Examples of quantitative universal approximation

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Abstract

Let $\mathcal{L} := (L_j)$ be a sequence of continuous maps from a complete metric space $(\mathcal{X}, d_\mathcal{X})$ to a separable metric space $(\mathcal{Y}, d_\mathcal{Y})$. An element $x \in \mathcal{X}$ is called $\mathcal{L}$-*universal* for a subset $\mathcal{M}$ of $\mathcal{Y}$ if $F(x, \mathcal{M}, \varepsilon) < \infty$ for all $\varepsilon > 0$, where

$$F(x, \mathcal{M}, \varepsilon) := \sup_{y \in \mathcal{M}} \inf \left\{ j \in \mathbb{N} \colon d_\mathcal{Y}(y, L_j x) < \varepsilon \right\}.$$

In this article we obtain quantitative estimates for $F(x, \mathcal{M}, \varepsilon)$ in a variety of examples arising in the theory of universal approximation.

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1 Introduction

Let $(\mathcal{X}, d_\mathcal{X})$ be a complete metric space, let $(\mathcal{Y}, d_\mathcal{Y})$ a separable metric space, and let $\mathcal{L} := (L_j)_{j \in \mathbb{N}}$ be a sequence of continuous mappings $L_j : \mathcal{X} \to \mathcal{Y}$. An element $x \in \mathcal{X}$ is called $\mathcal{L}$-*universal* if

$$\forall n \in \mathbb{N}\, \forall y \in \mathcal{Y}\, \exists N \in \mathbb{N} \colon\; d_\mathcal{Y}(y, L_N x) < \frac{1}{n}.$$

We denote the set of all $\mathcal{L}$-universal elements by $\mathcal{U}(\mathcal{L})$. It is a G_δ-set, due to the separability of $\mathcal{Y}$. The sequence $\mathcal{L}$ is called *universal* if $\mathcal{U}(\mathcal{L}) \neq \emptyset$.

Given a subset $\mathcal{M} \subset \mathcal{Y}$, one might ask how fast the elements of $\mathcal{M}$ can be approximated by some $\mathcal{L}$-universal element x, that is, how many elements of the sequence $(L_j x)_{j \in \mathbb{N}}$ are needed to cover $\mathcal{M}$ by $B(L_j x, \varepsilon)$, $j = 1, \ldots, N$, the

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ε-balls around $L_j x$? Evidently, the answer will be expressed in terms of the numbers:

$$F(x,\mathcal{M},\varepsilon) = F(x,\varepsilon) := \sup_{y\in\mathcal{M}} \inf \left\{ j \in \mathbb{N} \colon d_{\mathcal{Y}}\big(y, L_j x\big) < \varepsilon \right\}.$$

Note that $F(x,\mathcal{M},\varepsilon)$ also depends on the metric $d_{\mathcal{Y}}$. Obviously, if $F(x,\mathcal{M},\varepsilon)$ is finite for every $\varepsilon > 0$, then $\mathcal{M}$ must be *totally bounded* (that is, $\mathcal{M}$ can be covered by a finite number of ε-balls for every $\varepsilon > 0$).

When $\mathcal{Y}$ is a Fréchet space, a natural metric to consider is

$$d_{\mathcal{Y}}(y,z) := \sup_{n\in\mathbb{N}} \left(\min\left\{ p_n(y-z), \frac{1}{n} \right\} \right), \tag{1.1}$$

where $(p_n)_{n\in\mathbb{N}}$ is an increasing sequence of seminorms defining the topology on $\mathcal{Y}$. In this case $d_{\mathcal{Y}}(y,z) < 1/n$ if and only if $p_n(y-z) < 1/n$. If $\mathcal{Y}$ is a Fréchet space, then the totally bounded subsets $\mathcal{M}$ of $\mathcal{Y}$ are precisely the relatively compact ones.

The above question was first studied in [10] for sequences of *composition* and *differentiation operators* on spaces $H(\Omega)$ of holomorphic functions on a simply connected domain Ω equipped with the compact-open topology. This is the Fréchet-space topology defined by the seminorms

$$p_n(f) := \|f - g\|_{K_n} := \max_{z\in K_n} |f(z) - g(z)|, \tag{1.2}$$

where $\mathcal{K} := (K_n)_{n\in\mathbb{N}}$ is a compact exhaustion of Ω, (i.e., $K_n \subseteq \Omega$ compact, K_n is contained in the interior of K_{n+1} for each $n \in \mathbb{N}$, and $\cup_{n\in\mathbb{N}} K_n = \Omega$). Recall that, in this situation, the totally bounded subsets of $H(\Omega)$ are exactly the *normal families*.

Consider the sequence $\mathcal{C} := (C_n)_{n\in\mathbb{N}}$ of composition operators, defined by

$$C_n : H(\Omega_2) \to H(\Omega_1),\ f \mapsto f \circ \varphi_n,$$

where $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of injective holomorphic mappings $\varphi_n \colon \Omega_1 \to \Omega_2$ between open subsets Ω_1, Ω_2 of $\mathbb{C}$. Recall that (φ_n) is called *runaway* if, for every pair of compact sets $K \subseteq \Omega_1, L \subseteq \Omega_2$, there exists an $N \in \mathbb{N}$ with $\varphi_N(K) \cap L = \emptyset$. This property characterizes the existence of $\mathcal{C}$-universal elements when $\Omega_1 = \Omega_2$ and Ω_1 is not conformally equivalent to $\mathbb{C}\backslash\{0\}$, cf. [3].

Now consider the sequence of differentiation operators $\mathcal{D} := (D^n)_{n\in\mathbb{N}}$, where

$$D \colon H(\Omega) \to H(\Omega),\ f \mapsto f'.$$

In this case, the existence of $\mathcal{D}$-universal elements is equivalent to Ω being simply connected, cf. [15].

In order to summarize the main results from [10] we introduce the following notation which will be used throughout this article. For a totally bounded subset $\mathcal{M}$ of an arbitrary metric space $\mathcal{Y}$ we define the *n-th covering number*

$$\lambda_n := \lambda_n(\mathcal{M}) := \min\Big\{ l \in \mathbb{N} \colon \exists\, y_1, \ldots, y_l \in \mathcal{Y} \colon \mathcal{M} \subseteq \bigcup_{j=1}^{l} B(y_j, 1/n) \Big\}.$$

Obviously, the sequence $(\lambda_n(\mathcal{M}))_{n\in\mathbb{N}}$ measures the size of $\mathcal{M}$ in a metrical sense.

For totally bounded subsets $\mathcal{M}$ of $\mathcal{Y} = H(\Omega)$, i.e. for normal families over Ω, we need two more sequences. The first one, $(\gamma_n)_{n\in\mathbb{N}} = (\gamma_n(\mathcal{M}))_{n\in\mathbb{N}}$ measures the approximative behavior of the Taylor/Faber expansions and is defined as the smallest integers with

$$\big\|T_{\gamma_n} f - f\big\|_{K_n} < \frac{1}{n} \quad \forall\, f \in \mathcal{M},$$

where $T_k f$ denotes the k-th Taylor/Faber polynomial on the compact set K_n. The second sequence $(\sigma_n)_{n\in\mathbb{N}} = (\sigma_n(\mathcal{M}))_{n\in\mathbb{N}}$ measures the speed of convergence of the anti-derivatives to 0 and is defined as the smallest integers with

$$\big\|(T_m f)^{(-j)}\big\|_{K_n} < \frac{1}{n^2} \quad \forall\, f \in \mathcal{M}, m \in \mathbb{N} \cup \{0\}, j \geq \sigma_n.$$

Using this notation, the main results in [10] are summarized in the following theorem.

Theorem 1. *(i) In case of $\mathcal{C}$ (composition operators): For any normal family $\mathcal{M}$, there exists a $\mathcal{C}$-universal function f with*

$$F(f, \mathcal{M}, 2/n) \leq n\big(\lambda_n + 1\big) \quad (n \in \mathbb{N}).$$

The set of all $\mathcal{C}$-universal functions satisfying the above estimate contains a G_δ-set, but is never dense. The set of $\mathcal{C}$-universal functions f satisfying

$$F(f, \mathcal{M}, 2/n) = O\big(n\lambda_n\big) \quad (n \to \infty)$$

is dense.

(ii) In case of $\mathcal{D}$ (differentiation operators): Let Ω be bounded. For any normal family $\mathcal{M}$, there exists a $\mathcal{D}$-universal function f with

$$F(f, \mathcal{M}, 3/n) \leq n\big(\lambda_n + 1\big)\big(\gamma_n + \sigma_{n(\lambda_n+1)}\big) \quad (n \in \mathbb{N}).$$

For $\Omega = \mathbb{D}$, the unit disk, $\gamma_n = O\big(n \log(nM_{2n+1})\big)$ and $\sigma_n = O\big(\log(n^2 M_{2n+1})\big)$ as $n \to \infty$, where $M_n := \sup_{f\in\mathcal{M}} \|f\|_{K_n}$. Hence, in this case,

$$F(f, \mathcal{M}, 1/n) = O\big(n^2 \lambda_{3n} \log(n\lambda_{3n} \max\{1, M_{12n\lambda_{3n}+1}\})\big) \quad (n \to \infty).$$

We introduce a special kind of fast approximating universal behavior.

Definition 2. A family of operators $\mathcal{L}$ is called *m-polynomial universal for $\mathcal{M}$* if there is a $\mathcal{L}$-universal element x such that

$$F(x, \mathcal{M}, 1/n) = O\big(n^m\big) \quad (n \to \infty).$$

For a totally bounded set $\mathcal{M} \subseteq \mathcal{Y}$ with covering numbers λ_n, i.e., λ_n functions $f_1^{(n)}, \ldots, f_{\lambda_n}^{(n)} \in \mathcal{Y}$ cover $\mathcal{M}$ with their $\frac{1}{n}$-neighborhoods, the set of all m-polynomial universal functions is given by

$$\bigcup_{c\in\mathbb{N}} \Big(\bigcap_{n\in\mathbb{N}} \bigcap_{j=1}^{\lambda_n} \bigcup_{N=1}^{c\cdot n^m} L_N^{-1}\big(B(f_j^{(n)}, 1/n)\big) \cap \mathcal{U}(\mathcal{L}) \Big).$$

This is a $G_{\delta\sigma}$-set. It is unknown if it is also a G_δ-set.

In Section 2 we consider the above question for sequences of composition operators on kernels of differential operators and obtain exactly the same estimates as in the holomorphic case (compare Theorem 1 and Theorem 5). Section 3 contains an investigation of similar questions for universal Taylor series and comparisons of the results with those from [10] for differentiation operators. Finally, in Section 4, we consider some classic examples of normal families, like the set of normalized univalent functions S, and their covering numbers.

2 Composition operators on kernels of differential operators

In this section, let $\Omega \subset \mathbb{R}^d$ be open and let $P \in \mathbb{C}[X_1, \ldots, X_d]$ be a non-zero polynomial. As usual, we equip $C^\infty(\Omega)$ with the Fréchet-space topology induced by the family of semi-norms

$$q_{K_n,n}(f) := \max_{x \in K_n, |\alpha| \leq n} |\partial^\alpha f(x)|,$$

where $(K_n)_{n\in\mathbb{N}}$ is a compact exhaustion of Ω. We denote this Fréchet space by $\mathscr{E}(\Omega)$ and the metric defined in (1.1) by d. As the differential operator $P(D)$ is continuous on $\mathscr{E}(\Omega)$, it follows that the kernel of $P(D)$ in $\mathscr{E}(\Omega)$, namely

$$\mathscr{N}_P(\Omega) := \{f \in \mathscr{E}(\Omega) : P(D)f = 0\},$$

is a closed subspace of $\mathscr{E}(\Omega)$, and hence is itself a Fréchet space in a natural way. As is well known, $\mathscr{E}(\Omega)$ is separable, so the same is true for $\mathscr{N}_P(\Omega)$.

In case of P being hypoelliptic, the above mentioned Fréchet-space topology of $\mathscr{N}_P(\Omega)$ is induced by the family of semi-norms $(q_{K_n,0})_{n\in\mathbb{N}}$, see for example [8, Theorem 4.4.2]. We denote the corresponding metric defined in (1.1) by d_0. In particular, when dealing with the Cauchy–Riemann operator or the Laplace operator, we consider the spaces of holomorphic functions and harmonic functions respectively, equipped with the compact-open topology. As is well known, $\mathscr{N}_P(\Omega)$ is a Montel space if P is hypoelliptic (this follows for example from [8, Theorem 4.4.2]), so in this case $\mathcal{M} \subset \mathscr{N}_P(\Omega)$ is relatively compact if and only if $\mathcal{M}$ is bounded, i.e., if and only if for every compact $K \subset \Omega$ we have

$$\sup_{f \in \mathcal{M}} q_{K,0}(f) < \infty.$$

Definition 3. (i) Let $\varphi : \Omega \to \Omega$ be a C^∞-diffeomorphism. Then P is called *φ-invariant* if, for any $f \in C^\infty(\Omega)$, we have $f \circ \varphi \in \mathscr{N}_P(\Omega)$ whenever $f \in \mathscr{N}_P(\Omega)$. If P is φ-invariant and φ^{-1}-invariant, then we call P *completely φ-invariant.*

(ii) An open subset $U \subset \Omega$ is called *P-approximable in* Ω if $\{f|_U : f \in \mathscr{N}_P(\Omega)\}$ is dense in $\mathscr{N}_P(U)$.

Remark 4. (i) If P is φ-invariant, then the mapping

$$C_\varphi : \mathscr{N}_P(\Omega) \to \mathscr{N}_P(\Omega), \quad f \mapsto f \circ \varphi$$

is well-defined and linear. Moreover, for compact $K \subset \mathbb{R}^d$ and $n \in \mathbb{N}_0$, we obviously have $q_{K,n}(C_\varphi f) \leq M q_{\varphi(K),n}(f)$ for all $f \in \mathscr{E}(\Omega)$, where $M > 0$ is a suitable constant depending on K and n. Thus C_φ is a continuous, linear operator on $\mathscr{N}_P(\Omega)$.

(ii) If, for the C^∞-diffeomorphism $\varphi : \Omega \to \Omega$, there is $g \in \mathscr{E}(\Omega)$ such that the set $\{x \in \Omega : g(x) = 0\}$ is nowhere dense in Ω and $P(D)(C_\varphi(f)) = g\, C_\varphi(P(D)f)$ for every $f \in \mathscr{E}(\Omega)$, then it follows immediately that P is completely φ-invariant. In case of $P(D)$ being the Cauchy–Riemann, Laplace or heat operator, it is shown in [9, Proposition 3.6] that this condition on φ is already necessary for P to be φ-invariant. Moreover, the same is true in case of $P(D)$ being the wave operator, under the mild additional assumption that φ does not mingle the time variable with the space variables and *vice versa*. It should be noted that in [9] the term "φ-invariance" is used for what we call complete φ-invariance here. Nevertheless, the proof of [9, Proposition 3.6] uses only that $f \circ \varphi \in \mathscr{N}_P(\Omega)$ for every $f \in \mathscr{N}_P(\Omega)$.

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of C^∞-diffeomorphisms of Ω such that P is completely φ_n-invariant for every $n \in \mathbb{N}$. There are several articles dealing with the existence of universal functions for $(C_{\varphi_n})_{n\in\mathbb{N}}$ for special $P(D)$, in particular for the Cauchy–Riemann or the Laplace operator, see e.g. [3], [4], [6]. For arbitrary P, a characterization is given in [9] for the case that Ω has convex components.

Our first result in this section is the following theorem.

Theorem 5. *Let $(\varphi_m)_{m\in\mathbb{N}}$ be a sequence of C^∞-diffeomorphisms on Ω such that P is completely φ_m-invariant for every m in $\mathbb{N}$. Assume that, for every compact subset K of Ω, there are a bounded open neighborhood $U \subset \Omega$ of K with $\overline{U} \subset \Omega$ and $m \in \mathbb{N}$ such that $\varphi_m(U) \cup U$ is P-approximable and $\varphi_m(U) \cap U = \emptyset$. Then there is a strictly increasing sequence of natural numbers $(m_n)_{n\in\mathbb{N}}$ such that, for any $\mathcal{M} \subset \mathscr{N}_P(\Omega)$ relatively compact, there is a universal function u for $(C_{\varphi_{m_n}})_{n\in\mathbb{N}}$ such that*

$$F(u, \mathcal{M}, 2/n) \le n(\lambda_n + 1) \quad \forall n \in \mathbb{N}.$$

In order to make the proof of the above theorem more transparent, we first prove the following lemma.

Lemma 6. *Under the hypotheses of Theorem 5, for any compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of Ω, there is a strictly increasing sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers such that, for any sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{N}_P(\Omega)$, there is $v \in \mathscr{N}_P(\Omega)$ with*

$$q_{K_n,n}(f_n - C_{\varphi_{m_n}}(v)) < \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

Proof. Fix a compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of Ω and a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{N}_P(\Omega)$. We simply write C_n in place of C_{φ_n}.

We start by constructing a sequence of bounded, open subsets $(U_n)_{n\in\mathbb{N}}$ of Ω, sequences of natural numbers $(m_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$, and a sequence $(M_n)_{n\in\mathbb{N}}$ in $(1, \infty)$, such that:

(i) $\forall\, n \in \mathbb{N} : K_n \subset U_n \subset \overline{U_n} \subset \Omega$,

(ii) $\forall\, n \in \mathbb{N} : \varphi_{m_n}(U_n) \cap U_n = \emptyset$ and $\varphi_{m_n}(U_n) \cup U_n$ is P-approximable in Ω,

(iii) $(m_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ are strictly increasing, with $r_n \ge n + 1$ for each $n \in \mathbb{N}$,

(iv) $(M_n)_{n\in\mathbb{N}}$ is non-decreasing,

(v) $\forall\, n \in \mathbb{N}, f \in \mathscr{N}_P(\Omega): \, q_{K_n,n}(C_{m_n}(f)) \le M_n q_{K_{r_n},n}(f)$,

(vi) $\forall\, n \in \mathbb{N}: \, K_{r_n} \subset U_{n+1}$.

By hypothesis, there exists a bounded open neighborhood $U_1 \subset \Omega$ of K_1 with $\overline{U_1} \subset \Omega$, and there exists $m_1 \in \mathbb{N}$ with $\varphi_{m_1}(U_1) \cap U_1 = \emptyset$ and $\varphi_{m_1}(U_1) \cup U_1$ being P-approximable in Ω. Moreover, by the continuity of C_{m_1}, there are $r_1 \in \mathbb{N}, r_1 \ge 2$ and $M_1 > 1$ with $q_{K_1,1}(C_{m_1}(f)) \le M_1 \, q_{K_{r_1},1}(f)$.

Assume that $U_1, \ldots, U_n, m_1, \ldots, m_n, r_1, \ldots, r_n$ and $M_1, \ldots, M_n$ have already been constructed. For the compact set

$$K := \overline{U_n} \cup K_{r_n+1} \cup \bigcup_{j=1}^{m_n} \varphi_j(\overline{U_n}),$$

there exist, by hypothesis, a bounded open neighborhood $U_{n+1} \subset \overline{U_{n+1}} \subset \Omega$ and $m_{n+1} \in \mathbb{N}$ with $U_{n+1} \cap \varphi_{m_{n+1}}(U_{n+1}) = \emptyset$ and $U_{n+1} \cup \varphi_{m_{n+1}}(U_{n+1})$ being P-approximable in Ω. From $U_{n+1} \cap \varphi_{m_{n+1}}(U_{n+1}) = \emptyset$ and the definition of K, it follows that $m_{n+1} > m_n$. By the continuity of $C_{m_{n+1}}$, there are M_{n+1} and r_{n+1} with

$$q_{K_{n+1},n+1}(C_{m_{n+1}}(f)) \le M_{n+1} q_{K_{r_{n+1}},n+1}(f)$$

for any $f \in \mathscr{N}_P(\Omega)$, where, without loss of generality, we may assume that $M_{n+1} \ge M_n$ and $r_{n+1} > \max\{r_n, n+2\}$.

We observe that, by (iii) and (vi), we have $K_{n+1} \subseteq K_{r_n} \subseteq U_{n+1}$ for every $n \in \mathbb{N}$.

Next, we recursively construct a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathscr{N}_P(\Omega)$ such that:

(a) $\forall\, n \in \mathbb{N}: \, q_{K_n,n}(f_n - C_{m_n}(v_n)) < \frac{1}{2n}$,

(b) $\forall\, n \in \mathbb{N}: \, q_{K_{r_n},n}(v_{n+1} - v_n) < \frac{1}{2^{n+1} M_{n+1}}$.

Indeed, for $n = 1$, consider

$$w_1 : U_1 \cup \varphi_{m_1}(U_1) \to \mathbb{C}, \quad w_1(x) := \begin{cases} 0, & \text{if } x \in U_1, \\ f_1(\varphi_{m_1}^{-1}(x)), & \text{if } x \in \varphi_{m_1}(U_1). \end{cases}$$

Since $U_1 \cap \varphi_{m_1}(U_1) = \emptyset$, the map w_1 is well-defined, and $w_1 \in \mathscr{N}_P(U_1 \cup \varphi_{m_1}(U_1))$ follows from the complete φ_{m_1}-invariance of P. Fix $\psi_1 \in \mathscr{D}(U_1)$ such that $\psi_1 = 1$ in a neighborhood of K_1. Obviously, $\psi_1 \circ \varphi_{m_1}^{-1} \in \mathscr{D}(\varphi_{m_1}(U_1))$, so that, for any $f \in \mathscr{N}_P(U_1 \cup \varphi_{m_1}(U_1))$, we have $(\psi_1 \circ \varphi_{m_1}^{-1}) f \in C^\infty(\Omega)$ in a natural way. Therefore,

$$p_1(f) := q_{K_{r_1},1}((\psi_1 \circ \varphi_{m_1}^{-1}) f)$$

defines a continuous semi-norm on $\mathscr{N}_P(U_1 \cup \varphi_{m_1}(U_1))$. The P-approximability of $U_1 \cup \varphi_{m_1}(U_1)$ in Ω and the continuity of the seminorm p_1 imply the existence of $v_1 \in \mathscr{N}_P(\Omega)$ with

$$p_1(v_1 - w_1) < \frac{1}{4M_1}.$$

Since, from the definition of w_1, we have $(\psi_1 \circ \varphi_{m_1}^{-1})w_1 = (\psi_1 \circ \varphi_{m_1}^{-1})(f_1 \circ \varphi_{m_1}^{-1})$, and because $\psi_1 = 1$ in a neighborhood of K_1, this implies

$$\begin{aligned} q_{K_1,1}(f_1 - C_{m_1}(v_1)) &= q_{K_1,1}\left(C_{m_1}((\psi_1 \circ \varphi_{m_1}^{-1})(f_1 \circ \varphi_{m_1}^{-1} - v_1))\right) \\ &\leq M_1 q_{K_{r_1},1}\left((\psi_1 \circ \varphi_{m_1}^{-1})(f_1 \circ \varphi_{m_1}^{-1} - v_1)\right) \\ &= M_1 p_1\left((w_1 - v_1)\right) < \frac{1}{4}, \end{aligned}$$

where we used (v) in the second step.

Assuming that $v_1, \ldots, v_n$ have already been constructed, we consider

$$\begin{aligned} &w_{n+1} : U_{n+1} \cup \varphi_{m_{n+1}}(U_{n+1}) \to \mathbb{C}, \\ &w_{n+1}(x) := \begin{cases} v_n(x), & \text{if } x \in U_{n+1}, \\ f_{n+1}(\varphi_{m_{n+1}}^{-1}(x)), & \text{if } x \in \varphi_{m_{n+1}}(U_{n+1}). \end{cases} \end{aligned}$$

Then, as for w_1, we have $w_{n+1} \in \mathscr{N}_P(U_{n+1} \cup \varphi_{m_{n+1}}(U_{n+1}))$. Fix $\psi_{n+1} \in \mathscr{D}(U_{n+1})$ such that $\psi_{n+1} = 1$ in a neighborhood of $K_{r_n} \supseteq K_{n+1}$. As above,

$$p_{n+1}(f) := q_{K_{r_{n+1}},n+1}((\psi_{n+1} \circ \varphi_{m_{n+1}}^{-1})f) + q_{K_{r_n},n}(f)$$

defines a continuous semi-norm on $\mathscr{N}_P(U_{n+1} \cup \varphi_{m_{n+1}}(U_{n+1}))$, so that the P-approximability of $U_{n+1} \cup \varphi_{m_{n+1}}(U_{n+1})$ in Ω yields $v_{n+1} \in \mathscr{N}_P(\Omega)$ with

$$p_{n+1}(v_{n+1} - w_{n+1}) < \frac{1}{2^{n+1}M_{n+1}}.$$

Again, since $(\psi_{n+1} \circ \varphi_{m_{n+1}}^{-1})w_{n+1} = (\psi_{n+1} \circ \varphi_{m_{n+1}}^{-1})(f_{n+1} \circ \varphi_{m_{n+1}}^{-1})$, and as $\psi_{m_{n+1}} = 1$ in a neighborhood of $K_{r_n} \supseteq K_{n+1}$, this implies

$$\begin{aligned} &q_{K_{n+1},n+1}(f_{n+1} - C_{m_{n+1}}(v_{n+1})) \\ &\quad = q_{K_{n+1},n+1}\Big(C_{m_{n+1}}((\psi_1 \circ \varphi_{m_{n+1}}^{-1})(f_{n+1} \circ \varphi_{m_{n+1}}^{-1} - v_{n+1}))\Big) \\ &\quad \leq M_{n+1} q_{K_{r_{n+1}},n+1}\Big((\psi_{n+1} \circ \varphi_{m_{n+1}}^{-1})(f_{n+1} \circ \varphi_{m_{n+1}}^{-1} - v_{n+1})\Big) \\ &\quad = M_{n+1} q_{K_{r_{n+1}},n+1}\Big((\psi_{n+1} \circ \varphi_{m_{n+1}}^{-1})(w_{n+1} - v_{n+1})\Big) \\ &\quad \leq M_{n+1} p_{n+1}(v_{n+1} - w_{n+1}) < \frac{1}{2^{n+1}} < \frac{1}{2(n+1)}, \end{aligned}$$

where we used (v) in the second step. Moreover, since $K_{r_n} \subset U_{n+1}$, and since, by definition, $v_n|_{U_{n+1}} = w_{n+1}|_{U_{n+1}}$, we obtain

$$q_{K_{r_n},n}(v_{n+1} - v_n) = q_{K_{r_n},n}(v_{n+1} - w_{n+1}) \leq p_{n+1}(v_{n+1} - w_{n+1}) < \frac{1}{2^{n+1}M_{n+1}},$$

thereby finishing the construction of $(v_n)_{n\in\mathbb{N}}$.

Because of the inclusion $K_{r_n} \supseteq K_n$, the fact that $M_n \geq 1$ and (b), we have

$$\forall\, n \in \mathbb{N} : \; q_{K_n,n}(v_{n+1} - v_n) < \frac{1}{2^{n+1}},$$

so that $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{N}_P(\Omega)$, and hence convergent. We set $v := \lim_{n\to\infty} v_n$, and observe that $v = v_n + \sum_{j=n}^{\infty}(v_{j+1} - v_j)$ for every $n \in \mathbb{N}$.

From the continuity of C_{m_n}, and using (a), (v), (b), and (iv), we finally get that, for $n \in \mathbb{N}$,

$$\begin{aligned} q_{K_n,n}(f_n - C_{m_n}(v)) &= q_{K_n,n}(f_n - C_{m_n}(v_n) - \sum_{j=n}^{\infty} C_{m_n}(v_{j+1} - v_j)) \\ &\le \frac{1}{2n} + \sum_{j=n}^{\infty} q_{K_n,n}(C_{m_n}(v_{j+1} - v_j)) \\ &\le \frac{1}{2n} + \sum_{j=n}^{\infty} M_n q_{K_{r_n},n}(v_{j+1} - v_j) \\ &\le \frac{1}{2n} + \sum_{j=n}^{\infty} M_n q_{K_{r_j},j}(v_{j+1} - v_j) \\ &\le \frac{1}{2n} + \sum_{j=n}^{\infty} \frac{M_n}{2^{j+1} M_{j+1}} < \frac{1}{n}. \end{aligned}$$

This completes the proof of the lemma. □

Proof of Theorem 5. Let $(K_n)_{n\in\mathbb{N}}$ be the compact exhaustion of Ω defining the metric d on $\mathscr{N}_P(\Omega)$. For $n \in \mathbb{N}$, let $f_1^{(n)}, \ldots, f_{\lambda_n}^{(n)} \in \mathscr{N}_P(\Omega)$ be such that

$$\mathcal{M} \subseteq \bigcup_{j=1}^{\lambda_n} B(f_j^{(n)}, 1/n),$$

and let $(g_n)_{n\in\mathbb{N}}$ be a dense sequence in $\mathscr{N}_P(\Omega)$. We define $(f_n)_{n\in\mathbb{N}}$ to be the sequence

$$f_1^{(1)}, \ldots, f_{\lambda_1}^{(1)}, g_1, f_1^{(2)}, \ldots, f_{\lambda_2}^{(2)}, g_2, f_1^{(3)}, \ldots, f_{\lambda_3}^{(3)}, g_3, \ldots$$

Applying Lemma 6 gives an increasing sequence of natural numbers $(m_n)_{n\in\mathbb{N}}$ and $u \in \mathscr{N}_P(\Omega)$ such that

$$q_{K_n,n}(f_n - C_{\varphi_{m_n}}(u)) < \frac{1}{n}.$$

Since $\{g_n : n \in \mathbb{N}\}$ is dense in $\mathscr{N}_P(\Omega)$, it follows that u is universal for $(C_{\varphi_{m_n}})_{n\in\mathbb{N}}$. Now fix $f \in \mathcal{M}$ and $n \in \mathbb{N}$. Then $d(f, f_j^{(n)}) < 1/n$ for some $1 \le j \le \lambda_n$. Because $f_j^{(n)} = f_N$ for some $n \le N \le \sum_{j=1}^{n}(\lambda_j + 1) \le n(\lambda_n + 1)$, and because

$$q_{K_N,N}(f_N - C_{\varphi_{M_N}}(u)) < \frac{1}{N},$$

that is

$$d(f_j^{(n)}, C_{\varphi_{m_N}}(u)) < \frac{1}{N},$$

the result follows. □

In order to verify the hypothesis of Theorem 5 in some concrete situations we recall the following results about approximation of zero solutions of differential equations. Part (i) of the next theorem is the Malgrange–Lax Theorem, cf. [8, Theorem 4.4.5], while part (ii) is due to Hörmander, see e.g. [8, Theorem 10.5.2].

Theorem 7. *Let $U \subseteq \Omega$ be open.*

(i) Assume that P is elliptic. If $\Omega \backslash U$ is not the disjoint union $F \cup K$, where K is compact and non-empty and F is closed in Ω, then U is P-approximable in Ω.

(ii) Suppose that every $\mu \in \mathscr{E}'(\overline{\Omega})$ with $\operatorname{supp} P(-D)\mu \subset U$ already belongs to $\mathscr{E}'(U)$. Then U is P-approximable in Ω.

Remark 8. (i) Let $\hat{\Omega}$ denote the one-point compactification of Ω. It is easily seen that the condition in (i) of Theorem 7 is equivalent to $\hat{\Omega} \backslash U$ being connected while part (ii) immediately implies the P-approximability in Ω of every $U \subset \Omega$ with convex components.

(ii) It is shown in [9, Proof of Corollary 4.6] that, if φ satisfies the condition under (ii) of Remark 4, and if $K \subset \Omega$ is compact, has only convex components and satisfies $\varphi(K) \cap K = \emptyset$, then $\varphi(K^\circ) \cup K^\circ$ is P-approximable in Ω, where K° denotes the interior of K.

(iii) Assume that Ω has only convex components and that every element of the sequence $(\varphi_m)_{m\in\mathbb{N}}$ of C^∞-diffeomorphisms satisfies the condition (ii) of Remark 4. Then it follows from (ii) above that the assumption of Theorem 5 is fulfilled if and only if, for every compact subset K of Ω, there is $m \in \mathbb{N}$ with $\varphi_m(K) \cap K = \emptyset$.

Corollary 9. *Let $(\varphi_m)_{m\in\mathbb{N}}$ be a sequence of C^∞-diffeomorphisms of Ω such that P is completely φ_m-invariant for every $m \in \mathbb{N}$. Assume, further, that for any compact subset $K \subset \Omega$, there is $m \in \mathbb{N}$ with $\varphi_m(K) \cap K = \emptyset$. Then there is an increasing sequence of natural numbers $(m_n)_{n\in\mathbb{N}}$ for which the following hold:*

- *(i) Assume that Ω is contractible and that Ω has the complementation property, i.e., given any compact subset $K \subset \Omega$, there is at most one component of $\Omega \backslash K$ whose closure in Ω is not compact. If P is elliptic, then, for any relatively compact subset $\mathcal{M}$ of $\mathscr{N}_P(\Omega)$, there is a universal function $u \in \mathscr{N}_P(\Omega)$ for $(C_{\varphi_{m_n}})_{n\in\mathbb{N}}$ such that $F(u, \mathcal{M}, 2/n) \leq n(\lambda_n + 1)$ for each $n \in \mathbb{N}$.*

- *(ii) If P is arbitrary, each φ_m satisfies the condition (ii) from Remark 4, and Ω has only convex components, then, for any relatively compact subset $\mathcal{M}$ of $\mathscr{N}_P(\Omega)$, there is a universal function $u \in \mathscr{N}_P(\Omega)$ for $(C_{\varphi_{m_n}})_{n\in\mathbb{N}}$ such that $F(u, \mathcal{M}, 2/n) \leq n(\lambda_n + 1)$ for each $n \in \mathbb{N}$.*

Proof. Part (ii) follows from the hypothesis, Remark 8, and Theorem 5.

In order to show (i), it is straightforward to verify that, with

$$U_n := \{x \in \Omega : |x| < n \text{ and } \operatorname{dist}(x, \Omega^c) > 1/n\},$$

the set $\Omega \backslash U_n$ is not the disjoint union of a non-empty, compact set K and a set F closed in Ω. As φ_m is a homeomorphism, the same holds for $\varphi_m(U_n)$ for arbitrary m. By hypothesis, there is m_0 such that $U_n \cap \varphi_{m_0}(U_n) = \emptyset$. The contractibility of Ω easily gives that every continuous mapping $g : \Omega \to S^1$ is homotopic to a constant. Together with the complementation property of Ω, this implies the unicoherence of $\hat{\Omega}$ (see e.g. [5, Theorem 4.12]), so that for the two connected and closed sets $\hat{\Omega} \backslash U_n$ and $\hat{\Omega} \backslash \varphi_{m_0}(U_n)$ covering $\hat{\Omega}$, their intersection $\hat{\Omega} \backslash (U_n \cup \varphi_{m_0}(U_n))$ is also connected. Therefore, $U_n \cup \varphi_{m_0}(U_n)$ is P-approximable in Ω, by Theorem 7 (i). Part (i) now follows from this and from Theorem 5. □

3 Universal Taylor series

Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain. For $L \subset \mathbb{C}\backslash\Omega$ compact with connected complement and $\zeta \in \Omega$, we consider the sequence $\mathcal{T}_L^\zeta = (T_{L,n}^\zeta)_{n\in\mathbb{N}}$ of linear operators

$$T_{L,n}^{(\zeta)} : H(\Omega) \to A(L), \qquad f \mapsto T_{L,n}^{(\zeta)} f(z) := T_n^\zeta f(z) := \sum_{\nu=0}^{n} a_\nu^{(\zeta)} (z-\zeta)^\nu,$$

where $a_\nu^{(\zeta)}$ denotes the ν-th Taylor coefficient of f expanded about ζ, and $A(L)$ denotes the space of all continuous functions on L that are holomorphic in the interior of L. Endowing $A(L)$ with the sup-norm $\|f\|_L$, it follows from Mergelyan's theorem that $\{f|_L : f \in A(\tilde{L})\}$ is dense in $A(L)$ for any compact superset $\tilde{L}$ of L.

As shown in [13, Lemma 2.1], there exists a sequence $(L_k)_{k\in\mathbb{N}}$ of compact sets $L_k \subset \mathbb{C}\setminus\Omega$ with connected complement such that, for every compact subset $L \subset \mathbb{C}\setminus\Omega$ with connected complement, there is $k_0 \in \mathbb{N}$ with $L \subset L_{k_0}$. The set of all universal Taylor series in the sense of [13] is then given by

$$\mathcal{U}(\zeta) := \bigcap_{k\in\mathbb{N}} \mathcal{U}(\mathcal{T}_{L_k}^{(\zeta)}),$$

and it is shown in [12, Theorem 2] that

$$\mathcal{U}(\zeta_1) = \mathcal{U}(\zeta_2)$$

for any $\zeta_1, \zeta_2 \in \Omega$. Abusing our former notation we simply write $\mathcal{U}(\mathcal{T})$ for these equal sets, that is, $f \in \mathcal{U}(\mathcal{T})$ if and only if the set of the Taylor polynomials of f expanded about an arbitrary $\zeta \in \Omega$ is dense in any $A(L)$, where $L \subset \mathbb{C}\backslash\Omega$ is compact and has connected complement.

Our first aim is to compare how fast a normal family $\mathcal{M}$ may be approximated by the partial sums of a universal Taylor series $f \in \mathcal{U}(\mathcal{T})$ with the speed of approximation by the derivatives of a function $g \in \mathcal{U}(\mathcal{D})$. In a second step we then estimate the possible speed of approximation for $f \in \mathcal{U}(\mathcal{T})$. To help us in pursuit of these goals, we introduce the following notion:

Definition 10. Let $\Omega \subseteq \mathbb{C}$ be open, let $L_k \subset \mathbb{C}\backslash\Omega$ be compact, and let $f_k \in A(L_k)$. We say $f \in H(\Omega)$ has a *uniformly universal power series in* $\zeta_1 \in \Omega$ *for* $(f_k, L_k)_{k\in\mathbb{N}}$ if there is a sequence of natural numbers $(N_k)_{k\in\mathbb{N}}$ such that

$$\forall\, 1 \le j \le k \;\; \exists\, 1 \le n \le N_k : \; \|f_j - T_n^{(\zeta_1)} f\|_{L_j} < \frac{1}{j^2}.$$

Let $\mathcal{P}_\mathbb{Q}$ be the set of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$, and let $\mathcal{K}_{\mathbb{C}\setminus\Omega} := (L_k)$ be a sequence of compact sets in $\mathbb{C}\setminus\Omega$ as above. If Ω is simply connected and if (f_k, L_k) contains each element $(p, L) \in \mathcal{P}_\mathbb{Q} \times \mathcal{K}_{\mathbb{C}\setminus\Omega}$ infinitely often, then a uniformly universal power series f for (f_k, L_k) is a universal Taylor series, i.e. $f \in \mathcal{U}(\mathcal{T})$.

Remark 11. (i) Let $\Omega = \mathbb{D}$, let f be a uniformly universal power series in $\zeta_1 = 0$ for (f_k, L_k), $k > 2$, with

$$f_k \equiv 0, \quad L_k := \Big\{z \colon \Big|z - \frac{3}{2}k\Big| \le k\Big\},$$

and let (N_k) be a sequence of numbers as in Definition 10. Assume only that $\|T_{N_k}^{(0)} f\|_{L_k} \leq 1$ for each $k > 2$. Then the Taylor coefficients satisfy

$$|a_\nu|^{1/\nu} \leq k^{\log \frac{5}{2} - 1} \quad \text{for all } \nu \text{ with } \tilde{N}_k := \left[\frac{N_k}{\log k}\right] + 1 \leq \nu \leq N_k,$$

cf. [7, p.84]. Thus approximation by partial sums occurs with rather large blocks of small coefficients. Assume, further, that $f \in \mathcal{U}(\mathcal{T})$, so in particular the radius of convergence of f is 1. Since

$$\limsup_{\substack{\nu \to \infty \\ \nu \in I}} |a_\nu|^{1/\nu} = 0, \quad I := \mathbb{N} \cap \bigcup_{k \in \mathbb{N}} [\tilde{N}_k, N_k],$$

for every $\varepsilon > 0$ the power series of f must also have infinitely many Taylor coefficients a_ν with $|a_\nu|^{1/\nu} \geq 1 - \varepsilon$, $\nu \in \mathbb{N} \backslash I$. More precisely, the set of indices

$$\kappa := \{k \in \mathbb{N} \colon \exists \nu \in (N_{k-1}, \tilde{N}_k) \text{ with } |a_\nu|^{1/\nu} > k^{\log \frac{5}{2} - 1}\} \subset \{k \in \mathbb{N} \colon N_{k-1} < \tilde{N}_k\}$$

is infinite. Thus, on the infinite set κ we have

$$\frac{N_k}{N_{k-1}} \geq \log k, \quad k \in \kappa.$$

The same holds if f has finite radius of convergence, without necessarily belonging to $\mathcal{U}(\mathcal{T})$.

(ii) We compare the above quotient N_k/N_{k-1} with a similar one for a function $g \in \mathcal{U}(\mathcal{D})$. In [10, Theorem 8], a function $g \in \mathcal{U}(\mathcal{D}) \cap H(\Omega)$ (where $H(\Omega)$ is endowed with the natural metric as in (1.1) and seminorms as in (1.2)) is constructed, which is fast approximating for a normal family $\mathcal{M}$. For appropriate functions f_j, $j = 1, \ldots, k$, define (N_k) to be a sequence of natural numbers with

$$\forall\, 1 \leq j \leq k \;\; \exists\, 1 \leq n \leq N_k : \; \|f_j - g^{(n)}\|_{K_j} < \frac{1}{j}.$$

For the constructed function $g \in \mathcal{U}(\mathcal{D})$, we obtain from [10, Proof of Theorem 8] that $N_k \leq N_{k-1} + \sigma_k + \gamma_k$, where σ_k and γ_k are defined as in the paragraph preceding Theorem 1. Considering $\mathcal{M} = \{0\}$, i.e. $f_j \equiv 0$, as in (i), $\sigma_k = \gamma_k := k$ is a possible choice, and so is $N_k := k(k+1)$. Hence

$$\frac{N_k}{N_{k-1}} = \frac{k+1}{k-1},$$

which is bounded, and not strictly increasing to ∞ on a subsequence κ, as is the case for $f \in \mathcal{U}(\mathcal{T})$.

This simple example already illustrates the tremendous difference between the speeds of approximation by $f \in \mathcal{U}(\mathcal{T})$ and $g \in \mathcal{U}(\mathcal{D})$. To elucidate this difference, we remark that successive derivatives of a function may change rather quickly, while in universal approximation successive partial sums change rather slowly, which is expressed by large blocks of rather small coefficients, namely so-called Ostrowski gaps, cf. [7]. Even the boundedness of the partial sums on a non-polar set $E \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ causes small coefficients, in this case so-called Hadamard–Ostrowski gaps, as recently shown in [2].

Nevertheless, we also want to give results in the other direction by showing which speeds of approximation are possible, this time by estimating possible upper bounds, not for $F(f,\mathcal{M},1/n)$, but for the numbers N_k as defined in Definition 10. In order to construct a universal function f with small $F(f,\mathcal{M},1/n)$, we first find a sequence (f_k) containing appropriately chosen centers, whose balls $B(f_k,1/n)$ cover $\mathcal{M}$. Their number is $\lambda_n(\mathcal{M})$, the n-covering number of $\mathcal{M}$. Then these centers f_k will be approximated by the first N_k Taylor polynomials of f, i.e., by $T_j^{(\zeta)}f$, $j \in \{1,\ldots,N_k\}$. Finally, $F(f,\mathcal{M},1/n)$ and N_k are connected, since $F(f,\mathcal{M},1/n) \le N_k$ for some k which may depend on $\lambda_n(\mathcal{M})$.

With regard to estimate N_k, we start by recalling some results on best polynomial approximation, cf. [1]. For a continuous complex-valued function f on a compact set K in the plane, let

$$d_n := d_n(f,K) := \inf\{\|f-p\|_K : p \in \mathcal{P}_n\},$$

where $\mathcal{P}_n$ is the vector space of complex polynomials of degree at most n. Recall that a Green's function g_K for $\mathbb{C}\setminus K$ is a continuous function $g_K : \mathbb{C} \to [0,+\infty)$ which is identically equal to zero on K, harmonic on $\mathbb{C}\setminus K$, and has a logarithmic singularity at infinity, in the sense that $g_K(z) - \log|z|$ is harmonic at infinity.

Theorem 12 (Walsh)**.** *Let K be a compact subset of the plane such that $\mathbb{C}\backslash K$ is connected and has a Green's function g_K. For $R>1$, let $D_R := \{z\in\mathbb{C} : g_K < \log R\}$. Let f be continuous on K. Then $\limsup_{n\to\infty} d_n(f,K)^{1/n} \le 1/R$ if and only if f is the restriction to K of a function holomorphic in D_R.*

The proof of the "if" part of this theorem for the case $K=[-1,1]$, given in [1, Section 2] by the use of duality theory, in fact provides the following result which will be crucial for our considerations. We include its proof here for the reader's convenience.

Lemma 13. *Let Ω be an open subset of $\mathbb{C}$, and let K be a compact subset of Ω such that $\mathbb{C}\backslash K$ is connected and has a Green's function g_K. Let $R>1$ be such that $\overline{D_R} \subset \Omega$. Then, for every $f \in H(\Omega)$, we have*

$$\forall\, 1<r<\rho<R: \quad d_n(f,K) \le \|f\|_{\partial D_R} \left(\frac{r}{\rho}\right)^n \frac{8\lambda(D_R\setminus D_\rho)}{\pi\ \mathrm{dist}(\partial D_R, D_\rho)\ \mathrm{dist}(\partial D_r, K)},$$

where λ denotes Lebesgue measure on $\mathbb{C}$.

Proof. Let $1<r<\rho<R$. Choose $\phi \in \mathscr{D}(\Omega)$ with supp $\phi \subseteq D_R$ and $\phi = 1$ in a neighborhood of D_ρ, and set $F := \phi f \in \mathscr{D}(\Omega) \subset \mathscr{D}(\mathbb{R}^2)$. Then it follows, as in [1, Section 2], that

$$d_n = d_n(f,K) = \int_{D_R\backslash D_\rho} \tilde{\mu}(z) \frac{\partial}{\partial \overline{z}} F(z)\, d\lambda(z), \tag{3.1}$$

where λ denotes Lebesgue measure on $\mathbb{C}$, and $\tilde{\mu} \in H(\mathbb{C}\backslash K)$ satisfies

$$\forall\, z \in \mathbb{C}\backslash D_r: \quad |\tilde{\mu}(z)| \le \frac{1}{\pi\ \mathrm{dist}(\partial D_r, K)} \big(\exp(\log r - g_K(z))\big)^n.$$

In particular, for all $z \in \mathbb{C}\backslash D_\rho$, we have

$$|\tilde{\mu}(z)| \le \frac{1}{\pi\ \mathrm{dist}(\partial D_r, K)} \big(\exp(\log r - \log\rho)\big)^n = \frac{1}{\pi\ \mathrm{dist}(\partial D_r, K)} \left(\frac{r}{\rho}\right)^n,$$

so that, by (3.1), by the identity $\frac{\partial}{\partial \overline{z}}F(z) = f(z)\frac{\partial}{\partial \overline{z}}\phi(z)$ and by the maximum principle applied to f, we have

$$\begin{aligned} d_n &\le \frac{1}{\pi \operatorname{dist}(\partial D_r, K)} \left(\frac{r}{\rho}\right)^n \int_{D_R\setminus D_\rho} \left| f(z)\frac{\partial}{\partial \overline{z}}\phi(z)\right| \, d\lambda(z) \\ &\le \|f\|_{\partial D_R} \left(\frac{r}{\rho}\right)^n \frac{1}{\pi \operatorname{dist}(\partial D_r, K)} \sup_{z\in D_R} \left|\frac{\partial}{\partial \overline{z}}\phi(z)\right| \lambda(D_R\setminus D_\rho). \end{aligned} \tag{3.2}$$

Let $\delta := \operatorname{dist}(\partial D_R, D_\rho)$ be the distance from D_ρ to the complement of D_R. According to [8, Proof of Theorem 1.4.2], we can choose ϕ with

$$\forall\, \alpha \in \mathbb{N}^2, |\alpha| = k, x \in \mathbb{R}^2 : \, |\partial^\alpha \phi(x)| \le 8^k/(\delta_1 \ldots \delta_k),$$

where $(\delta_j)_{j\in\mathbb{N}}$ is any decreasing sequence of positive numbers with $\sum_{j=1}^\infty \delta_j < \delta$. In particular, we can choose ϕ such that

$$\forall\, z \in \mathbb{C} : \quad \left|\frac{\partial}{\partial \overline{z}}\phi(z)\right| \le \frac{8}{\delta_1},$$

with $0 < \delta_1 < \delta$ arbitrary. Combining this with (3.2) gives

$$\forall\, 0 < \delta_1 < \delta : \quad d_n \le \|f\|_{\partial D_R} \left(\frac{r}{\rho}\right)^n \frac{8}{\pi \operatorname{dist}(\partial D_r, K)\, \delta_1} \lambda(D_R\setminus D_\rho),$$

and, letting δ_1 tend to δ, we have

$$\forall\, 1 < r < \rho < R : \quad d_n \le \|f\|_{\partial D_R} \left(\frac{r}{\rho}\right)^n \frac{8\lambda(D_R\setminus D_\rho)}{\pi \operatorname{dist}(\partial D_R, D_\rho) \operatorname{dist}(\partial D_r, K)}.$$

This completes the proof. □

To formulate our next result conveniently, we introduce the following notion. Let K, L be two non-empty, disjoint, compact subsets of $\mathbb{C}$ such that $\mathbb{C}\setminus(K \cup L)$ has a Green's function g. We call $R > 1$ *separating for K and L* if no component of $D_R := \{z \in \mathbb{C} : g(z) < \log R\}$ contains elements of both K and L. That is, if U_R is the union of the components of D_R intersecting K, and if $V_R := D_R\setminus U_R$, then U_R, V_R are open, disjoint neighborhoods of K, L, respectively with $U_R \cup V_R = D_R$.

Proposition 14. *Let $\Omega \subseteq \mathbb{C}$ be open and simply connected, let $\zeta \in \Omega$, and let $(K_k)_{k\in\mathbb{N}}$ be a compact exhaustion of Ω such that $\zeta \in K_1$ and $\mathbb{C}\setminus K_k$ is connected for every $k \in \mathbb{N}$. Also, for $k \in \mathbb{N}$, let $\Omega_k \subset \mathbb{C}$ be open, let $L_k \subset \Omega_k$ be compact, and let $f_k \in H(\Omega_k)$. Assume that $\mathbb{C}\setminus L_k$ is connected, that $K_k \cap L_k = \emptyset$, and that $\mathbb{C}\setminus(K_k \cup L_k)$ has a Green's function g_k for every $k \in \mathbb{N}$. Let $R_k > 1$ be separating for K_k, L_k, and suppose further that $D_k := D_{R_k} = \{z \in \mathbb{C} : g_k(z) < \log R_k\} \subset \Omega \cup \Omega_k$.*

Then, for every choice of $1 < r_k < \rho_k < R_k$ $(k \in \mathbb{N})$, there exists $f \in H(\Omega)$ with uniformly universal power series in ζ for $(f_k, L_k)_{k\in\mathbb{N}}$ such that

$$\forall\, k \ge 2 : \quad N_k < N_{k-1} + \frac{\log^+\left(k^2\|f_k - T^{(\zeta)}_{N_{k-1}} f\|_{\overline{V_k}}\, q_k^{N_{k-1}} C_k\right)}{\log\left(\dfrac{\rho_k}{r_k}\right)} + 1,$$

where

$$V_k := V_{R_k}, \quad q_k := \frac{\operatorname{diam}(K_k \cup L_k)}{\operatorname{dist}(K_k, \overline{V_k})},$$

and

$$C_k := C(r_k, \rho_k, R_k) := \frac{8\lambda(D_{R_k} \setminus D_{\rho_k})}{\pi \operatorname{dist}(\partial D_{R_k}, D_{\rho_k}) \operatorname{dist}(\partial D_{r_k}, K_k \cup L_k)}.$$

Proof. Like in [11, Proof of Theorem 2] we begin by constructing a sequence of polynomials $(P_k)_{k\in\mathbb{N}_0}$ and a strictly increasing sequence of integers $(N_k)_{k\in\mathbb{N}_0}$ with the following properties: the degree of P_k satisfies $\deg P_k = N_k$, the point ζ is a zero of P_k of multiplicity at least N_{k-1}, and

$$\forall\, k \geq 1 : \big\|P_k\big\|_{K_k} < \frac{1}{k^2}, \tag{3.3}$$

as well as

$$\forall\, k \geq 1 : \Big\|\sum_{\nu=0}^{k} P_\nu - f_k\Big\|_{L_k} < \frac{1}{k^2}. \tag{3.4}$$

We set $P_0(z) \equiv 1$ and $N_0 = 0$. Suppose that, for some $k \in \mathbb{N}$, the polynomials $P_0, \ldots, P_{k-1}$ and the integers $N_0, \ldots, N_{k-1}$ have already been determined. Because R_k is separating for K_k and L_k, we have, with $U_k := U_{R_k}$ and $V_k := V_{R_k}$, disjoint open neighborhoods of K_k and L_k with $U_k \cup V_k = D_k$. Consider the function

$$h_k : U_k \cup V_k \to \mathbb{C}, \qquad z \mapsto \begin{cases} 0, & \text{if } z \in U_k, \\ \dfrac{f_k(z) - \sum_{\nu=0}^{k-1} P_\nu(z)}{(z-\zeta)^{N_{k-1}}}, & \text{if } z \in V_k, \end{cases}$$

which is well-defined and holomorphic. From Lemma 13, we obtain that

$$d_n(h_k, K_k \cup L_k) \leq \|h_k\|_{\partial D_k} \left(\frac{r_k}{\rho_k}\right)^n C_k \leq \|h_k\|_{\overline{V_k}} \left(\frac{r_k}{\rho_k}\right)^n C_k,$$

where, in the last step, we used the maximum principle and the fact that $h_k|_{U_k} = 0$. Hence, in order to have

$$d_n(h_k, K_k \cup L_k) < \frac{1}{k^2 \max\limits_{K_k \cup L_k} |z-\zeta|^{N_{k-1}}}, \tag{3.5}$$

it suffices that

$$k^2 \max_{K_k \cup L_k} |z-\zeta|^{N_{k-1}} \big\|h_k\big\|_{\overline{V_k}} C_k < \left(\frac{\rho_k}{r_k}\right)^n.$$

The latter is obviously the case if

$$k^2 \Big\| f_k - \sum_{\nu=0}^{k-1} P_\nu \Big\|_{\overline{V_k}} \left(\frac{\max\limits_{K_k \cup L_k} |z-\zeta|}{\min\limits_{\overline{V_k}} |z-\zeta|} \right)^{N_{k-1}} C_k < \left(\frac{\rho_k}{r_k}\right)^n.$$

Moreover, $\min_{\overline{V_k}} |z-\zeta| \geq \operatorname{dist}(K_k, \overline{V_k})$ and $\max_{K_k \cup L_k} |z-\zeta| \leq \operatorname{diam}(K_k \cup L_k)$, the diameter of $K_k \cup L_k$, so that (3.5) is satisfied if

$$n \geq \frac{\log^+\left(k^2 \|f_k - \sum_{\nu=0}^{k-1} P_\nu\|_{\overline{V_k}}\, q_k^{N_{k-1}} C_k\right)}{\log\left(\dfrac{\rho_k}{r_k}\right)} =: c(k). \tag{3.6}$$

By the above, if we fix $n \in \mathbb{N} \cap [c(k), c(k)+1]$, then there is $\Pi_n \in \mathcal{P}_n$ satisfying

$$\|\Pi_n\|_{K_k} < \frac{1}{k^2 \cdot \max\limits_{K_k \cup L_k} |z-\zeta|^{N_{k-1}}}$$

and

$$\left\|\Pi_n - \frac{f_k - \sum_{\nu=0}^{k-1} P_\nu}{(z-\zeta)^{N_{k-1}}}\right\|_{L_k} < \frac{1}{k^2 \cdot \max\limits_{K_k \cup L_k} |z-\zeta|^{N_{k-1}}}.$$

By adding a sufficiently small multiple of the identity to Π_n, we can assume without loss of generality that $\deg \Pi_n \geq 1$. Setting $P_k(z) := (z-\zeta)^{N_{k-1}} \Pi_n(z)$, we thus obtain that ζ is a zero of P_k of multiplicity at least N_{k-1}, that $N_k := \deg P_k \leq N_{k-1} + n$ and $\deg P_k > N_{k-1}$, and that P_k fulfils (3.3) and (3.4).

With the P_k constructed, we now define $f : \Omega \to \mathbb{C},\ z \mapsto \sum_{k=0}^{\infty} P_k(z)$. Because of (3.3), the function f is well-defined and holomorphic in Ω. Since $\deg P_k = N_k$ and $P_k(z) = (z-\zeta)^{N_{k-1}} \Pi_k(z)$ for some polynomial Π_k of strictly positive degree, it follows that $T_{N_k}^{(\zeta)} f = \sum_{\nu=0}^{k} P_\nu$ for every $k \in \mathbb{N}$. On the one hand, by (3.4), this implies that

$$\forall\, k \geq 1: \quad \|f_k - T_{N_k}^{(\zeta)} f\|_{L_k} < \frac{1}{k^2}, \tag{3.7}$$

and on the other hand, by (3.6) and the maximum principle, we have

$$\begin{aligned} N_k &= \deg P_k \leq N_{k-1} + n \\ &\leq N_{k-1} + \frac{\log^+\left(k^2 \|f_k - T_{N_{k-1}}^{(\zeta)} f\|_{\overline{V_k}}\, q_k^{N_{k-1}} C_k\right)}{\log\left(\dfrac{\rho_k}{r_k}\right)} + 1. \end{aligned}$$

Thus f has all the required properties. □

Obviously, the result stated in Proposition 14 contains too many unknown quantities in order to allow an explicit (non-recursive) estimate for the growth of N_k. But nevertheless, in the general context, we already see that the N_k grow slower if L_k is farther away from Ω (respectively K_k), since q_k is smaller then.

Let us say that $f \in H(\mathbb{D})$ has a *universal Taylor series in 0 for* $H(\Omega)$ if the Taylor polynomials of f about 0 are dense in $H(\Omega)$, where $\Omega \subset \mathbb{C} \setminus \mathbb{D}$ is open. Instead of constructing a holomorphic function f with a universal Taylor series about the origin in the sense of [13], we construct $f \in H(\mathbb{D})$ having a universal Taylor series in 0 for $H(c+\mathbb{D})$ for some $c \in \mathbb{C}$, and we investigate how fast the elements of a given normal family $\mathcal{M}$ in $H(c+\mathbb{D})$ can be approximated by the Taylor polynomials of f.

Also in this situation the N_k grow slower, as we will see later, since the sets L_k and the functions f_k to approximate on L_k can be chosen closer to their predecessors L_{k-1} and f_{k-1}. Indeed, by (3.7), $T^{(\zeta)}_{N_{k-1}} f$ is close to f_{k-1} on L_{k-1}. If additionally L_k is close to L_{k-1} and f_k is close to f_{k-1}, then $\|f_k - T^{(\zeta)}_{N_{k-1}} f\|_{\overline{V_k}}$ remains rather small.

We consider the standard compact exhaustions of $\mathbb{D}$ and $c+\mathbb{D}$, respectively, that is $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ and $\mathcal{K}_c = (K_{c,n})_{n\in\mathbb{N}}$, where

$$K_n := \frac{n}{n+1}\overline{\mathbb{D}}, \quad K_{c,n} := c + K_n, \quad n \in \mathbb{N}. \tag{3.8}$$

Since we are now dealing with disks, we have the following approximation result at our disposal, which will replace the use of Lemma 13.

Lemma 15. *Let $L = L_1 \cup L_2 := \overline{D}(a_1, R_1) \cup \overline{D}(a_2, R_2)$ be the union of two disjoint closed disks. Let $K = K_1 \cup K_2 := \overline{D}(a_1, r_1) \cup \overline{D}(a_2, r_2)$, where $0 < r_j < R_j$ $(j = 1, 2)$. Given f holomorphic on a neighborhood of L and $n \geq 1$, there exists a polynomial p such that $\deg p < 2n$ and*

$$\|f - p\|_K \leq \|f\|_L \frac{2\alpha^n}{(1-\alpha)} \left(\frac{\operatorname{diam}(K)}{\operatorname{dist}(L_1, L_2)}\right)^n,$$

where $\alpha := \max\{r_1, r_2\}/\min\{R_1, R_2\}$.

Proof. Set $q(z) := (z-a_1)^n (z-a_2)^n$. We consider the special kind of Hermite interpolation polynomial

$$p(w) := \frac{1}{2\pi i}\int_{\partial L} \frac{f(z)}{q(z)} \frac{q(z)-q(w)}{z-w}\, dz,$$

and we shall show that this works.

Since $(q(z)-q(w))/(z-w)$ is a polynomial in z, w of degree at most $2n-1$ in each variable, it follows that $p(w)$ is a polynomial of degree at most $2n-1$.

Also, by Cauchy's integral formula, if $w \in K$, then

$$f(w) = \frac{1}{2\pi i}\int_{\partial L} \frac{f(z)}{z-w}\, dz,$$

and so

$$f(w) - p(w) = \frac{1}{2\pi i}\int_{\partial L} \frac{f(z)}{z-w} \frac{q(w)}{q(z)}\, dz.$$

It follows that

$$\|f-p\|_K \leq \frac{\|f\|_L \|q\|_K}{2\pi \operatorname{dist}(\partial L, K)} \int_{\partial L} \frac{|dz|}{|q(z)|}.$$

Now, if $w \in K_1$, then $|q(w)| \leq r_1^n (\operatorname{diam} K)^n$. An analogous estimate holds for $w \in K_2$. Hence

$$\|q\|_K \leq \max\{r_1, r_2\}^n (\operatorname{diam} K)^n.$$

Also, we clearly have

$$\operatorname{dist}(\partial L, K) \geq \min\{R_1 - r_1, R_2 - r_2\}.$$

Further, if $z \in \partial L_1$, then $|q(z)| \geq R_1^n \operatorname{dist}(L_1, L_2)^n$. Hence

$$\int_{\partial L_1} \frac{|dz|}{|q(z)|} \leq \frac{2\pi R_1}{R_1^n \operatorname{dist}(L_1, L_2)^n}.$$

An analogous estimate holds for the integral over ∂L_2. Putting together these estimates, we get

$$\|f - p\|_K \leq \|f\|_L \frac{\max\{r_1, r_2\}^n (\operatorname{diam} K)^n}{\min\{R_1 - r_1, R_2 - r_2\} \operatorname{dist}(L_1, L_2)^n} \left(\frac{1}{R_1^{n-1}} + \frac{1}{R_2^{n-1}} \right).$$

If we set $r := \max\{r_1, r_2\}$ and $R := \min\{R_1, R_2\}$, then we obtain

$$\|f - p\|_K \leq \|f\|_L \frac{r^n (\operatorname{diam} K)^n}{(R - r) \operatorname{dist}(L_1, L_2)^n} \left(\frac{2}{R^{n-1}} \right).$$

The result follows from this. □

In the situation of the above lemma, let us choose $K = K_k \cup K_{c,k}$, that is $a_1 = 0$, $a_2 = c \in \mathbb{C}$, $r_1 = r_2 = \frac{k}{k+1} < 1$, and let $R_1 = R_2 =: R > 1$. The disks L_1, L_2 are disjoint as long as $\operatorname{dist}(L_1, L_2) = |c| - 2R > 0$. Further, we obtain

$$\alpha \frac{\operatorname{diam}(K)}{\operatorname{dist}(L_1, L_2)} < \frac{1}{R} \frac{|c| + 2}{|c| - 2R} =: q(R, c),$$

and the inequality in Lemma 15 reads

$$\|f - p\|_K \leq \|f\|_L \frac{2R}{R - 1} q(R, c)^n. \tag{3.9}$$

Lemma 16. *Let $R > 1$ and $c \in \mathbb{C}$ with $|c| > 2R$ and $q(R, c) < 1$. For any sequence of polynomials $(f_k)_{k \in \mathbb{N}}$, there is $f \in H(\mathbb{D})$ having a uniformly universal Taylor series in the origin for $(f_k, K_{c,k})_{k \in \mathbb{N}}$ such that the corresponding sequence $(N_k)_{k \in \mathbb{N}}$ is strictly increasing, $T_{N_k}^{(0)} f$ and N_k depend only on $f_1, \ldots, f_{k-1}$ and, for $k \geq 7$, we have*

$$N_k \leq N_{k-1} + 1 + \frac{2 \log^+(A)}{\log(q(R, c)^{-1})}, \tag{3.10}$$

where

$$A := \frac{2R}{R - 1} (2R)^{\max\{\deg f_k, N_{k-1}\}} \left(\frac{2|c| + 2}{|c| - R} \right)^{N_{k-1}} \left(\|f_k - f_{k-1}\|_{K_{c,k-1}} + \frac{1}{(k-1)^2} \right).$$

Proof. The proof is very similar to that of Proposition 14, only we use Lemma 15 instead of Lemma 13.

As in the proof of Proposition 14, we construct recursively a sequence of polynomials $(P_k)_{k \in \mathbb{N}_0}$ and a strictly increasing sequence of integers $(N_k)_{k \in \mathbb{N}_0}$ such that the degree of P_k satisfies $\deg P_k = N_k$, the origin is a zero of P_k of multiplicity at least N_{k-1}, and

$$\forall\, k \geq 1: \quad \big\|P_k\big\|_{K_k} < \frac{1}{k^2} \text{ and } \Big\|\sum_{\nu=0}^{k} P_\nu - f_k\Big\|_{K_{c,k}} < \frac{1}{k^2}. \tag{3.11}$$

Let $P_0(z) \equiv 1$, and $N_0 := 0$. If, for some $k \in \mathbb{N}$, the polynomials $P_0, \ldots, P_{k-1}$ and the integers $N_0, \ldots, N_{k-1}$ have already been constructed, then we consider

$$h_k : \overline{B(0,R)} \cup \overline{B(c,R)} \to \mathbb{C}, \quad z \mapsto \begin{cases} 0, & \text{if } z \in \overline{B(0,R)} \\ \dfrac{f_k(z) - \sum\limits_{\nu=0}^{k-1} P_\nu(z)}{z^{N_{k-1}}}, & \text{if } z \in \overline{B(c,R)}, \end{cases}$$

which is well-defined and holomorphic in a neighborhood of $\overline{B(0,R)} \cup \overline{B(c,R)}$, since $|c| > 2R$. Lemma 15 and inequality (3.9) yield, for any $n \in \mathbb{N}$, the existence of a polynomial Π_n of degree not exceeding $2n-1$, with

$$\|h_k - \Pi_n\|_{K_k \cup K_{c,k}} \leq \|h_k\|_{\overline{B(0,R)} \cup \overline{B(c,R)}} \frac{2R}{R-1} q(R,c)^n = \|h_k\|_{\overline{B(c,R)}} \frac{2R}{R-1} q(R,c)^n,$$

where we used $h_k|_{\overline{B(0,R)}} = 0$. As $\max_{K_k \cup K_{c,k}} |z| = \max_{K_{c,k}} |z| = |c| + \frac{k}{k+1} < |c| + 1$, as well as $\min_{\overline{B(c,R)}} |z| = |c| - R$, as in the proof of Proposition 14 we obtain that, in order to have

$$\|h_k - \Pi_n\|_{K_k \cup K_{c,k}} < \frac{1}{k^2 \max\limits_{K_k \cup K_{c,k}} |z|^{N_{k-1}}},$$

it is sufficient to have

$$k^2 \Big\| f_k - \sum_{\nu=0}^{k-1} P_\nu \Big\|_{\overline{B(c,R)}} \left(\frac{|c|+1}{|c|-R} \right)^{N_{k-1}} \frac{2R}{R-1} < q(R,c)^{-n}. \tag{3.12}$$

An application of Bernstein's lemma, cf. [14, Theorem 5.5.7], and (3.11) yields

$$\begin{aligned} \Big\| f_k - \sum_{\nu=0}^{k-1} P_\nu \Big\|_{\overline{B(c,R)}} &\leq \Big\| f_k - \sum_{\nu=0}^{k-1} P_\nu \Big\|_{K_{c,k-1}} \left(\frac{Rk}{k-1} \right)^{\max\{\deg f_k, N_{k-1}\}} \\ &\leq \left(\|f_k - f_{k-1}\|_{K_{c,k-1}} + \frac{1}{(k-1)^2} \right) (2R)^{\max\{\deg f_k, N_{k-1}\}}. \end{aligned}$$

Therefore, $n \in \mathbb{N}$ satisfies (3.12) if $n > \alpha(k)$, where $\alpha(k)$ is given by

$$\frac{\log^+ \left(\frac{2Rk^2}{R-1} (2R)^{\max\{\deg f_k, N_{k-1}\}} \left(\frac{|c|+1}{|c|-R} \right)^{N_{k-1}} \left(\|f_k - f_{k-1}\|_{K_{c,k-1}} + \frac{1}{(k-1)^2} \right) \right)}{\log(q(R,c)^{-1})}. \tag{3.13}$$

Fixing $n \in \mathbb{N} \cap [\alpha(k), 1 + \alpha(k)]$, we continue as in the proof of Proposition 14, to construct P_k and $N_k := \deg P_k \leq N_{k-1} + 2n - 1$.

As in the proof of Proposition 14, it follows that $f : \mathbb{D} \to \mathbb{C}$, $z \mapsto \sum_{k=0}^{\infty} P_k(z)$ is holomorphic and has a uniformly universal Taylor series in 0 for $(f_k, K_{c,k})_{k \in \mathbb{N}}$. For the corresponding sequence $(N_k)_{k \in \mathbb{N}}$, we have

$$N_k = \deg P_k \leq N_{k-1} + 2n - 1 \leq N_{k-1} + 1 + 2\alpha(k).$$

If $k \geq 7$, then, because $k^2 \leq 2^{k-1} (\leq 2^{N_{k-1}})$, we obtain from (3.13) that $\alpha(k)$ is majorized by

$$\frac{\log^+ \left(\frac{2R}{R-1} (2R)^{\max\{\deg f_k, N_{k-1}\}} \left(\frac{2|c|+2}{|c|-R} \right)^{N_{k-1}} \left(\|f_k - f_{k-1}\|_{K_{c,k-1}} + \frac{1}{(k-1)^2} \right) \right)}{\log(q(R,c)^{-1})}.$$

This completes the proof. ☐

Remark 17. Proposition 14 and Lemma 16 show how small the values of the sequence $(N_k)_{k\in\mathbb{N}}$ can be chosen. Nevertheless, inspection of their proofs gives that, at each step, N_k can be chosen arbitrarily large.

Lemma 18. *Suppose that $\mathcal{M}$ be a normal family in $H(c+\mathbb{D})$. Let $M_n := 1+\sup_{f\in\mathcal{M}}\|f\|_{K_{c,n}}$ and let $(\lambda_n)_{n\in\mathbb{N}}$ be the covering numbers of $\mathcal{M}$, i.e., there are functions $f_1^{(n)},\ldots,f_{\lambda_n}^{(n)}\in H(c+\mathbb{D})$ whose $\frac{1}{n}$-neighborhoods cover $\mathcal{M}$. If*

$$m\geq \tau_{n+1}(\mathcal{M}) := (n+1)^2\log\big((n+1)^3 M_{n+1}\big),$$

then

$$\|g-T_m^{(c)}g\|_{K_{c,n}} < \frac{1}{n}\quad \forall\, g\in\mathcal{M}\cup\{f_1^{(n+1)},\ldots,f_{\lambda_{n+1}}^{(n+1)}\}.$$

Proof. Let $g=f_j^{(n+1)}$ for some $1\leq j\leq\lambda_{n+1}$. Then there is $f\in\mathcal{M}$ such that $\|g-f\|_{K_{c,n+1}}\leq\frac{1}{n+1}$, which implies

$$\|g\|_{K_{c,n+1}}\leq\|f\|_{K_{c,n+1}}+\frac{1}{n+1}\leq M_{n+1}.$$

This last estimate obviously also holds for $g\in\mathcal{M}$. By Cauchy's estimate, we get

$$\|g-T_m^{(c)}g\|_{K_{c,n}}\leq M_{n+1}\sum_{\nu=m+1}^{\infty}\left(\frac{n(n+2)}{(n+1)^2}\right)^{\nu} = M_{n+1}\,n(n+2)\,\left(\frac{n(n+2)}{(n+1)^2}\right)^{m}.$$

The last term is less than $1/n$ provided that

$$m\geq (n+1)^2\log\big((n+1)^3\,M_{n+1}\big)\geq\frac{\log\big(n^2(n+2)\,M_{n+1}\big)}{\log\Big(\frac{(n+1)^2}{n(n+2)}\Big)}.$$

This completes the proof. ☐

By choosing an appropriate sequence of polynomials (f_k) in Lemma 16, we can construct a function $f\in H(\mathbb{D})$ which has a universal Taylor series in 0 for $H(c+\mathbb{D})$ for some center $c\in\mathbb{C}$, which is fast approximating for a normal family $\mathcal{M}\subset H(c+\mathbb{D})$, and such that the corresponding sequence (N_k) has bounded quotients N_k/N_{k-1} (compare with Remark 11).

Theorem 19. *Let $c\in\mathbb{C}$ with $|c|>10$. Let $\mathcal{M}$ be a normal family in $H(c+\mathbb{D})$ with covering numbers (λ_n), and suppose that $M_n=O(\exp(n^l))$ for some $l\in\mathbb{N}\cup\{0\}$. Then there exists a function $f\in H(\mathbb{D})$ which has a universal Taylor series for $H(c+\mathbb{D})$, which is fast approximating for $\mathcal{M}$ in the sense that*

$$F(f,\mathcal{M},2/n)\leq N_{(n+1)(\lambda_{n+1}+1)},$$

and which has bounded quotients N_k/N_{k-1}.

Proof. For $R = 2$, we have $|c| > 10 > 2R$ and $q(2,c) < 1$. Let $f_1^{(n)}, \dots, f_{\lambda_n}^{(n)} \in H(c+\mathbb{D})$ be functions whose $\frac{1}{n}$-neighborhoods cover $\mathcal{M}$. Let (q_n) be a sequence of polynomials which is dense in $H(c+\mathbb{D})$, and consider the sequence (g_k) given by

$$f_1^{(1)}, \dots, f_{\lambda_1}^{(1)}, q_1, f_1^{(2)}, \dots, f_{\lambda_2}^{(2)}, q_2, \dots, f_1^{(n)}, \dots, f_{\lambda_n}^{(n)}, q_n, \dots$$

If $g_k := f_j^{(n)}$ for some j, n, then n is uniquely determined by k, and we have $n \leq k$. In this case, we set $f_k := T^{(c)}_{\tau_n(\mathcal{M})} g_k$. Using Lemma 18, we deduce

$$\deg f_k = n^2 \log\left(n^3 M_n\right) \quad \text{and} \quad \|g_k - f_k\|_{K_{c,n-1}} < \frac{1}{n-1}. \tag{3.14}$$

If, on the other hand, g_k is one of the (q_n), then we define $f_k := g_k$. Without loss of generality $\deg f_k \leq k$ and $\|f_k\|_{K_{c,k}} \leq k$.

Now let $f \in H(\mathbb{D})$ be the function constructed in Lemma 16, that is, f has a uniformly universal Taylor series in the origin for $(f_k, K_{c,k})$ with the corresponding sequence (N_k) satisfying (3.10). In view of (3.10), we can estimate the degree of f_k in case $g_k = f_j^{(n)}$:

$$\deg f_k = n^2 \log\left(n^3 M_n\right) \leq k^2 \log\left(k^3 M_k\right) = O(k^{l+3}),$$

since $n \leq k$ and $M_k = O(\exp(k^l))$. Hence, by Bernstein's lemma, cf. [14, Theorem 5.5.7],

$$\|f_k\|_{K_{c,k}} \leq \|f_k\|_{K_{c,n}} \left(\frac{k}{k+1}\frac{n+1}{n}\right)^{\deg f_k} \leq M_n\, e^{n^2 \log(n^3 M_n)}.$$

Thus, independently of whether $g_k = f_j^{(n)}$ or $g_k = q_n$, we obtain

$$\log\left(\|f_k - f_{k-1}\|_{K_{c,k-1}} + \frac{1}{(k-1)^2}\right) = O(k^{l+3}).$$

Hence, using the fact that $\max\{\deg f_k, N_{k-1}\} \leq \deg f_k + N_{k-1}$, inequality (3.10) reads

$$N_k \leq \alpha_1 N_{k-1} + \alpha_2 k^{l+3},$$

where α_1, α_2 are constants. As mentioned in Remark 17, it is possible to increase N_k at each step. So, let us choose

$$N_k := \max\left\{\alpha_1 N_{k-1} + \alpha_2 k^{l+3}, (k+1)^{l+3}\right\}, \tag{3.15}$$

which guarantees $N_{k-1} \geq k^{l+3}$ for every $k \in \mathbb{N}$. Depending on where the maximum in (3.15) is attained, N_k/N_{k-1} is either bounded by the constant $\alpha_1 + \alpha_2$ or by $\left(\frac{k+1}{k}\right)^{l+3}$. Either way, N_k/N_{k-1} remains bounded as $k \to \infty$.

To each $g \in \mathcal{M}$ is associated $g_k = f_j^{(n+1)}$ with $\|g_k - g\|_{K_{c,n+1}} < \frac{1}{n+1}$. Using (3.14), we have $\|f_k - g_k\|_{K_{c,n}} < 1/n$, which implies $\|f_k - g\|_{K_{c,n}} \leq 2/n$. By the choice of f and the sequences $(f_k), (g_k)$, we obtain $k \leq (n+1)(\lambda_{n+1}+1)$ and hence $F(f, \mathcal{M}, 2/n) \leq N_k$.

Since (q_n) is dense in $H(c+\mathbb{D})$, by construction so are the Taylor polynomials of f about 0, and hence f has a universal Taylor series for $H(c+\mathbb{D})$. □

In the next section we shall encounter several examples of normal families $\mathcal{M}$ for which $M_n = O(1)$, and so Theorem 19 is applicable with $l = 0$.

4 Normal families and covering numbers

In order to get some impression of the speed of approximation, we conclude with some examples of normal families in $H(\mathbb{D})$. Throughout this section, we suppose ε to be an arbitrarily small positive number, and consider the standard compact exhaustion (3.8) of $\mathbb{D}$ to define the seminorms in (1.2) and hence the natural metric on $H(\mathbb{D})$ in (1.1).

Let $E := \{f_1, \dots, f_k\}$ be a finite subset of $H(\mathbb{D})$, let $B^\infty := \{f \in H(\mathbb{D}) : \sup_{\mathbb{D}} |f| \le 1\}$, and let

$$S := \{f \in H(\mathbb{D})\colon f \text{ one-to-one}, f(0) = 0, f'(0) = 1\}.$$

For each of these three normal families, we obtain the existence of $\mathcal{C}$- or $\mathcal{D}$-universal functions f with rates of approximation as follows:

$\mathcal{M}$	$F(f, \mathcal{M}, 1/n)$ (for $\mathcal{C}$)	$F(f, \mathcal{M}, 1/n)$ (for $\mathcal{D}$)
E	$O\big(n\big)$	$O\big(n^2 \log(n \max\{1, M_{12kn+1}\})\big)$
B^∞	$O\big(n\lambda_{2n}\big)$	$O\big(n(n\lambda_{3n})^{1+\varepsilon}\big)$
S	$O\big(n\lambda_{2n}\big)$	$O\big(n^2 \lambda_{3n} \log(n\lambda_{3n})\big)$

where $M_n := \sup_{f\in\mathcal{M}} \|f\|_{K_n}$ and λ_n denotes the n-th covering number of $\mathcal{M}$.

Furthermore, $\mathcal{C}$ is 8- and $\mathcal{D}$ is $(9+\varepsilon)$-polynomial universal for the automorphism group

$$\operatorname{Aut}(\mathbb{D}) = \Big\{ f_{\gamma,a}(z) := e^{i\gamma} \frac{z-a}{1-\bar{a}z} \colon \gamma \in [0, 2\pi), a \in \mathbb{D} \Big\},$$

and $\mathcal{C}$ is $(2+\varepsilon)$- and $\mathcal{D}$ $(4+\varepsilon)$-polynomial universal for the set of all Koebe extremal functions

$$K := \Big\{ f_\alpha = e^{-i\alpha} f_0(e^{i\alpha} z) \colon \alpha \in [0, 2\pi) \Big\} \subseteq S, \quad f_0(z) = \frac{z}{(1-z)^2}.$$

For all this and further details, see [10].

In this context, the question arises, interesting in its own right, to estimate the n-th covering number λ_n for S, or, going back one step, to estimate the minimal number $N(\delta)$ of balls of radius δ required to cover S. The following theorem provides upper and lower bounds for $N(\delta)$.

Theorem 20. *There exist constants $c, C > 0$ such that*

$$e^{c/\sqrt{\delta}} \le N(\delta) \le e^{(C/\delta)\log^2(1/\delta)}.$$

In particular, $N(\delta)$ grows faster than any power of $1/\delta$ as $\delta \to 0$. The proof of the upper bound is given in [10]. We give here the proof of the lower bound. It is based on an elementary lemma.

Lemma 21. *Let $f(z) = z + \sum_{k=2}^\infty a_k z^k$, where $\sum_{k=2}^\infty k|a_k| < 1$. Then $f \in S$.*

Proof. Let $z, w \in \mathbb{D}$. Then $|z^k - w^k| \le k|z-w|$ for all k, so

$$|f(z) - f(w)| \ge |z-w| - \sum_{k=2}^\infty |a_k|\,|z^k - w^k| \ge |z-w| \Big(1 - \sum_{k=2}^\infty k|a_k|\Big).$$

It follows that f is injective. Thus $f \in S$. □

Proof of the lower bound. Let $f, g \in H(\mathbb{D})$, say $f(z) = \sum_{k=0}^{\infty} a_k z^k$, and $g(z) = \sum_{k=0}^{\infty} b_k z^k$. By the maximum principle and the standard Cauchy estimates, for each $k \in \mathbb{N}$ we have

$$\|f - g\|_{K_k} = \max_{|z|=k/(k+1)} |f(z) - g(z)| \geq |a_k - b_k| \left(\frac{k}{k+1}\right)^k \geq \frac{|a_k - b_k|}{4},$$

and consequently

$$d(f, g) \geq \sup_{k\in\mathbb{N}} \min\left(\frac{|a_k - b_k|}{4}, \frac{1}{k}\right). \tag{4.1}$$

Now let $n \geq 2$, and consider the family $\mathcal{F}_n$ of polynomials f of the form

$$f(z) := z + \frac{1}{n^2} \sum_{k=2}^{n} \varepsilon_k z^k, \quad (\varepsilon_k \in \{-1, 1\},\ k = 2, \ldots, n).$$

By Lemma 21, we clearly have $\mathcal{F}_n \subset S$. Also, by (4.1), the distance between distinct polynomials $f, g \in \mathcal{F}_n$ is at least $1/(2n^2)$. Thus, in any covering by balls of radius $1/(4n^2)$, each of the polynomials of $\mathcal{F}_n$ must belong to a different ball. There are 2^{n-1} elements in the family $\mathcal{F}_n$. Therefore

$$N(1/4n^2) \geq 2^{n-1}.$$

As this holds for each $n \geq 2$, the lower bound follows. □

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