Cor. 3] states that a power series $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ which converges on $\mathbb{D}$ and is analytically continuable to a domain $\Omega \supset \mathbb{D}$, but not to a neighbourhood of a given point $\xi \in \partial\mathbb{D}$, has no Hadamard-Ostrowski gaps, if $\mathbb{C} \setminus \Omega$ is thin at ξ. In particular, it follows that all functions holomorphic in $\mathbb{D}$ which have an isolated singularity at some point ξ on the boundary of $\mathbb{D}$ do not have Hadamard-Ostrowski gaps. It is easily seen that the same is true for all (repeated) antiderivatives of such functions. Thus, for a large class of functions holomorphic in $\mathbb{D}$ this turns out to be the case.

Our main result is the following. The proof is given in the next section.

Theorem 1.4 *Let $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ be a power series with radius of convergence one and without Hadamard-Ostrowski gaps. Then $S_n f \to \infty$ locally in capacity on $\mathbb{C} \setminus \overline{\mathbb{D}}$.*

Under the conditions of the theorem, $S_n f$ does not need to tend to ∞ (globally) in capacity on any open annulus $\{z\colon 1 < |z| < R\}$, where $R > 1$. As an example, on the semi-circular arcs

$$B_n := \{z \in \mathbb{C}\colon \operatorname{Re} z \leq 0, |z| = 1 + 1/n\}$$

the partial sums $S_n f(z) = (z^{n+1} - 1)(z - 1)^{-1}$ of the geometric series f are uniformly bounded, more precisely

$$\max_{z \in B_n} |S_n f(z)| \leq 5/\sqrt{2},$$

while $\operatorname{cap}(B_n) = (1+1/n)\operatorname{cap}(B) \geq \operatorname{cap}(B)$, where B is the corresponding semi-circular arc on the unit circle. This also shows that pointwise convergence to ∞ on the annulus does not imply convergence in capacity there.

Remark 1.5 Let h_n be a sequence of meromorphic functions in the plane. According to [11, Thm. 2], convergence in capacity of $h_n \to h$ on a bounded set $D \subset \mathbb{C}$ implies that there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ in $\mathbb{N}$ such that $h_{n_k} \to h$ *quasi-everywhere* on D, i.e.

$$\operatorname{cap}\big(\{z \in D\colon h_{n_k}(z) \not\to h(z)\}\big) = 0.$$

Corollary 1.6 *Let $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ be a power series with radius of convergence one and without Hadamard-Ostrowski gaps. Then every subsequence $(S_{n_k} f)_{k \in \mathbb{N}}$ of $(S_n f)_{n \in \mathbb{N}}$ contains a subsequence $(S_{n_{k_j}} f)_{j \in \mathbb{N}}$ with $1/S_{n_{k_j}} f \to 0$ quasi-everywhere on $\mathbb{C} \setminus \overline{\mathbb{D}}$.*

The proof follows with a standard diagonal sequence argument from Theorem 1.4:

The annuli $K_l := \{z : 1+1/l \leq |z| \leq 1+l\}$ $(l \in \mathbb{N})$ form a compact exhaustion of $\mathbb{C} \setminus \overline{\mathbb{D}}$. Let $(S_{n_k} f)_{k \in \mathbb{N}}$ be any subsequence of $(S_n f)_{n \in \mathbb{N}}$. By Theorem 1.4 and Remark 1.5, there exists a subsequence $(n_k^{(1)})_{k \in \mathbb{N}}$ of $(n_k)_{k \in \mathbb{N}}$ such that

$$1/S_{n_k^{(1)}} f \to 0 \text{ quasi-everywhere on } K_1.$$

Preceding in this way, there exists a subsequence $(n_k^{(2)})_{k\in\mathbb{N}}$ of $(n_k^{(1)})_{k\in\mathbb{N}}$ such that

$$1/S_{n_k^{(2)}} f \to 0 \text{ quasi-everywhere on } K_2,$$

and so on. The diagonal sequence $(n_k^{(k)})_{k\in\mathbb{N}}$ is a subsequence of $(n_k)_{k\in\mathbb{N}}$ for which $\mathrm{cap}\big(\{z \in K_l \colon 1/S_{n_k^{(k)}} f \not\to 0\}\big) = 0$ for all $l \in \mathbb{N}$. Since countable unions of polar sets are polar, this implies $\mathrm{cap}\big(\{|z| > 1 \colon 1/S_{n_k^{(k)}} f \not\to 0\}\big) = 0$.

2 Proof of the main theorem

For $r > 1$ we set $\mathbb{D}_r := \{\zeta \in \mathbb{C} : |\zeta| < r\}$. The following purely potential theoretic result will play the key role for our argumentation.

Lemma 2.1 *Suppose that (u_j) is a sequence of upper bounded and subharmonic functions on $\mathbb{D}$. If*

$$\limsup_{j\to\infty} \sup_{\mathbb{D}} u_j \le 0$$

and if there is a sequence (E_j) of compact subsets of $\mathbb{D}_\delta$ for some $\delta < 1$ with cap$(E_j) \ge \alpha > 0$ and

$$\beta := \limsup_{j\to\infty} \max_{E_j} u_j < 0$$

then, for all $r \in (\delta, 1)$, a positive constant $c(\delta, r)$ exists so that

$$\limsup_{j\to\infty} \max_{\overline{\mathbb{D}_r}} u_j \le \beta \frac{c(\delta, r)}{\log(1/\alpha)} .$$

Proof. In view of the maximum principle (for subharmonic functions) we may suppose that the sets $\mathbb{C} \setminus E_j$ are connected. Let $\varepsilon > 0$ be fixed. Then there exists an integer $j_0 = j_0(\varepsilon)$ with

$$\sup_{\mathbb{D}} u_j < \varepsilon \quad \text{and} \quad \max_{E_j} u_j < \beta + \varepsilon \quad (j \ge j_0) .$$

Let $\omega_{\mathbb{D}\setminus E_j}$ denote the harmonic measure of $\mathbb{D} \setminus E_j$. According to the two-constant-theorem (see, e.g. [13, Theorem 4.3.7]) we get, for $\zeta \in \mathbb{D} \setminus E_j$ and $j \ge j_0$,

$$u_j(\zeta) \le (\beta + \varepsilon)\omega_{\mathbb{D}\setminus E_j}(\zeta, E_j) + \varepsilon(1 - \omega_{\mathbb{D}\setminus E_j}(\zeta, E_j)) = \beta\omega_{\mathbb{D}\setminus E_j}(\zeta, E_j) + \varepsilon .$$

Moreover, from a result in [4, p.123], we obtain the existence of a positive constant $c(\delta, r)$ with

$$\max_{|\zeta|=r} \omega_{\mathbb{D}\setminus E_j}(\zeta, E_j) \ge \frac{c(\delta, r)}{\log(1/\mathrm{cap}(E_j))} \ge \frac{c(\delta, r)}{\log(1/\alpha)}$$

and thus the maximum principle yields

$$\sup_{\overline{\mathbb{D}_r}} u_j \le \beta \frac{c(\delta, r)}{\log(1/\alpha)} + \varepsilon \quad (j > j_0) .$$

Since $\varepsilon > 0$ was arbitrary, the conclusion follows. $\square$

Now, we are prepared for the *Proof of Theorem 1.4*:

1. We show in a first step that $S_n f(1/\zeta) \to \infty$ locally in capacity on $\mathbb{D}$.

For this purpose, let $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ be a power series with radius of convergence one. Assume that $(\zeta \mapsto S_n f(1/\zeta))_{n\in\mathbb{N}}$ does not tend to ∞ in capacity on a non-polar compact set $E \subset \mathbb{D}$. We have to show that f has Hadamard-Ostrowski gaps.

By assumption, there exist $M > 0$, $\alpha > 0$, and a sequence $(n_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ tending to infinity such that the compact sets

$$E_j := \{\zeta \in E \colon |S_{n_j} f(1/\zeta)| \le M\} \quad (j \in \mathbb{N})$$

all satisfy $\operatorname{cap}(E_j) \ge \alpha$. We define

$$u_j(\zeta) := \log |S_{n_j} f(1/\zeta)\zeta^{n_j}|^{1/n_j} = \log|\zeta| + \frac{1}{n_j}\log|S_{n_j} f(1/\zeta)| \quad (\zeta \in \mathbb{D}) .$$

Then the u_j are subharmonic in $\mathbb{C}$ (note that $S_n f(1/\zeta)\zeta^n$ is a polynomial in ζ). Moreover, (2) implies

$$\limsup_{n\to\infty} \max_{|\zeta|=1} |S_n f(1/\zeta)|^{1/n} = 1$$

and therefore (by the maximum principle)

$$\limsup_{j\to\infty} \sup_{\mathbb{D}} u_j \le 0 .$$

Since $E_j \subset E \subset \mathbb{D}_\delta$ for some $\delta < 1$, we further obtain

$$\max_{E_j} u_j \le \log(\delta) + \frac{1}{n_j}\log M$$

and thus

$$\limsup_{j\to\infty} \max_{E_j} u_j \le \log(\delta) < 0 . \tag{3}$$

Since $\operatorname{cap}(E_j) \ge \alpha$, Lemma 2.1 yields

$$\limsup_{j\to\infty} \max_{\overline{\mathbb{D}_r}} u_j \le \log(\delta)\frac{c(\delta, r)}{\log(1/\alpha)} < 0$$

for $\delta < r < 1$ (note that $\alpha \le \operatorname{cap}(E_j) \le \operatorname{cap}(\mathbb{D}_\delta) = \delta < 1$). If we fix such an $r \in (\delta, 1)$, for $R := 1/r \in (1, 1/\delta)$ this estimate implies

$$\limsup_{j\to\infty} \max_{|z|=R} |S_{n_j} f(z)|^{1/n_j} < R.$$

Hence, there is $\gamma < 1$ so that

$$\max_{|z|=R} |S_{n_j} f(z)| < (\gamma R)^{n_j}$$

for j sufficiently large. For every $\nu \le n_j$, Cauchy's formula implies

$$|a_\nu|^{\frac{1}{\nu}} = \left| \frac{1}{2\pi i} \int\limits_{|z|=R} \frac{S_{n_j} f(z)}{z^{\nu+1}} dz \right|^{\frac{1}{\nu}} \le \gamma^{\frac{n_j}{\nu}} \cdot R^{\frac{n_j}{\nu}-1} \le \gamma \cdot R^{\frac{n_j}{\nu}-1}.$$

Now, if $q \in (0,1)$ is arbitrary and $qn_j \le \nu \le n_j$, we obtain

$$|a_\nu|^{\frac{1}{\nu}} \le \gamma \cdot R^{\frac{1}{q}-1},$$

and hence

$$\limsup_{j\to\infty} \max_{qn_j \le \nu \le n_j} |a_\nu|^{\frac{1}{\nu}} \le \gamma \cdot R^{\frac{1}{q}-1}.$$

Finally, we can choose $q < 1$ close enough to 1 so that $\gamma R^{\frac{1}{q}-1} < 1$ which gives

$$\limsup_{j\to\infty} \max_{qn_j \le \nu \le n_j} |a_\nu|^{\frac{1}{\nu}} < 1.$$

This implies that f has Hadamard-Ostrowski gaps.

2. From [13, Theorem 5.3.1] it is easily seen that convergence in capacity is preserved under bi-Lipschitz mappings: Let $K \subset \mathbb{C}$ be a compact set and $\varphi : K \to \mathbb{C}$ bi-Lipschitz. For any sequence (h_n) of Borel-measurable functions on $\varphi(K)$ we have $h_n \to \infty$ in capacity on $\varphi(K)$ if and only if $h_n \circ \varphi \to \infty$ in capacity on K. If we apply this for arbitrary compact subsets K of $\mathbb{C} \setminus \overline{\mathbb{D}}$ and $\varphi(z) = 1/z$ on K, the conclusion follows from the first part of the proof with $h_n(\zeta) = S_n f(1/\zeta)$. □

3 Further Remarks

1. A closer inspection of the proof of our main theorem shows that the existence of a sequence of compact subsets K_j of an annulus $\{z : R \le |z| \le S\}$, where $1 < R \le S$, with $\inf_{j\in\mathbb{N}} \operatorname{cap}(K_j) > 0$ and

$$\limsup_{j\to\infty} \max_{z\in K_j} \frac{1}{|z|} |S_{n_j} f(z)|^{1/n_j} < 1 \tag{4}$$

implies that f has Hadamard-Ostrowski gaps (in this case we still have

$$\limsup_{j\to\infty} \max_{E_j} u_j < 0$$

in the estimate (3), where again $E_j := 1/K_j$). In particular, (4) is satisfied if

$$\limsup_{j\to\infty} \max_{z\in K_j} |S_{n_j} f(z)|^{1/n_j} < R\,.$$

With regard to (1) and (2), this means that the subsequence (S_{n_j}) has a reduced growth compared to the full sequence on the compact sets K_j.

2. Consider a power series f of radius of convergence one and being analytically continuable to a domain Ω strictly larger than $\mathbb{D}$. From a classical result of Ostrowski on overconvergence (see e.g. [7, Thm. 16.7.1]), it follows that the conclusion of Corollary 1.6 does no longer hold if f has Hadamard-Ostrowski gaps.
More precisely, let $\xi \in \Omega \cap \partial\mathbb{D}$. If $(p_k)_{k\in\mathbb{N}}$ and $(q_k)_{k\in\mathbb{N}}$ are as in Definition 1.2, the subsequence $(S_{p_k}f)_{k\in\mathbb{N}}$ converges uniformly on some open disk U with centre ξ (to the continuation of f). Thus, in particular, no subsequence of $(S_{p_k}f)_{k\in\mathbb{N}}$ can tend to infinity quasi-everywhere on $U \setminus \overline{\mathbb{D}}$.
In contrast, there exist f having $\partial\mathbb{D}$ as its natural boundary and having Hadamard-Ostrowski gaps so that $(S_n f)(z) \to \infty$ for all $|z| > 1$. A simple example is given by the gap series

$$f(z) = \sum_{k=0}^{\infty} z^{2^k},$$

where one may choose $p_k = 2^k$ and $q_k = p_{k+1}$.

3. In [2] it was shown that, for functions f having no Hadamard-Ostrowski gaps, pointwise (finite) limit functions of $(S_n f)_{n\in\mathbb{N}}$ on sets $E \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ can only exist if E is polar. This also appears now as consequence of Corollary 1.6.

4. Let H_0 denote the space of all functions holomorphic in the punctured sphere $\mathbb{C}_\infty \setminus \{1\}$ (where $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$) and vanishing at infinity. From the result of Gardiner mentioned in Remark 1.3, it follows that functions in $H_0\setminus\{0\}$ do not have Hadamard-Ostrowski gaps. According to 3., finite limit functions can only exist on polar sets $E \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. As is shown by the geometric series f, finite limit functions may exist on sets of positive capacity on the boundary $\partial\mathbb{D}$ of the unit disk. Indeed, if $K \subset \partial\mathbb{D} \setminus \{1\}$ is a Dirichlet set (see e.g. [8]), then a subsequence of

$$S_n f(z) = (z^{n+1} - 1)(z - 1)^{-1}$$

tends to 0 uniformly on K. It is known that Dirichlet sets of Hausdorff dimension 1 exist. Polar sets, however, necessarily have vanishing Hausdorff dimension.

On the other hand, from a result of Melas [9], it easily follows that for all countable sets $E \subset \mathbb{C}\setminus\mathbb{D}$ there is a residual set in H_0 (where H_0 is endowed with the topology of locally uniform convergence) consisting of functions which are universal on E in the sense that for each function $h\colon E \to \mathbb{C}$ a subsequence of $(S_n f)_{n\in\mathbb{N}}$ tends to h pointwise on E. Moreover, [2, Thm. 2] shows that there is a residual set of functions in H_0 so that $\{S_n f\colon n \in \mathbb{N}\}$ is (uniformly) dense in the space $C(K)$ of continuous functions on K for "many" perfect sets $K \subset \mathbb{C} \setminus \mathbb{D}$. This shows in particular that, for a residual set of functions in H_0, uncountable exceptional sets $E \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ exist on which the sequence $(S_n f)_{n\in\mathbb{N}}$ turns out to be far away from tending pointwise to ∞.

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