HYPERCYCLIC C_0-SEMIGROUPS AND EVOLUTION FAMILIES GENERATED BY FIRST ORDER DIFFERENTIAL OPERATORS

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Abstract. We show that C_0-semigroups generated by first order partial differential operators on $L^p(\Omega,\mu)$ and $C_{0,\rho}(\Omega)$, respectively, are hypercyclic if and only if they are weakly mixing, where $\Omega\subset\mathbb{R}^d$ is open. In the case of $d=1$ we give an easy to check characterization of when this happens. Furthermore, we give an example of a hypercyclic evolution family such that not each of the operators of the family are hypercyclic themselves. This stands in complete contrast to hypercyclic C_0-semigroups.

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1. Introduction

A continuous linear operator T on a separable Banach space X is called *hypercyclic* if there is a hypercyclic vector $x\in X$ which means that $\{T^nx;\,n\in\mathbb{N}\}$ is dense in X. There are a number of articles dealing with hypercyclic operators, for a survey see e.g. [10], [11].

Analogously, a C_0-semigroup $T=(T(t))_{t\geq 0}$ on a Banach space X, or more generally a family $(T(\iota))_{\iota\in I}$ of continuous linear operators on X, is called *hypercyclic* if there exists an element $x\in X$ such that $\{T(t)x;\,t\geq 0\}$, resp. $\{T(\iota)x;\,\iota\in I\}$ is dense in X. In this case x is again called *hypercyclic vector* for the semigroup T, or for the family $(T(\iota))_{\iota\in I}$, respectively.

The first example of a hypercyclic C_0-semigroup was given by Rolewicz [14], already in 1969. A systematic study of hypercyclic C_0-semigroup was initiated by Desch, Schappacher, and Webb [7]. Since then, various authors contributed to this subject, see e.g. [2], [4], [3], [12], [1], [6], [5].

A notion closely related to hypercyclicity is that of *transitivity*. A C_0-semigroup, or more generally a family of continuous linear operators $(T_\iota)_{\iota\in I}$ on a Banach space X is called *transitive* if for each pair of non-empty, open subsets U,V of X there is $\iota\in I$ such that $T_\iota^{-1}(U)\cap V\neq\emptyset$. It was shown by Grosse-Erdmann that $(T_\iota)_{\iota\in I}$ is transitive if and only if $(T_\iota)_{\iota\in I}$ is hypercyclic and the set of hypercyclic vectors is dense [9, Satz 1.2.2 and its proof]. Moreover, Peris proved that a commuting family of continuous linear operators $(T_\iota)_{\iota\in I}$ for which each T_ι has dense range is hypercyclic if and only if the set of hypercyclic vectors is dense [13]. In particular, an arbitrary commuting family of continuous linear operators $(T_\iota)_{\iota\in I}$ for which each T_ι has dense range is hypercyclic if and only if it is transitive. A C_0-semigroup is hypercyclic if and only if it is transitive (see e.g. [7, Theorem 2.2]). A family of continuous linear operators $(T_\iota)_{\iota\in I}$ on a Banach space X is called *weakly mixing* if $(T_\iota\oplus T_\iota)_{\iota\in I}$ is transitive on $X\oplus X$.

In [12], we gave a condition characterizing when C_0-semigroups on spaces of integrable and continuous functions, respectively, generated by first order partial

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differential operators are hypercyclic. In section 3 below we show that these semigroups are hypercyclic if and only if they are weakly mixing. However, for a given C_0-semigroup of the form discussed here, it might be difficult to check whether these conditions hold, because they involve sequences of integrals with respect to certain measures, and it has to be verified that these sequences tend to zero. Therefore, we simplify the condition characterizing hypercyclicity considerably in the case of one spacial dimension. This gives a characterization which is easy to check in concrete examples. Before we do this, in section 2 we deal with families of certain weighted composition operators on the aforementioned spaces and prove that they are transitive if and only if they are weakly mixing.

In section 4 we use the results of section 2 to show that for a hypercyclic evolution family $U = (U(s,t))_{s\in\mathbb{R},t\geq s}$ on a Banach space it may happen that some of the operators $U(s,t)$ are not hypercyclic. This stands in complete contrast to C_0-semigroups, which can be viewed as special evolution families. In [4], Conejero, Müller, and Peris showed that for a hypercyclic C_0-semigroup T not only each operator $T(t), t > 0$, has to be hypercyclic but also that the set of the respective hypercyclic vectors coincide.

2. Families of weighted composition operators

In this section we show that certain families of weighted composition operators on spaces of measurable or continuous functions, respectively, are transitive if and only if they are weakly mixing and give a characterization of when this happens.

Let Ω be an open subset of $\mathbb{R}^d$, $I \neq \emptyset$ a set, and let $\varphi : I \times \Omega \to \Omega$ be a mapping such that $\varphi(\iota, \cdot)$ is injective and continuous for all $\iota \in I$. Typically, I will be either $\mathbb{N}_0, [0,\infty)$, or $\mathbb{R}$.

Furthermore, let ρ be a positive function on Ω, i.e. $\rho(x) > 0,\ x \in \Omega$. We will consider spaces of continuous functions $C_{0,\rho}(\Omega,\mathbb{C})$ and $C_{0,\rho}(\Omega,\mathbb{R})$, where $C_{0,\rho}(\Omega,\mathbb{K}) := \{f : X \to \mathbb{K} \text{ continuous}; \forall \varepsilon > 0 : \{x \in \Omega;\ |f(x)|\rho(x) \geq \varepsilon\} \text{ is compact}\}$ is equipped with the norm $\|f\| := \sup_{x\in\Omega} |f(x)|\rho(x)$.

Moreover, we consider spaces of p-integrable functions. Let μ be a (positive) locally finite Borel measure on Ω. In particular, μ is σ-finite. For $1 \leq p < \infty$ let $L^p(\mu,\mathbb{R}), L^p(\mu,\mathbb{C})$ be as usual. Since in most occasions it will not matter whether the considered functions are real or complex valued, we will write $L^p(\mu)$ and $C_{0,\rho}(\Omega)$ for brevity. Since μ is locally finite, the set of compactly supported, continuous functions $C_c(\Omega)$ is dense in $L^p(\mu)$ (cf. [15, Theorem 3.14]) and obviously $C_c(\Omega)$ is dense in $C_{0,\rho}(\Omega)$, too. $L^p(\mu)$ as well as $C_{0,\rho}(\Omega)$ is a separable Banach space.

Let $w : I \times \Omega \to (0,\infty)$ be such that for every $\iota \in I$ we have $1/w(\iota,\cdot) \in L^\infty_{loc}(\mu)$. We assume that for $\iota \in I$ the mapping $T_{w,\varphi}(\iota) : L^p(\mu) \to L^p(\mu), f \mapsto w(\iota,\cdot)f(\varphi(\iota,\cdot))$ is a well-defined continuous operator on $L^p(\mu)$ and if all $w(\iota,\cdot)$ are continuous that the same is true for $T_{w,\varphi}(\iota) : C_{0,\rho}(\Omega) \to C_{0,\rho}(\Omega), f \mapsto w(\iota,\cdot)f(\varphi(\iota,\cdot))$ (see e.g. [12, Theorem 4.1, Theorem 4.2] for conditions ensuring this). $T_{w,\varphi}(\iota)$ is called a *weighted composition operator*.

In [12] we characterized when the family $T_{w,\varphi}$ is transitive on $L^p(\mu)$, respectively on $C_{0,\rho}(\Omega)$. We now show that this is the case if and only if $T_{w,\varphi}$ is weakly mixing. In order to formulate our theorems we make the following observation.

Since $\varphi(\iota,\cdot)$ is one-to-one for every $\iota \in I$ it has an inverse mapping from $\varphi(\iota,\Omega)$ to Ω which is denoted by $\varphi(-\iota,\cdot)$. In case of $(I,+)$ being a group we want to emphasis that in general $\varphi(-\iota,\cdot)$ is different from $\varphi(\kappa,\cdot)$ for κ being the additive inverse of ι in $(I,+)$. Nevertheless, in the most important case when $\varphi(\iota_1+\iota_2,\cdot) = \varphi(\iota_1,\cdot)\circ\varphi(\iota_2,\cdot)$ for all $\iota_1,\iota_2 \in I$ and $\varphi(e,\cdot) = id_X$ with e being the unit in $(I,+)$, we have $\varphi(-\iota,\cdot) = \varphi(\kappa,\cdot)$.

Being an open subset of $\mathbb{R}^d$, Ω is σ-compact. Hence, by the continuity and injectivity of $\varphi(\iota,\cdot)$ it follows that for each closed subset C of Ω the image $\varphi(\iota, C)$ is an F_σ-set, and in particular, Borel measurable. Since the closed subsets of Ω generate the Borel σ-algebra over Ω, the injectivity of $\varphi(\iota,\cdot)$ now implies that $\varphi(\iota, B)$ is a Borel subset of Ω whenever B is. So by setting

$$\nu_{p,\iota}(B) := \int_{\varphi(\iota,\cdot)^{-1}(B)} w(\iota,\cdot)^p \, d\mu$$

and

$$\nu_{p,-\iota}(B) := \int_{\varphi(\iota,B)} 1/w(\iota,\varphi(-\iota,\cdot))^p \, d\mu$$

for $p \in [1,\infty)$ and $\iota \in I$ we obtain a family of well-defined Borel-measures on Ω.

Obviously, for a non-negative, measurable function f one has

$$\int f \, d\nu_{p,\iota} = \int w(\iota,\cdot)^p f(\varphi(\iota,\cdot)) \, d\mu$$

and

$$\int f \, d\nu_{p,-\iota} = \int \chi_{\varphi(\iota,\Omega)} f(\varphi(-\iota,\cdot))/w(\iota,\varphi(-\iota,\cdot))^p \, d\mu.$$

Theorem 1. *Under the above hypotheses on φ and w, the following are equivalent.*

i) *$T_{w,\varphi}$ is weakly mixing on $L^p(\mu)$.*
ii) *$T_{w,\varphi}$ is transitive on $L^p(\mu)$.*
iii) *For every compact subset K of Ω there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a sequence $(\iota_n)_{n\in\mathbb{N}}$ in I such that*

$$\lim_{n\to\infty} \mu(K \backslash L_n) = 0$$

as well as

$$\lim_{n\to\infty} \nu_{p,\iota_n}(L_n) = \lim_{n\to\infty} \nu_{p,-\iota_n}(L_n) = 0.$$

PROOF: That *i)* implies *ii)* is obvious. That *ii)* implies *iii)* is shown in [12, Theorem 4.3]. To show that *iii)* implies *i)* we write T instead of $T_{w,\varphi}$ for brevity.

Let $U_i, V_i, i = 1, 2$, be non-empty open subsets of $L^p(\mu)$ and $f_i, g_i \in C_c(\Omega)$ be such that $f_i \in U_i, g_i \in V_i, i = 1, 2$. Set $K := \operatorname{supp} f_1 \cup \operatorname{supp} f_2 \cup \operatorname{supp} g_1 \cup \operatorname{supp} g_2$ which is a compact subset of Ω. Let $(L_n)_{n\in\mathbb{N}}$ and $(\iota_n)_{n\in\mathbb{N}}$ be as in iii) for K.

For $n \in \mathbb{N}$ we set

$$v_n := \left(\frac{g_1(\varphi(-\iota_n,\cdot))}{w(\iota_n,(\varphi(-\iota_n,\cdot)))}\right)\chi_{\varphi(\iota_n,L_n)}$$

and

$$\tilde{v}_n := \left(\frac{g_2(\varphi(-\iota_n,\cdot))}{w(\iota_n,(\varphi(-\iota_n,\cdot)))}\right)\chi_{\varphi(\iota_n,L_n)}$$

which are measurable and because of $1/w(\iota,\cdot) \in L^\infty_{loc}(\mu)$ in $L^p(\mu)$. Denoting the sup-norm by $\|\cdot\|_\infty$ we then have

$$\|v_n\|^p \le \|g_1\|_\infty^p \nu_{p,-\iota_n}(L_n)$$

so that $(f_1\chi_{L_n} + v_n)_{n\in\mathbb{N}}$ converges to f_1. Furthermore

$$\|T(\iota_n)(f_1\chi_{L_n})\|^p \le \|f_1\|_\infty^p \nu_{p,\iota_n}(L_n),$$

so that $(T(\iota_n)(f_1\chi_{L_n}))_{n\in\mathbb{N}}$ converges to 0. Since $T(\iota_n)v_n - g_1 = -g_1\chi_{K\backslash L_n}$ we see that $(T(\iota_n)(f_1\chi_{L_n} + v_n))_{n\in\mathbb{N}}$ converges to g_1. The same arguments yield that $(f_2\chi_{L_n} + \tilde{v}_n)_{n\in\mathbb{N}}$ converges to f_2 and $(T(\iota_n)(f_2\chi_{L_n} + \tilde{v}_n))_{n\in\mathbb{N}}$ converges to g_2. Thus, $T(\iota_n)(U_i) \cap V_i \neq \emptyset, i = 1, 2$, for sufficiently large n, so that *i)* follows. □

The same kind of arguments as above show that *iii*) implies *i*) in the next theorem, except that one needs Brouwer's Theorem. By Brouwer's Theorem, $\varphi(\iota,\Omega)$ is an open subset of Ω since $\varphi(\iota,\cdot)$ is injective. Therefore, $C_c(\varphi(\iota,\Omega))$ can be identified with a subspace of $C_c(\Omega)$, so that $g(\varphi(-\iota_n,\cdot))\in C_c(\Omega)$ for $g\in C_c(\Omega)$. That *ii*) implies *iii*) is shown in [12, Theorem 4.5].

Theorem 2. *Under the above hypotheses on φ and w and the additional assumption that for all compact subsets K of Ω we have $\inf_{x\in K}\rho(x)>0$, the following are equivalent*

i) *$T_{w,\varphi}$ is weakly mixing on $C_{0,\rho}(\Omega)$.*
ii) *$T_{w,\varphi}$ is topologically transitive on $C_{0,\rho}(\Omega)$*
iii) *For every compact subset K of Ω we can find a sequence $(\iota_n)_{n\in\mathbb{N}}$ in I such that*

$$\lim_{n\to\infty}\sup_{x\in\varphi(\iota_n,\cdot)^{-1}(K)} w(\iota_n,x)\rho(x)=\lim_{n\to\infty}\sup_{x\in\varphi(\iota_n,K)}\frac{\rho(x)}{w(\iota_n,\varphi(-\iota_n,x))}=0.$$

3. Hypercyclic C_0-semigroups

In this section we use the results of the previous one to show that certain C_0-semigroups generated by first order differential operators are hypercyclic if and only if they are weakly mixing.

Let $\Omega\subset\mathbb{R}^d$ be open and $F:\Omega\to\mathbb{R}^d$ be a C^1-vector field. We make the general assumption that for every $x_0\in\Omega$ the unique solution $\varphi(\cdot,x_0)$ of the initial value problem

$$\dot{x}=F(x),\ x(0)=x_0$$

is defined on $[0,\infty)$. Since F is a C^1-vector field it is well-known that the same is true for $\varphi(t,\cdot)$ for every $t\geq 0$. For a continuous function $h:\Omega\to\mathbb{R}$ we define for $t\geq 0$

$$h_t:\Omega\to(0,\infty),x\mapsto\exp(\int_0^t h(\varphi(s,x))\,ds).$$

It is easily seen, that because of $\varphi(t+s,x)=\varphi(t,\varphi(s,x))$ we have $h_{t+s}(x)=h_t(x)h_s(\varphi(t,x))$ for all $s,t\geq 0$ and $x\in\Omega$.

As in section 2, let μ be a locally finite Borel measure on Ω and ρ a positive function on Ω. We want to define a C_0-semigroup T on $L^p(\mu)$, resp. $C_{0,\rho}(\Omega)$, via $(T(t)f)(x):=h_t(x)f(\varphi(t,x))$. In order to do so we recall the definition of the following Borel measures on Ω from section 2. For $1\leq p<\infty$ and $t\geq 0$

$$\nu_{p,t}(B):=\int_{\varphi(t,\cdot)^{-1}(B)} h_t^p\,d\mu$$

and

$$\nu_{p,-t}(B):=\int_{\varphi(t,B)} 1/h_t(\varphi(t,\cdot))^p\,d\mu.$$

Recall that $\varphi(t,\cdot),t\geq 0$, is one-to-one and that we denote its inverse mapping from $\varphi(t,\Omega)$ to Ω by $\varphi(-t,\cdot)$. Obviously, if $\varphi(\cdot,x)$ is well-defined in $-t<0$ it follows that the two notions of $\varphi(-t,x)$ coincide, hence there is no problem of consistency.

Recall that for a positive measurable function f on Ω one has

$$\int f\,d\nu_{p,t}=\int h_t^p(x)f(\varphi(t,x))\,d\mu(x)$$

and

$$\int f\,d\nu_{p,-t}=\int\chi_{\varphi(t,\Omega)}(x)f(\varphi(-t,x))/h_t(\varphi(-t,x))^p\,d\mu(x).$$

For a general characterization in terms of the measures $(\nu_{p,t})_{t\geq 0}$ of when the operators $T(t)f = h_t f(\varphi(t,\cdot)), t \geq 0$, are well-defined on $L^p(\mu)$ and form a C_0-semigroup, see [12, Theorem 4.7]. If the so defined operators form a C_0-semigroup on $L^p(\mu)$ we call the measure μ *p-admissible for F and h*.

In the special case when the measure μ has a positive Lebesgue density ρ, one has the following characterization of when μ is p-admissible for F and h. For its proof see [12, Proposition 4.12].

Proposition 3. *Let μ be a locally finite Borel measure on Ω admitting a positive Lebesgue density ρ. Then, the following are equivalent.*

i) *μ is p-admissible for F and h.*
ii) *There are $M \geq 1, \omega \in \mathbb{R}$ such that for $t \geq 0$ and λ^d-almost all $x \in \Omega$*

$$h_t^p(x)\rho(x) \leq Me^{\omega t}\rho(\varphi(t,x))|\det D_x\varphi(t,x)|,$$

where λ^d denotes d-dimensional Lebesgue measure and $D_x\varphi(t,x)$ the Jacobian of $x \mapsto \varphi(t,x)$. If ii) holds then $\|T(t)\| \leq Me^{t\frac{\omega}{p}}$.

For the case of continuous functions one has the following theorem. For its proof see [12, Theorem 4.8].

Theorem 4. *Let ρ be a positive, upper semicontinuous function on Ω. The following are equivalent.*

i) *The family of mappings $T(t) : C_{0,\rho}(\Omega) \to C_{0,\rho}(\Omega), f \mapsto h_t(\cdot)f(\varphi(t,\cdot))$ is well-defined and a C_0-semigroup on $C_{0,\rho}(\Omega)$.*
ii) a) *There are constants $M \geq 1, \omega \in \mathbb{R}$ such that for all $t \geq 0, x \in \Omega$ one has $h_t(x)\rho(x) \leq Me^{\omega t}\rho(\varphi(t,x))$.*
    b) *For every compact subset K of Ω and every $\delta \geq 0$ the sets $\varphi^{-1}(t,\cdot)(K)\cap \{x \in \Omega;\ h_t(x)\rho(x) \geq \delta\}$ are compact for every $t \geq 0$.*

We call a positive function ρ on Ω *C_0-admissible for F and h* if via $(T(t)f)(x) = h_t(x)f(\varphi(t,x))$ a C_0-semigroup is defined on $C_{0,\rho}(\Omega)$.

Under the assumption that h is continuously differentiable, it is shown in [12] that for μ p-admissible for F and h, resp. ρ C_0-admissible for F and h, the generator of our C_0-semigroup is given by the closure of the operator

$$C_c^1(\Omega) \to X, f \mapsto \sum_{j=1}^d F_j\partial_j f + hf$$

in $X = L^p(\mu)$, resp. in $X = C_{0,\rho}(\Omega)$, with $C_c^1(\Omega) = C_c(\Omega) \cap C^1(\Omega)$.

Taking into account that h is continuous, hence $1/h_t \in L^\infty(\mu)$, the following theorems are immediate consequences of Theorem 1 and Theorem 2, respectively.

Theorem 5. *Let μ be a locally finite, p-admissible Borel measure for F and h on Ω. Then the following are equivalent.*

i) *The C_0-semigroup defined by $(T(t)f)(x) = h_t(x)f(\varphi(t,x))$ is weakly mixing on $L^p(\mu)$.*
ii) *The C_0-semigroup defined by $(T(t)f)(x) = h_t(x)f(\varphi(t,x))$ is hypercyclic on $L^p(\mu)$.*
iii) *For every compact subset K of Ω there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \mu(K\backslash L_n) = 0$$

as well as

$$\lim_{n\to\infty}\nu_{p,t_n}(L_n)=\lim_{n\to\infty}\nu_{p,-t_n}(L_n)=0.$$

Theorem 6. *Let ρ be a positive C_0-admissible function for F and h on Ω such that $\inf_{x\in K}\rho(x)>0$ for every compact subset K of Ω. Then the following are equivalent.*

i) *The C_0-semigroup defined by $(T(t)f)(x)=h_t(x)f(\varphi(t,x))$ is weakly mixing on $C_{0,\rho}(\Omega)$.*

ii) *The C_0-semigroup defined by $(T(t)f)(x)=h_t(x)f(\varphi(t,x))$ is hypercyclic on $C_{0,\rho}(\Omega)$.*

iii) *For every compact subset K of Ω there is a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty}\sup_{x\in K}h_{t_n}(\varphi(-t_n,x))\rho(\varphi(-t_n,x))\chi_{\varphi(t_n,\Omega)}(x)=\lim_{n\to\infty}\sup_{x\in K}\frac{\rho(\varphi(t_n,x))}{h_{t_n}(x)}=0.$$

It seems that the conditions for hypercyclicity given by the above theorems might be hard to evaluate for a given C_0-semigroup. It turns out that in the one dimensional case, i.e. $d=1$, the above characterizations can be considerably simplified, provided that μ admits a positive Lebesgue density.

The reason for this is that for $d=1$ a compact subset which is contained in a single component of $\Omega\backslash\{F=0\}$ is already contained in the trajectory $\{\varphi(t,x);\,t\geq 0\}$ of a single point $x\in\Omega$, simply because the trajectories are either one-point sets (in case of the point being a zero of F) or open intervals (which then do not contain any zero of F and are trajectories without double points). So, for a compact subset K of a component of $\Omega\backslash\{F=0\}$ there is a compact interval $[a,b]$ containing K which is contained in the same component of $\Omega\backslash\{F=0\}$ as K such that either $F_{|[a,b]}>\gamma>0$ or $F_{|[a,b]}<\gamma'<0$ for some suitable γ,γ'. In the first case, it follows that $\varphi(t,a)\geq a+t\gamma>b$ for all $t>(b-a)/\gamma$, while in the second case $\varphi(t,b)\leq b+t\gamma'<a$ for all $t>(a-b)/\gamma'$. So in both cases $[a,b]$, hence K, is contained in a trajectory of a single point.

Using this idea, we first prove a technical tool.

Lemma 7. *Let $\Omega\subset\mathbb{R}$ be open and $[a,b]\subset\{F\neq 0\}$. Assume that $\rho:\Omega\to(0,\infty)$ is measurable and satisfies $h_t^p(x)\rho(x)\leq Me^{\omega t}\rho(\varphi(t,x))|\partial_2\varphi(t,x)|$ for some constants $M\geq 1,\omega\geq 0$ and for every $t\geq 0,x\in[a,b]$.*

Then there is $C>0$ such that $1/C<\rho(y)<C$ for all $y\in[a,b]$ and

$$\begin{aligned}&h_t^p(\varphi(-t,c))\rho(\varphi(-t,c))|\partial_2\varphi(-t,c)|\chi_{\varphi(t,\Omega)}(c)\\ \leq\;& Ch_t^p(\varphi(-t,y))\rho(\varphi(-t,y))|\partial_2\varphi(-t,y)|\chi_{\varphi(t,\Omega)}(y)\\ \leq\;& C^2h_t^p(\varphi(-t,d))\rho(\varphi(-t,d))|\partial_2\varphi(-t,d)|\chi_{\varphi(t,\Omega)}(d)\end{aligned}$$

as well as

$$\begin{aligned}h_t^{-p}(c)\rho(\varphi(t,c))|\partial_2\varphi(t,c)|\;&\leq\; Ch_t^{-p}(y)\rho(\varphi(t,y))|\partial_2\varphi(t,y)|\\ &\leq\; C^2h_t^{-p}(d)\rho(\varphi(t,d))|\partial_2\varphi(t,d)|.\end{aligned}$$

for all $t\geq 0$, where $c:=a,d:=b$ if $F_{|[a,b]}>0$, respectively $c:=b,d:=a$ if $F_{|[a,b]}<0$.

Proof: Let $[a,b]\subset\{F\neq 0\}$. We define

$$c:=\begin{cases}a & ,\quad F_{|[a,b]}>0\\ b & ,\quad F_{|[a,b]}<0\end{cases}$$

and

$$d := \begin{cases} a \quad , \quad F_{|[a,b]} < 0 \\ b \quad , \quad F_{|[a,b]} > 0. \end{cases}$$

Then there is $r > 0$ such that for every $y \in [a,b]$ there are $s_y, t_y \in [0,r]$ such that $\varphi(s_y, y) = d$ and $\varphi(t_y, c) = y$. It follows that for all $y \in [a,b]$

$$\begin{aligned} h^p_{t_y}(c)\rho(c) &\leq Me^{\omega t_y}\rho(\varphi(t_y,c))|\partial_2\varphi(t_y,c)| \\ &\leq Me^{\omega r}\rho(y)|\partial_2\varphi(t_y,c)| \end{aligned}$$

and

$$\begin{aligned} h^p_{s_y}(y)\rho(y) &\leq Me^{\omega s_y}\rho(\varphi(s_y,y))|\partial_2\varphi(s_y,y)| \\ &\leq Me^{\omega r}\rho(d)|\partial_2\varphi(s_y,y)|. \end{aligned}$$

The continuity of the mappings $(t,x) \mapsto \partial_2\varphi(t,x)$ and $(t,x) \mapsto h_t(x)$, and the fact that for fixed t the map $x \mapsto \partial_2\varphi(t,x)$ has no zeros, now imply the first part of the lemma.

Observe that we have $\varphi(s+t,y) = \varphi(s,\varphi(t,y))$ for all $s,t \in \mathbb{R}$ for which the involved quantities are defined. From this it follows that

$$\partial_2\varphi(s,\varphi(t,y))\partial_2\varphi(t,y) = \partial_2\varphi(t,\varphi(s,y))\partial_2\varphi(s,y).$$

In particular

$$\begin{aligned} \partial_2\varphi(s_y,\varphi(-t,y))\partial_2\varphi(-t,y) &= \partial_2\varphi(-t,\varphi(s_y,y))\partial_2\varphi(s_y,y) \\ &= \partial_2\varphi(-t,d)\partial_2\varphi(s_y,y), \end{aligned} \tag{1}$$

for $y \in \varphi(t,\Omega)$, as well as

$$\begin{aligned} \partial_2\varphi(s_y,\varphi(t,y))\partial_2\varphi(t,y) &= \partial_2\varphi(t,\varphi(s_y,y))\partial_2\varphi(s_y,y) \\ &= \partial_2\varphi(t,d)\partial_2\varphi(s_y,y). \end{aligned} \tag{2}$$

Moreover, $h_t(y)h_s(\varphi(t,y)) = h_{t+s}(y) = h_s(y)h_t(\varphi(s,y))$ again for all $s,t \in \mathbb{R}$ for which the involved quantities are defined. Hence,

$$\begin{aligned} h_t(\varphi(-t,y))h_{s_y}(\varphi(t,\varphi(-t,y))) &= h_{s_y}(\varphi(-t,y))h_t(\varphi(-t,\varphi(s_y,y))) \\ &= h_{s_y}(\varphi(-t,y))h_t(\varphi(t,d)), \end{aligned} \tag{3}$$

for $y \in \varphi(t,\Omega)$, as well as

$$h_t(y)h_{s_y}(\varphi(t,y)) = h_{s_y}(y)h_t(\varphi(s_y,y)) = h_{s_y}(y)h_t(d). \tag{4}$$

Clearly, if $y \in \varphi(t,\Omega)$ and $\varphi(s_y,y) = d$ it follows that $d \in \varphi(t,\Omega)$, hence $\chi_{\varphi(t,\Omega)}(y) \leq \chi_{\varphi(t,\Omega)}(d)$. From this we get for $t \geq 0$ and all $y \in [a,b]$

$$\begin{aligned} & h^p_t(\varphi(-t,y))\rho(\varphi(-t,y))|\partial_2\varphi(-t,y)|\chi_{\varphi(t,\Omega)}(y) \\ = \; & \frac{h^p_t(\varphi(-t,y))\, h^p_{s_y}(\varphi(-t,y))\rho(\varphi(-t,y))\, |\partial_2\varphi(-t,y)|}{h^p_{s_y}(\varphi(-t,y))}\chi_{\varphi(t,\Omega)}(y) \\ \leq \; & \frac{h^p_t(\varphi(-t,y))Me^{\omega r}\rho(\varphi(s_y,\varphi(-t,y))|\partial_2\varphi(s_y,\varphi(-t,y))\partial_2\varphi(-t,y)|}{h^p_{s_y}(\varphi(-t,y))}\chi_{\varphi(t,\Omega)}(y) \\ = \; & \frac{h^p_t(\varphi(-t,y))Me^{\omega r}\rho(\varphi(-t,d))|\partial_2\varphi(-t,d)\partial_2\varphi(s_y,y)|h^p_t(\varphi(-t,d))}{h^p_{s_y}(\varphi(-t,y))h^p_t(\varphi(-t,d))}\chi_{\varphi(t,\Omega)}(y) \\ = \; & \frac{h^p_t(\varphi(-t,y))Me^{\omega r}\rho(\varphi(-t,d))|\partial_2\varphi(-t,d)\partial_2\varphi(s_y,y)|h^p_t(\varphi(-t,d))}{h^p_t(\varphi(-t,y))h^p_{s_y}(\varphi(t,\varphi(-t,y)))}\chi_{\varphi(t,\Omega)}(y) \\ \leq \; & \frac{Me^{\omega r}|\partial_2\varphi(s_y,y)|}{h^p_{s_y}(y)}\, h^p_t(\varphi(-t,d))\rho(\varphi(-t,d))|\partial_2\varphi(-t,d)|\chi_{\varphi(t,\Omega)}(d), \end{aligned}$$

where we applied (1) in the fourth and (3) in the fifth line. Using the continuity of the mappings $(t,x)\mapsto\partial_2\varphi(t,x)$ and $(t,x)\mapsto h_t(x)$ and the compactness of $[0,r]\times[a,b]$, we get a constant $K>0$ such that for all $y\in[a,b]$

$$\begin{aligned}&h_t^p(\varphi(-t,y))\rho(\varphi(-t,y))|\partial_2\varphi(-t,y)|\chi_{\varphi(t,\Omega)}(y)\\ \leq\ & Kh_t^p(\varphi(-t,d))\rho(\varphi(-t,d))|\partial_2\varphi(-t,d)|\chi_{\varphi(t,\Omega)}(d).\end{aligned}$$

In the same way one shows that for all $y\in[a,b]$

$$\begin{aligned}&\frac{h_t^p(\varphi(-t,c))h_{t_y}^p(\varphi(-t,c))\rho(\varphi(-t,c))|\partial_2\varphi(-t,c)|}{h_{t_y}^p(\varphi(-t,c))}\chi_{\varphi(t,\Omega)}(c)\\ \leq\ & \frac{Me^{\omega r}|\partial_2\varphi(t_y,c)|}{h_{t_y}^p(c)}\,h_t^p(\varphi(-t,y))\rho(\varphi(-t,y))|\partial_2\varphi(-t,y)|\chi_{\varphi(t,\Omega)}(y)\end{aligned}$$

giving

$$\begin{aligned}&h_t^p(\varphi(-t,c))\rho(\varphi(-t,c))|\partial_2\varphi(-t,c)|\chi_{\varphi(t,\Omega)}(c)\\ \leq\ & Kh_t^p(\varphi(-t,y))\rho(\varphi(-t,y))|\partial_2\varphi(-t,y)|\chi_{\varphi(t,\Omega)}(y)\end{aligned}$$

for all $y\in[a,b]$.

Furthermore, for $t\geq 0$ and all $y\in[a,b]$

$$\begin{aligned}h_t^{-p}(y)\rho(\varphi(t,y))|\partial_2\varphi(t,y)| &= \frac{h_{s_y}^p(\varphi(t,y))}{h_{s_y}^p(\varphi(t,y))}\,h_t^{-p}(y)\rho(\varphi(t,y))|\partial_2\varphi(t,y)|\\ &\leq \frac{Me^{\omega s_y}\rho(\varphi(s_y,\varphi(t,y))|\partial_2\varphi(s_y,\varphi(t,y))\partial_2\varphi(t,y)|}{h_{s_y}^p(\varphi(t,y))h_t^p(y)}\\ &= \frac{Me^{\omega s_y}\rho(\varphi(t,d))|\partial_2\varphi(t,d)\partial_2\varphi(s_y,y)|}{h_{s_y}^p(y)h_t^p(d)}\\ &\leq \frac{Me^{\omega r}|\partial_2\varphi(s_y,y)|}{h_{s_y}^p(y)}h_t^{-p}(d)\rho(\varphi(t,d))|\partial_2\varphi(t,d)|,\end{aligned}$$

where we used (2) and (4) in the third line. Again the continuity of the mappings $(t,x)\mapsto\partial_2\varphi(t,x)$ and $(t,x)\mapsto h_t(x)$ and the compactness of $[0,r]\times[a,b]$ give a constant $L>0$ such that for all $y\in[a,b]$

$$h_t^{-p}(y)\rho(\varphi(t,y))|\partial_2\varphi(t,y)|\leq Lh_t^{-p}(d)\rho(\varphi(t,d))|\partial_2\varphi(t,d)|.$$

In the same way as above one shows that for all $y\in[a,b]$

$$h_t^{-p}(c)\rho(\varphi(t,c))|\partial_2\varphi(t,c)|\leq Lh_t^{-p}(y)\rho(\varphi(t,y))|\partial_2\varphi(t,y)|,$$

proving the lemma. □

As a second tool, we need the following proposition. For its proof, see [12, Proposition 4.12].

Proposition 8. *Let μ be p-admissible for F and h with Lebesgue density ρ. Then, a μ-density of $\nu_{p,t}$, respectively $\nu_{p,-t}$, is given by*

$$\chi_{\varphi(t,\Omega_t)}\frac{h_t^p(\varphi(-t,\cdot))\rho(\varphi(-t,\cdot))|\det D_x\varphi(-t,\cdot)|}{\rho},$$

respectively

$$\frac{h_t^{-p}\rho(\varphi(t,\cdot))|\det D_x\varphi(t,\cdot)|}{\rho}.$$

Recall that we denote d-dimensional Lebesgue measure by λ^d.

Theorem 9. *Let $\Omega\subset\mathbb{R}$ be open and assume the locally finite p-admissible measure μ has a positive Lebesgue density ρ. Then, the following are equivalent.*

i) *The C_0-semigroup T defined via $(T(t)f)(x) = h_t(x)f(\varphi(t,x))$ is hypercyclic on $L^p(\mu)$.*
ii) *The C_0-semigroup T defined via $(T(t)f)(x) = h_t(x)f(\varphi(t,x))$ is weakly mixing on $L^p(\mu)$.*
iii) *$\lambda^1(F=0) = 0$ and for every $m \in \mathbb{N}$ for which there are m different components $C_1, \ldots, C_m$ of $\Omega\backslash\{F=0\}$, for λ^m-almost all choices of $x_j \in C_j, j = 1, \ldots, m$, there is a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that*

$$\lim_{n\to\infty} h_{t_n}^{-p}(x_j)\rho(\varphi(t_n,x_j))\partial_2\varphi(t_n,x_j) = 0$$

as well as

$$\lim_{n\to\infty} h_{t_n}^{p}(\varphi(-t_n,x_j))\rho(\varphi(-t_n,x_j))\partial_2\varphi(-t_n,x_j)\chi_{\varphi(t_n,\Omega)}(x_j) = 0$$

for $j = 1, \ldots, m$.

Proof: That $i)$ implies $ii)$ follows from Theorem 5. In order to show that $ii)$ implies $iii)$ observe that $\varphi(t,x) = x$ if $F(x) = 0$ so that $h_t(x)f(\varphi(t,x)) = \exp(th(x))f(x)$ for every $f \in L^p(\mu)$ on $\{F=0\}$. From this it follows easily that T cannot be hypercyclic if $\lambda^1(F=0) > 0$. Let $x_1, \ldots, x_m$ be from different components of $\Omega\backslash\{F=0\}$ which, by Proposition 3, we assume without loss of generality to satisfy $h_t(x_j)\rho(x_j) \leq Me^{\omega t}\rho(\varphi(t,x_j))|\partial_2\varphi(t,x_j)|$ for all $t \geq 0, j = 1, \ldots, m$. Since Ω is open there is $r < 0$ such that $\varphi(t,x_j)$ is well-defined for all $t \in [r,\infty), j = 1, \ldots, m$ and the aforementioned inequality is valid for $\varphi(r,x_j)$ in place of x_j, too. For $j = 1, \ldots, m$ we define $K_j := \{\varphi(t,x_j); 0 \leq t \leq 1\}$ if $F(x_j) > 0$, respectively $K_j := \{\varphi(t,x_j); r \leq t \leq 0\}$ if $F(x_j) < 0$. Then the K_j's are compact intervals contained in $\Omega\backslash\{F=0\}$ satisfying $\lambda(K_j) > 0$, since $F(x_j) \neq 0$, and $K_j = [x_j, \varphi(1,x_j)]$ if $F(x_j) > 0$, respectively $K_j = [x_j, \varphi(r,x_j)]$ if $F(x_j) < 0$. In particular $\mu(K_j) > 0$. From Theorem 5 it follows that for the compact subset $K := \cup_{1\leq j\leq m}K_j$ of Ω there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\mu(K\cap L_n^c) = \lim_{n\to\infty}\nu_{p,t_n}(L_n) = \lim_{n\to\infty}\nu_{p,-t_n}(L_n) = 0.$$

Since T is weakly mixing, it follows from Proposition 3 that $\omega > 0$, because otherwise $\{\|T(t)\|; t \geq 0\}$ was bounded, implying the boundedness of each orbit under T. Defining $L_{n,j} := L_n \cap K_j, n \in \mathbb{N}, 1 \leq j \leq m$ we obtain from Proposition 8 and Lemma 7 that for some constant $C_j > 0$

$$\begin{aligned}\nu_{p,-t_n}(L_{n,j}) &= \int_{L_{n,j}} \frac{h_{t_n}^{-p}(y)\rho(\varphi(t_n,y))|\partial_2\varphi(t_n,y)|}{\rho(y)}\,d\mu(y)\\ &\geq C_j h_{t_n}^{-p}(x_j)\rho(\varphi(t_n,x_j))|\partial_2\varphi(t_n,x_j)|\mu(L_{n,j}).\end{aligned}$$

Because $\lim_{n\to\infty}\mu(L_{n,j}) = \mu(K_j) > 0$ it follows from $\lim_{n\to\infty}\nu_{p,t_n}(L_{n,j}) = 0$ that

$$\lim_{n\to\infty} h_{t_n}^{-p}(x_j)\rho(\varphi(t_n,x_j))|\partial_2\varphi(t_n,x_j)| = 0$$

for all $j = 1, \ldots, m$ and the continuity of $(s,y) \mapsto h_s(y), \varphi$, and $\partial_2\varphi$ together with Lemma 7 imply that $(t_n)_{n\in\mathbb{N}}$ has to converge to infinity.

Furthermore, we get from Proposition 8 and Lemma 7

$$\begin{aligned}\nu_{p,t_n}(L_{n,j}) &= \int_{L_{n,j}} \frac{h_{t_n}^{p}(\varphi(-t_n,y))\rho(\varphi(-t_n,y))|\partial_2\varphi(-t_n,y)|}{\rho(y)}\chi_{\varphi(t_n,\Omega)}(y)\,d\mu(y)\\ &\geq C_j h_{t_n}^{p}(\varphi(-t_n,x_j))\rho(\varphi(-t_n,x_j))|\partial_2\varphi(-t_n,x_j)|\chi_{\varphi(t_n,\Omega)}(x_j)\mu(L_{n,j})\end{aligned}$$

which shows by the same arguments as above that

$$\lim_{n\to\infty} h^p_{t_n}(\varphi(-t_n,x_j))\rho(\varphi(-t_n,x_j))|\partial_2\varphi(-t_n,x_j)|\chi_{\varphi(t_n,\Omega)}(x_j)=0.$$

In order to show that *iii*) implies *i*) let K be a compact subset of Ω. Since obviously $L^p(\Omega,\mu)=L^p(\Omega\backslash\{F=0\},\mu)$ and $\varphi(t,\Omega\backslash\{F=0\})\subset\Omega\backslash\{F=0\}$ for all $t\geq 0$ we can assume without loss of generality that $K\subset\Omega\backslash\{F=0\}$.

Therefore, there are finitely many intervals $[a_j,b_j]\subset\Omega\backslash\{F=0\}$ such that each $[a_j,b_j]$ is contained in a different component of $\Omega\backslash\{F=0\}$ and $K\subset\cup_{1\leq j\leq m}[a_j,b_j]$. We define $x_j:=a_j$ if $F_{|[a_j,b_j]}>0$, respectively $x_j:=b_j$ if $F_{|[a_j,b_j]}<0$, where without loss of generality we assume *iii*) to be true for $x_1,\ldots,x_m$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers according to *iii*) for $x_1,\ldots,x_m$. From Lemma 7 it follows that for some $C_j>0$

$$\begin{aligned}\nu_{p,-t_n}(K) &\leq \sum_{j=1}^m \nu_{p,t_n}([a_j,b_j])=\sum_{j=1}^m\int_{[a_j,b_j]}\frac{h_t^{-p}(y)\rho(\varphi(t_n,y))|\partial_2\varphi(t_n,y)|}{\rho(y)}\,d\mu(y)\\ &\leq \sum_{j=1}^m C_j\mu([a_j,b_j])h_{t_n}^{-p}(x_j)\rho(\varphi(t_n,x_j))|\partial_2\varphi(t_n,x_j)|\end{aligned}$$

so that $\lim_{n\to\infty}\nu_{p,t_n}(K)=0$.

Analogously, one shows that $\lim_{n\to\infty}\nu_{p,t_n}(K)=0$ as well. Now *i*) follows from Theorem 5. □

In order to prove an analogue of Theorem 9 for spaces of continuous functions, we need the following lemma. Its proof is so similar to that of Lemma 7 that we omit it.

Lemma 10. *Let $\Omega\subset\mathbb{R}$ be open and $[a,b]\subset\{F\neq 0\}$. Assume that $\rho:\Omega\to(0,\infty)$ satisfies $h_t(x)\rho(x)\leq Me^{\omega t}\rho(\varphi(t,x))$ for some $M\geq 1,\omega\in\mathbb{R}$ and all $x\in[a,b],t\geq 0$.*

Then there is $C>0$ such that $1/C<\rho(y)<C$ for all $y\in[a,b]$ and

$$\begin{aligned}h_t^p(\varphi(-t,c))\rho(\varphi(-t,c))\chi_{\varphi(t,\Omega)}(c) &\leq Ch_t^p(\varphi(-t,y))\rho(\varphi(-t,y))\chi_{\varphi(t,\Omega)}(y)\\ &\leq C^2h_t^p(\varphi(-t,d))\rho(\varphi(-t,d))\chi_{\varphi(t,\Omega)}(d)\end{aligned}$$

as well as

$$\begin{aligned}h_t^{-p}(c)\rho(\varphi(t,c)) &\leq Ch_t^{-p}(y)\rho(\varphi(t,y))\\ &\leq C^2h_t^{-p}(d)\rho(\varphi(t,d)).\end{aligned}$$

for all $t\geq 0$ and all $y\in[a,b]$, where $c:=a,d:=b$ if $F_{|[a,b]}>0$, respectively $c:=b,d:=a$ if $F_{|[a,b]}<0$.

Using the above lemma we can now prove the following theorem.

Theorem 11. *Let $\Omega\subset\mathbb{R}$ be open and assume that the function $\rho:\Omega\to(0,\infty)$ is C_0-admissible for F and h, and satisfies $\inf_{x\in K}\rho(x)>0$ for every $K\subset\Omega$ compact, as well as $h_t(x)\rho(x)\leq Me^{\omega t}\rho(\varphi(t,x))$ for some $M\geq 1,\omega\in\mathbb{R}$ and all $x\in\Omega,t\geq 0$.*

The following are equivalent.

i) *The C_0-semigroup T defined via $(T(t)f)(x)=h_t(x)f(\varphi(t,x))$ is hypercyclic on $C_{0,\rho}(\Omega)$.*
ii) *The C_0-semigroup T is weakly mixing on $C_{0,\rho}(\Omega)$.*
iii) *$\{F=0\}=\emptyset$ and for every $x\in\Omega$ there is a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty}\frac{\rho(\varphi(t_n,x))}{h_{t_n}(x)}=0$$

as well as

$$\lim_{n\to\infty} h_{t_n}(\varphi(-t_n,x))\rho(\varphi(-t_n,x))\chi_{\varphi(t_n,\Omega)}(x)=0.$$

PROOF: Obviously, *i*) implies *ii*) by Theorem 6. To show that *ii*) implies *iii*) observe that $\{F=0\}=\emptyset$ by the same reasoning as in the proof of Theorem 9. Let $x\in\Omega$. We define $K:=\{\varphi(t,x);0\leq t\leq 1\}$. Then K is a compact interval contained in Ω. From Theorem 6 it follows that for K there is a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\sup_{y\in K} h_{t_n}(\varphi(-t_n,y))\rho(\varphi(-t_n,y))\chi_{\varphi(t_n,\Omega)}(y)=\lim_{n\to\infty}\sup_{y\in K}\frac{\rho(\varphi(t_n,y))}{h_{t_n}(y)}=0.$$

From this, *iii*) follows immediately, since $x\in K$.

In order to show that *iii*) implies *i*) let K be a compact subset of Ω and $[a,b]\subset\Omega$ such that $K\subset[a,b]$. We define $x:=a$ if $F>0$, respectively $x:=b$ if $F<0$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers according to *iii*) for x. Now it follows with Lemma 10 as in the proof of Theorem 9 that

$$\lim_{n\to\infty}\sup_{y\in K} h_{t_n}(\varphi(-t_n,y))\rho(\varphi(-t_n,y))\chi_{\varphi(t_n,\Omega)}(y)=\lim_{n\to\infty}\sup_{y\in K}\frac{\rho(\varphi(t_n,y))}{h_{t_n}(y)}=0.$$

Theorem 6 now implies *i*). □

Remark 12. *Combining Lemma 7, resp. Lemma 10, with* [12, Theorem 5.1]*, one obtains an easy characterization of when T is mixing, i.e. for every pair of non-empty open subsets U,V there is $t_0>0$ such that $T(t)(U)\cap V\neq\emptyset$ for all $t\geq t_0$.*

Analogously to Theorem 9, one can prove for $\Omega\subset\mathbb{R}$ open, F continuously differentiable, and p-admissible measure μ with positive Lebesgue density ρ that $T(t)f=h_tf(\varphi(t,\cdot))$ defines a mixing C_0-semigroup on $L^p(\mu)$ if and only if $\lambda^1(F=0)=0$ and for λ^1-almost every $x\in\Omega\backslash\{F=0\}$ one has

$$\lim_{t\to\infty} h_t^{-p}(x)\rho(\varphi(t,x))\partial_2\varphi(t,x)=0$$

as well as

$$\lim_{t\to\infty} h_t^p(\varphi(-t,x))\rho(\varphi(-t,x))\partial_2\varphi(-t,x)\chi_{\varphi(t,\Omega)}(x)=0.$$

Analogously to Theorem 11, one proves that for $\Omega\subset\mathbb{R}$ open, ρ C_0-admissible satisfying $h_t(x)\rho(x)\leq Me^{\omega t}\rho(\varphi(t,x))$, that $T(t)f=h_tf(\varphi(t,\cdot))$ defines a mixing C_0-semigroup on $C_{0,\rho}(\Omega)$ if and only if $\{F=0\}=\emptyset$ and for every $x\in\Omega$

$$\lim_{t\to\infty}\frac{\rho(\varphi(t,x))}{h_t(x)}=0$$

as well as

$$\lim_{t\to\infty} h_t(\varphi(-t,x))\rho(\varphi(-t,x))\chi_{\varphi(t,\Omega)}(x)=0.$$

The special case when $F\equiv 1$ and $h\equiv 0$ gives $\varphi(t,x)=x+t$ and $h_t\equiv 1$ so that $(T(t)f)(x)=f(x+t)$, i.e. the so called *left translation semigroup*. From $h_t\equiv 1$ it follows from Proposition 3 that for a locally finite measure μ on $\Omega\in\{\mathbb{R},[0,\infty)\}$ with positive Lebesgue density ρ, that μ is p-admissible for some $p\in[1,\infty)$ if and only if it is p-admissible for all $p\in[1,\infty)$ if and only if there are constants $M\geq 1$ and $\omega\in\mathbb{R}$ such that $\rho(x+t)\leq Me^{\omega t}\rho(x)$ for all $t\geq 0$ and almost all $x\in\Omega$. Moreover, in case of continuous functions, the left translation semigroup is a well-defined C_0-semigroup on $C_{0,\rho}(\Omega)$ if and only if there are constants $M\geq 1$ and $\omega\in\mathbb{R}$ such that $\rho(x+t)\leq Me^{\omega t}\rho(x)$ for all $t\geq 0$ and $x\in\Omega$.

Hypercyclicity and mixing of the left translation semigroup was characterized in [7, Theorems 4.7 and 4.8], respectively [2, Theorem 4.3]. We obtain these results as corollaries of Theorems 9, 11, and Remark 12, respectively. Note however, that in case of continuous functions the aforementioned results deal with $C_{0,\rho}([0,\infty))$ instead of $C_{0,\rho}((0,\infty))$. It is shown in [12, Example 3.16] that when regarding hypercyclicity, resp. mixing, of the left translation semigroup there is no difference between $C_{0,\rho}([0,\infty))$ and $C_{0,\rho}((0,\infty))$. Obviously the spaces $L^p((0,\infty),\rho\,d\lambda)$ and $L^p([0,\infty),\rho\,d\lambda)$ are the same.

Corollary 13. ([7, Theorems 4.7 and 4.8], [2, Theorem 4.3]) *Let $\Omega\in\{\mathbb{R},[0,\infty)\}$ and $\rho:\Omega\to(0,\infty)$ be such that $\rho(x+t)\leq Me^{\omega t}\rho(x)$ for all $t\geq 0$ and for (almost) all $x\in\Omega$ with constants $M\geq 1$ and $\omega\in\mathbb{R}$.*

a) *The following are equivalent.*
   i) *The left translation semigroup is hypercyclic on $L^p(\mu)$, resp. $C_{0,\rho}(\Omega)$.*
   ii) *For every $x\in\Omega$ there is a sequence of positive numbers (tending to infinity) $(t_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty}\rho(x+t_n)=\lim_{n\to\infty}\rho(x-t_n)\chi_\Omega(x-t_n)=0.$$

b) *The following are equivalent.*
   i) *The left translation semigroup is mixing on $L^p(\mu)$ resp. $C_{0,\rho}(\Omega)$.*
   ii) *We have $\lim_{t\to\infty}\rho(t)=\lim_{t\to\infty}\rho(-t)\chi_\Omega(-t)=0$.*

Proof: We only consider the L^p-case. The case of continuous functions is dealt with in the same way. In order to prove *a)* just observe that $\Omega\backslash\{F=0\}=\Omega$ has only one component so that *a)* follows from Theorem 9. In order to prove *b)* observe that for every $x\in\Omega$ we have $\lim_{t\to\infty}\rho(t)=\lim_{t\to\infty}\rho(x+t)$ and $\lim_{t\to\infty}\rho(-t)\chi_\Omega(-t)=\lim_{t\to\infty}\rho(x-t)\chi_\Omega(x-t)$ so that *b)* follows from remark 12. □

4. Hypercyclic evolution families

As C_0-semigroups are related to autonomous abstract Cauchy problems, evolution families are related to non-autonomous abstract Cauchy problems.

Recall that a mapping $U:\{(t,s)\in\mathbb{R}^2;t\geq s\}\to L(X)$, where X is a Banach space, is called an *evolution family*, if it satisfies

i) $U(s,s)=id$ for all $s\in\mathbb{R}$
ii) $U(t,r)\circ U(r,s)=U(t,s)$ for all $s\leq r\leq t$
iii) U is continuous when we equip $L(X)$ with the strong operator topology.

If T is a C_0-semigroup it is trivial to observe that $U(t,s):=T(t-s),s\in\mathbb{R},t\geq s$ defines an evolution family.

Furthermore, let $D(A(t))\subset X$ be dense subspaces and $A(t):D(A(t))\to X$ be linear mappings, $t\in\mathbb{R}$. The evolution family U is said to *solve the non-autonomous Cauchy problem*

$$\text{(nCP)}\qquad \begin{aligned}\frac{d}{dt}u(t)&=A(t)u(t)\\ u(s)&=x,\ x\in X,t\geq s\end{aligned}$$

(on the spaces X_t) if there are dense subspaces $(X_t)_{t\in\mathbb{R}}$ of X such that $U(t,s)X_s\subset X_t\subset D(A(t)),t\geq s$, and the function $t\mapsto U(t,s)x$ solves (nCP) for fixed $s\in\mathbb{R},x\in X_s$ (cf. [8, Chapter VI.9]).

The results of section 2 can also be used to characterize when the evolution family $U = (U(t,s))_{s\in\mathbb{R},t\geq s}$ of non-autonomous Cauchy problems of the form

$$\begin{aligned}\frac{\partial}{\partial t}u(t,s,x) &= \langle F(t,x), \nabla_x u(t,s,x)\rangle + h(t,x)u(t,s,x), t \geq s, x \in \Omega \\ u(s,s,x) &= u_s(x), s \in \mathbb{R},\end{aligned}$$

is transitive, where again $\Omega \subset \mathbb{R}^d$ is open, $F : \mathbb{R} \times \Omega \to \Omega$ is locally Lipschitz continuous with respect to x and $h : \mathbb{R} \times \Omega \to \mathbb{R}$ is continuous.

We assume that for every $s \in \mathbb{R}$ and every $x_0 \in \Omega$ the unique solution $\varphi(\cdot, s, x_0)$ of the initial value problem

$$\dot{x}(t) = F(t, x(t)),\ x(s) = x_0$$

exists on all $\mathbb{R}$. The uniqueness of the solution implies that $\varphi(t,s,\cdot)$ is a bijective mapping on Ω and that its inverse mapping is given by $\varphi(s,t,\cdot)$.

Let μ again be a locally finite Borel measure on Ω. We define $h_{t,s}(x) := \exp(\int_s^t h(r, \varphi(r,s,x))\,dr), t \geq s$, and observe that $h_{t,s}$ is measurable, real-valued and satisfies $1/h_{t,s} \in L^\infty_{loc}(\mu)$.

We assume that the mappings $U(t,s)f := h_{t,s}(\cdot)f(\varphi(t,s,\cdot)), t \geq s$, are well-defined linear operators on $L^p(\mu)$ for all $s \in \mathbb{R}$ and $t \geq s$. It is obvious, that $U(s,s)f = f$ for all $s \in \mathbb{R}$ and that $U(t,r)(U(r,s)f) = U(t,s)f$ for all $s \leq r \leq t$, so that U is an evolution family if and only if the mapping $\{(u,v) \in \mathbb{R}^2; u \geq v\} \to L^p(\mu), (t,s) \mapsto U(t,s)f$ is continuous for every $f \in L^p(\mu)$.

For $s \in \mathbb{R}$ and $t \geq s$ we define as in section 2 the Borel measures

$$\nu_{p,(t,s)}(A) := \int_{\varphi(s,t,A)} h^p_{t,s}\,d\mu$$

and

$$\nu_{p,-(t,s)}(A) := \int_{\varphi(t,s,A)} 1/h^p_{t,s}(\varphi(s,t,\cdot))\,d\mu.$$

Theorem 14. *Assume that $(U(t,s)f)(x) := h_{t,s}(x)f(\varphi(t,s,x))$ defines a continuous linear operator on $L^p(\mu)$ for every $(t,s) \in \{(u,v) \in \mathbb{R}^2; u \geq v\}$.*

a) *For the family $U := (U(t,s))_{s\in\mathbb{R},t\geq s}$ of linear operators on $L^p(\mu)$ the following are equivalent.*
   i) *U is weakly mixing on $L^p(\mu)$.*
   ii) *U is transitive on $L^p(\mu)$.*
   iii) *For every compact subset K of Ω there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a sequence $((t_n,s_n))_{n\in\mathbb{N}}$ in $\{(u,v) \in \mathbb{R}^2; u \geq v\}$ such that $\lim_{n\to\infty} \mu(K\backslash L_n) = 0$ as well as*

$$\lim_{n\to\infty} \nu_{p,(t_n,s_n)}(L_n) = \lim_{n\to\infty} \nu_{p,-(t_n,s_n)}(L_n) = 0.$$

b) *For a fixed $s \in \mathbb{R}$ the following are equivalent.*
   i) *$\{U(t,s); t \geq s\}$ is weakly mixing on $L^p(\mu)$.*
   ii) *$\{U(t,s); t \geq s\}$ is transitive on $L^p(\mu)$.*
   iii) *For every compact subset K of Ω there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a sequence $(t_n)_{n\in\mathbb{N}}$ in $[s,\infty)$ such that $\lim_{n\to\infty} \mu(K\backslash L_n) = 0$ as well as*

$$\lim_{n\to\infty} \nu_{p,(t_n,s)}(L_n) = \lim_{n\to\infty} \nu_{p,-(t_n,s)}(L_n) = 0.$$

PROOF: Having in mind that $\varphi(t,s,\Omega) = \Omega$ and $\varphi(t,s,\cdot)^{-1} = \varphi(s,t,\cdot)$ for all $t,s \in R$ the theorem is a direct consequence of Theorem 1. □

If F and h are continuously differentiable, it is straight forward to show that in case of U being an evolution family, it solves the non-autonomous Cauchy problem

$$\begin{aligned}\frac{\partial}{\partial t}u(t,s,\cdot) &= \langle F(t,\cdot),\nabla_x u(t,s,\cdot)\rangle + h(t,\cdot)u(t,s,\cdot),\ t\geq s,\\ u(s,s,\cdot) &= f_0\end{aligned}$$

on the spaces $X_t := C_c^1(\Omega)$, $t\in\mathbb{R}$, in $L^p(\mu)$.

Example. Let $\Omega=\mathbb{R}$, $F\equiv 1$, $h(t,x):=2ct$ where $c\in\mathbb{R}$ and let μ be the finite Borel measure with Lebesgue density $\rho(x):=e^{-|x|}$. It then follows that $\varphi(t,s,x)=x+(t-s)$, $h_{t,s}(x)=\exp(c(t^2-s^2))$ and obviously $U(t,s)f:=h_{t,s}(\cdot)f(\varphi(t,s,\cdot))$ defines an evolution family on $L^p(\mu)$. Moreover, each operator $U(t,s)$ has dense range, since $U(t,s)(C_c(\Omega))=C_c(\Omega)$ and the operators of U commute so that U is transitive if and only if U is hypercyclic by the results of Peris [13] mentioned in the introduction. The same holds obviously for the family of operators $\{U(t,s);\, t\geq s\}$ for fixed $s\in\mathbb{R}$.

It is readily seen that $\nu_{p,(t,s)}$ has Lebesgue density $\exp(pc(t^2-s^2)-|\cdot-(t-s)|)$ while $\nu_{p,-(t,s)}$ has Lebesgue density $\exp(-pc(t^2-s^2)-|\cdot+(t-s)|)$ for all $s\in\mathbb{R}, t\geq s$.

For a bounded, measurable subset B of $\mathbb{R}$ with $\lambda^1(B)>0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in $[s,\infty)$ we have $\lim_{n\to\infty}\nu_{p,(t_n,s)}(B)=0$ if and only if $(t_n)_{n\in\mathbb{N}}$ converges to infinity and $pc\leq 0$ because for sufficiently large t_n

$$\nu_{p,(t_n,s)}(B)=\exp(pc(t_n^2-s^2)-(t_n-s))\int_B \exp(x)\,dx.$$

Furthermore, we have $\lim_{n\to\infty}\nu_{p,-(t_n,s)}(B)=0$ if and only if $(t_n)_{n\in\mathbb{N}}$ converges to infinity and $pc\geq 0$ because for sufficiently large t_n

$$\nu_{p,-(t_n,s)}(B)=\exp(-pc(t_n^2-s^2)-(t_n-s))\int_B \exp(-x)\,dx.$$

From this and Theorem 14 b) it follows that for fixed $s\in\mathbb{R}$ the family of operators $(U(t,s))_{t\geq s}$ is hypercyclic on $L^p(\mu)$ if and only if $c=0$.

On the other hand, if $t>0$ is sufficiently large and $s:=-t$ we have

$$\nu_{p,(t,-t)}(B)=\exp(-2t)\int_B \exp(x)\,dx$$

and

$$\nu_{p,-(t,-t)}(B)=\exp(-2t)\int_B \exp(-x)\,dx,$$

which both converge to 0 when t tends to infinity, so that by Theorem 14 a) the family $(U(t,s))_{s\in\mathbb{R},t\geq s}$ is hypercyclic on $L^p(\mu)$ for all values of c.

Finally, for fixed $(t,s)\in\{(u,v)\in\mathbb{R}^2;\, u>v\}$ we have for each $n\in\mathbb{N}$ that $U(t,s)^n f=\exp(cn(t^2-s^2))f(\cdot+n(t-s))$. Setting $h_n:=\exp(cn(t^2-s^2))$ and $\varphi(n,x)=x+n(t-s)$ and adapting the notation from section 2 we get for sufficiently large n

$$\nu_{p,n}(B)=\exp(n(cp(t^2-s^2)-(t-s)))\int_B \exp(x)\,dx,$$

and

$$\nu_{p,-n}(B)=\exp(-n(cp(t^2-s^2)+(t-s)))\int_B \exp(-x)\,dx.$$

Therefore, for a sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers both $\nu_{p,n_k}(B)$ and $\nu_{p,-n_k}(B)$ converge to 0 as k tends to infinity if and only if $(n_k)_{k\in\mathbb{N}}$ converges to infinity and $cp(t^2-s^2)-(t-s)<0$ as well as $cp(t^2-s^2)+(t-s)>0$. Hence, by Theorem 1, $U(t,s)$ is a hypercyclic operator on $L^p(\mu)$ if and only if $|c(t+s)|<1/p$.

So by fixing $c = 1$ this gives an example of an evolution family $(U(t,s))_{s\in\mathbb{R},t\geq s}$ on $L^p(\mu)$ which is hypercyclic but for which for fixed s none of the families $(U(t,s))_{t\geq s}$ is hypercyclic and for which a single operator $U(t,s), t > s$, is hypercyclic on $L^p(\mu)$ if and only if $|t+s| < 1/p$.

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