15

Localization and Delocalization for Nonstationary Models

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15.1 Introduction: Leaving stationarity

In recent years there has been considerable progress concerning mathematically rigorous results on the phenomenon of **localization**. We refer to the bibliography where we chose some classics, some recent articles as well as books on the subject. However, all these results provide only one part of the picture that is accepted since the groundbreaking work [4, 79] by Anderson, Mott and Twose: one expects a metal insulator transition. This effect is supposed to depend upon the dimension and the general picture is as follows: Once translated into the language of spectral theory there is a transition from

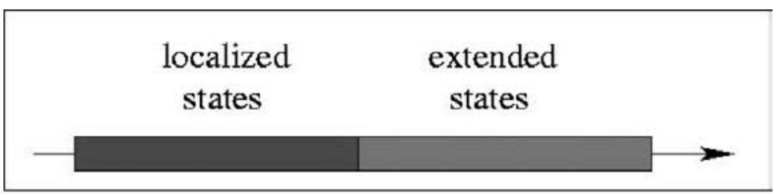

Fig. 15.1. Metal insulator transition

a **localized phase** that exhibits pure point spectrum (= only bound states = no transport) to a **delocalized phase** with absolutely continuous spectrum (= scattering states = transport). What has been proven so far is the occurrence of the former phase, well known under the name of localization. The missing part, delocalization, has not been settled for genuine random models.

There is need for an immediate disclaimer or, put differently, for an explanation of what I mean by "genuine".

An instance where a metal insulator transition has been verified rigorously is supplied by the **almost Mathieu operator**, a model with modest disorder

Au: Clarify "true evergreen"

for which the parameter that triggers the transition is the strength of the coupling. As references let us mention [6, 40, 57, 58, 59, 73] where the reader can find more about the literature on this true evergreen. Quite recently it has attracted a lot of interest especially among harmonic analysts; see [7, 8, 9, 10, 11, 12, 13, 14, 15, 44, 82]

The underlying Hilbert space is $l^2(\mathbb{Z})$. Consider parameters $\alpha, \lambda, \theta \in \mathbb{R}$ and define the self-adjoint, bounded operator $h_{\alpha,\lambda,\theta}$ by

$$(h_{\alpha,\lambda,\theta}u)(n) = u(n+1) + u(n-1) + \lambda\cos(2\pi(\alpha n + \theta))u(n),$$

for $u = (u(n))_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$.

Note that this operator is a discrete Schrödinger operator with a potential term with the coupling constant λ in front and the discrete analog of the Laplacian. For irrational α the potential term is an almost periodic function on $\mathbb{Z}$.

Basically, there is a metal insulator transition at the critical value 2 for the coupling constant λ. Since these operators are very close to being periodic, one can fairly label them as poorly disordered. Moreover, the proof of delocalization boils down to the proof of localization for a "dual operator" that happens to have the same form. In this sense, the almost Mathieu operator is not a genuine random model.

A second instance, where a delocalized phase is proven to exist is the Bethe lattice. See Klein's paper [61].

Quite recently, an order parameter has been introduced by Germinet and Klein to characterize the range of energies where a multiscale scenario provides a proof of a localized regime, [42]. In their work the important parameter is the energy.

However, as we already pointed out above, for genuine random models, there is no rigorous proof of the existence of a transition or even of the appearance of spectral components other than pure point, so far. This is a quite strange situation: the unperturbed problem exhibits extended states and purely a.c. spectrum but for the perturbed problem one can prove the opposite spectral behavior only.

In this survey we are dealing with models that are not transitive in the sense that the influence of the random potential is not uniform in space. The precise meaning of this admittedly vague description differs from case to case but will be clear for each of them.

15.2 Sparse random potentials

The term sparse potentials is mostly known for potentials that have been introduced in the 1970s by Pearson [81] to construct Schrödinger operators on the line with singular continuous spectrum. To use similar geometries to obtain a metal insulator transition can be traced back to Molchanov, Molchanov and

Vainberg [77, 78] and Krishna [70, 71], see also [63, 64, 72]. We have been strongly influenced by the paper [49] from which we take the random operator in $L^2(\mathbb{R}^d)$, **Model I**:

$$H(\omega) = -\Delta + V_\omega, \text{ where } V_\omega(x) = \sum_{k\in\mathbb{Z}^m} \xi_k(\omega) f(x-k),$$

$f \le 0$ is a compactly supported single site potential and the ξ_k are independent Bernoulli variables with $p_k := \mathbb{P}\{\xi_k = 1\}$.

To understand the appearance of a metallic regime, we recall the following facts from scattering theory:

We write $-\Delta = H_0$ so that the operators we are interested in can be written as $H = H_0 + V$. By $\sigma_{ac}(H)$ we denote the absolutely continuous spectrum, related to delocalized states.

Theorem 1 (Cooks criterion) *If for some $T_0 > 0$ and all ϕ in a dense set*

$$\int_{T_0}^{\infty} \|Ve^{-itH_0}\phi\| dt < \infty, \qquad (*)$$

then $\Omega_- := lim_{t\to\infty} e^{itH}e^{-itH_0}$ exists and, consequently, $[0,\infty) \subset \sigma_{ac}(H)$, i.e., there are scattering states for H and any nonnegative energy.

The typical application rests on the fact that $(*)$ is satisfied if

$$|V(x)| \le C(1+|x|)^{-(1+\epsilon)}, \qquad (**)$$

a condition that obviously fails to hold for almost every V_ω provided the p_k are not summable. However, the following nice result holds; see Hundertmark and Kirsch [49] who also provided the absolutely correct name:

Theorem 2 (Almost surely free lunch theorem) *Assume that*

$$W(x) := \left(\mathbb{E}(V_\omega(x)^2)\right)^{\frac{1}{2}} \overset{!}{\le} C(1+|x|)^{-(1+\epsilon)}.$$

Then V_ω satisfies Cook's criterion for a.e. ω.

The proof is so elegant and short that we can not resist to reproduce it here.

Proof.

$$\begin{aligned} &\mathbb{E}\left(\int_{T_0}^{\infty} \|V_\omega e^{-itH_0}\phi\| dt\right) \\ &= \int_{T_0}^{\infty} \mathbb{E}\left(\int V_\omega(x)^2 |e^{-itH_0}\phi(x)|^2 dx\right)^{\frac{1}{2}} dt \\ &= \int_{T_0}^{\infty} \left(\mathbb{E}\int V_\omega(x)^2 |e^{-itH_0}\phi(x)|^2 dx\right)^{\frac{1}{2}} dt \end{aligned}$$

$$\leq \int_{T_0}^{\infty} \left(\int \mathbb{E}(V_\omega(x)^2) |e^{-itH_0}\phi(x)|^2 dx] \right)^{\frac{1}{2}} dt$$

$$= \int_{T_0}^{\infty} \|W(x)e^{-itH_0}\phi\| dt \qquad \square$$

One can apply this result if the p_k decay fast enough to guarantee sufficient decay of $W(x)$. On the other hand one wants to have that $\sum_k p_k = \infty$, since otherwise V_ω has compact support a.s. by the Borel–Cantelli Lemma.

For fixed $d \geq 3$ and $\frac{d}{2} + \frac{1}{2} < \alpha < d$ and $p_k \sim k^{-\alpha}$ one can moreover control the essential spectrum below 0 as done in [49]: the negative essential spectrum consists of a sequence of energies that can at most accumulate at 0. Therefore, the negative spectrum is pure point. This can be summarized in the following picture:

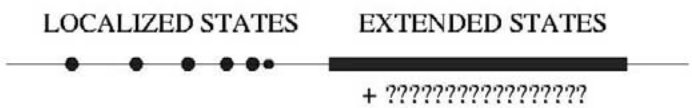

Fig. 15.2. The spectral picture for the sparse model I

We refer the reader to [49] for more on sparse random potentials, especially for models for which the negative spectrum has a richer structure and contains intervals.

Remarks 3 *In [17] we prove absence of an (absolutely) continuous spectrum outside the spectrum of the unperturbed operator for certain random sparse models reminiscent of Model I above and Model II from [49] but considerably more general. We use the techniques from [52, 90, 91].*

15.3 Random surface models

Consider the following self-adjoint random operator in $L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$, $\mathbb{R}^d = \mathbb{R}^m \times \mathbb{R}^{d-m}$:

$$H(\omega) = -\Delta + V_\omega, \text{ where } V_\omega(x) = \sum_{k \in \mathbb{Z}^m} q_k(\omega) f(x - (k, 0)),$$

the q_k are i.i.d. random variables and $f \geq 0$ is a single site potential that satisfies certain technical assumptions. This leads to the following geometry characterizing random surface models. Sometimes the upper half-plane is considered only.

There is a lot of literature, mostly on the discrete case, using a decomposition into a bulk and a surface term see; [5, 18, 21, 46, 50, 51, 54, 53, 55, 56].

The moral of the story is the appearance of a metal insulator transition at the edges of the unperturbed operator. We now concentrate on the continuum case, where we only know of [16, 49] as references. The existence of an a.c. component is proven in [49]. In the following, we present the result from [16], giving strong dynamical localization. Similar but somewhat different results have been announced in [49]. As discussed there, an additional Dirichlet boundary condition "stabilizes" the spectrum so that the appearance of a negative spectrum requires a certain strength of the random perturbation. Therefore, proving localization at negative energies is easier (compared to the case without Dirichlet boundary conditions) since one is automatically dealing with a "large coupling" regime.

In [16], no use is made of an additional Dirichlet b.c. and we have the following picture:

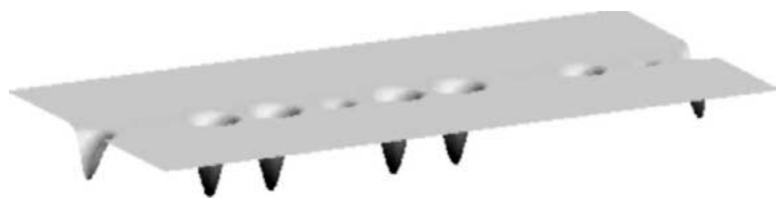

Fig. 15.3. A typical realization of a continuum random surface potential

It is not hard to see that

$$\sigma(H(\omega)) = [E_0, \infty) \text{ where } E_0 = \inf \sigma(-\Delta + q_{\min} \cdot f^{\mathrm{per}}),$$

and

$$f^{\mathrm{per}} = \sum_{k \in \mathbb{Z}^m} f(x - (k, 0))$$

denotes the periodic continuation of f along the surface. Near the bottom of the spectrum E_0 one expects localization, i.e., suppression of transport as is typical for insulators. For nonnegative energies one expects extended states. To stress the existence of a metallic phase let us cite Theorem 4.3 of [49] t

Theorem 4 *Let $H(\omega)$ be as below. Then we have, for every $\omega \in \Omega$:* $[0, \infty) \subset \sigma_{\mathrm{ac}}(H(\omega))$.

The idea of the proof is that a wave packet with velocity pointing away from the surface will escape the influence of the surface potential and is asymptotically free. The rigorous implementation of this idea uses Enss' technique from scattering theory.

The model *(1)* $0 < m < d$ *and points in* $\mathbb{R}^d = \mathbb{R}^m \times \mathbb{R}^{d-m}$ *are written as pairs, if convenient;*

(2) The single site potential $f \geq 0$, $f \in L^p(\mathbb{R}^d)$ *where* $p \geq 2$ *if* $d \leq 3$ *and* $p > d/2$ *if* $d > 3$, *and* $f \geq \sigma > 0$ *on some open set* $U \neq \emptyset$ *for some* $\sigma > 0$.

(3) The q_k *are i.i.d. random variables distributed with respect to a probability measure* μ *on* $\mathbb{R}$, *such that* $\operatorname{supp}\mu = [q_{min}, 0]$ *with* $q_{min} < 0$.

We will sometimes need further assumptions on the single site distribution μ:

(4) μ *is* Hölder continuous, *i.e., there are constants* $C, \alpha > 0$ *such that*

$$\mu[a,b] \leq C(b-a)^\alpha \text{ for } q_{min} \leq a \leq b \leq 0.$$

(5) Disorder assumption*: there exist* $C, \tau > 0$ *such that*

$$\mu[q_{min}, q_{min} + \varepsilon] \leq C \cdot \varepsilon^\tau \text{ for } \varepsilon > 0.$$

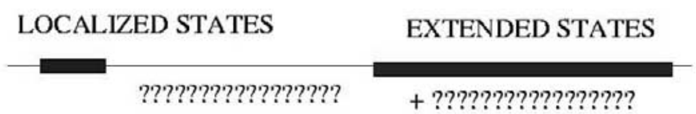

Fig. 15.4. Conclusion and open problems for the continuum surface model

What follows is the main result of [16].

Theorem 5 *Let* $H(\omega)$ *be as above with* $\tau > d/2$ *and assume that* $E_0 < 0$.

(a) *There exists an* $\varepsilon > 0$ *such that in* $[E_0, E_0 + \varepsilon]$ *the spectrum of* $H(\omega)$ *is pure point for almost every* $\omega \in \Omega$, *with exponentially decaying eigenfunctions.*

(b) *Assume that* $p < 2(2\tau - m)$. *Then there exists an* $\varepsilon > 0$ *such that in* $[E_0, E_0 + \varepsilon] = I$ *we have strong dynamical localization in the sense that for every compact set* $K \subset \mathbb{R}^d$:

$$\mathbb{E}\{\sup_{t>0} \| |X|^p e^{-itH(\omega)} P_I(H(\omega)) \chi_K \| \} < \infty\,.$$

A consequence is a pure point spectrum in the interval $[E_0, E_0 + \varepsilon] = I$. Together with the previous result on extended states we get the picture from Figure 15.4 that still leaves open some important questions.

Remarks 6 *(1) That we have to assume* $E_0 < 0$ *was pointed out to us by J. Voigt. Since we allow arbitrary* m *and* $d - m$, *a negative perturbation will not automatically create a negative spectrum.*

(2) In [17] we will present results that cover the negative spectrum for the model above using techniques from [52, 90, 91]. So far this works only for $m = 1$ *but the proofs allow quite arbitrary background perturbations.*

References

1. M.Aizenman and S.A.Molchanov: Localization at large disorder and at extreme energies: An elementary derivation. *Commun. Math. Phys.*, **157**, 245–278 (1993).
2. M.Aizenman and G.M.Graf: Localization bounds for an electron gas. *J. Phys. A: Math. Gen.* **31**, 6783 – 6806 (1998).
3. M.Aizenman, A. Elgart, S. Naboko, J.H. Schenker, G. Stolz: Moment Analysis for Localization in Random Schroedinger Operators, `eprint, arXiv, math-ph/0308023`
4. P.W. Anderson: Absence of diffusion in certain random lattices. *Phys. Rev.* **109**, 1492–1505 (1958).
5. Bentosela, F., Briet, Ph., Pastur, L.: On the spectral and wave propagation properties of the surface Maryland model. *J. Math. Phys.* **44** (2003), no. 1, 1–35.
6. J. Bourgain, M. Goldstein: On nonperturbative localization with quasiperiodic potential. *Ann. of Math.*, **152**, 835–879 (2000).
7. J. Bourgain, On the spectrum of lattice Schrödinger operators with deterministic potential. II. Dedicated to the memory of Tom Wolff. *J. Anal. Math.* **88** (2002), 221–254.
8. J. Bourgain, M. Goldstein and W. Schlag, Anderson localization for Schrödinger operators on $\mathbf{Z}^2$ with quasi-periodic potential. *Acta Math.* **188** (2002), no. 1, 41–86.
9. J. Bourgain, New results on the spectrum of lattice Schrödinger operators and applications. In *Mathematical results in quantum mechanics (Taxco, 2001)*, 27–38, Contemp. Math., 307, Amer. Math. Soc., Providence, RI, 2002.
10. J. Bourgain, On the spectrum of lattice Schrödinger operators with deterministic potential. Dedicated to the memory of Thomas H. Wolff. *J. Anal. Math.* **87** (2002), 37–75.
11. J. Bourgain, Exposants de Lyapounov pour opèrateurs de Schrödinger discrétes quasi-pèriodiques. *C. R. Math. Acad. Sci. Paris* **335** (2002), no. 6, 529–531.
12. J. Bourgain and S. Jitomirskaya, Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays. *J. Statist. Phys.* **108** (2002), no. 5-6, 1203–1218
13. J. Bourgain and S. Jitomirskaya, Absolutely continuous spectrum for 1D quasiperiodic operators. *Invent. Math.* **148** (2002), no. 3, 453–463.
14. J. Bourgain, On random Schrödinger operators on $\mathbb{Z}^2$. *Discrete Contin. Dyn. Syst.* **8** (2002), no. 1, 1–15.
15. J. Bourgain, M. Goldstein and W. Schlag, Anderson localization for Schrödinger operators on $\mathbb{Z}$ with potentials given by the skew-shift. *Comm. Math. Phys.* **220** (2001), no. 3, 583–621.
16. A. Boutet de Monvel, P. Stollmann: Dynamical localization for continuum random surface models, *Arch. Math.*, **80**, 87–97 (2003).
17. A. Boutet de Monvel, P. Stollmann and G. Stolz: Absence of (absolutely) continuous spectrum for certain nonstationary random models. In preparation
18. A.Boutet de Monvel, A. Surkova: Localisation des états de surface pour une classe d'opérateurs de Schrödinger discrets à potentiels de surface quasi-périodiques, *Helv. Phys. Acta* **71**(5), 459–490 (1998).

19. R.Carmona and J.Lacroix: *Spectral Theory of Random Schrödinger Operators.* Birkhäuser, Boston (1990).
20. R.Carmona and A.Klein and F.Martinelli: Anderson Localization for Bernoulli and Other Singular Potentials. *Commun. Math. Phys.*, **108**, 41–66 (1987).
21. Chahrour, A. and Sahbani, J.: On the spectral and scattering theory of the Schrödinger operator with surface potential. *Rev. Math. Phys.* **12** (2000), no. 4, 561–573
22. J.M.Combes and P.D.Hislop: Localization for some continuous, random Hamiltonians in d-dimensions, *J. Funct. Anal.*, **124**, 149–180 (1994).
23. J.M.Combes, and P.D.Hislop: Landau Hamiltonians with random potentials: Localization and density of states. *Commun. Math. Phys.*, **177**, 603–630 (1996).
24. D.Damanik and P.Stollmann: Multi-scale analysis implies strong dynamical localization. GAFA, *Geom. funct. anal.* **11**, 11–29 (2001).
25. R.del Rio, S.Jitomirskaya, Y.Last and B.Simon: What is localization? *Phys. Rev. Letters* **75**, 117–119 (1995).
26. R.del Rio, S.Jitomirskaya, Y.Last and B.Simon: Operators with singular continuous spectrum IV: Hausdorff dimensions, rank one perturbations and localization. *J. d'Analyse Math.* **69**, 153–200 (1996).
27. F.Delyon, Y.Levy and B.Souillard: Anderson localization for multidimensional systems at large disorder or low energy. *Commun. Math. Phys.* **100**, 463–470 (1985).
28. F.Delyon, Y.Levy and B.Souillard: Approach à la Borland to multidimensional localisation. *Phys. Rev. Lett.* **55**, 618–621 (1985).
29. T.C.Dorlas, N.Macris and J.V.Pulé: Localization in a single-band approximation to random Schrödinger operators in a magnetic field. *Helv. Phys. Acta*, **68**, 329–364 (1995).
30. H.von Dreifus: On the effects of randomness in ferromagnetic models and Schrödinger operators. NYU, Ph. D. Thesis (1987).
31. H.von Dreifus and A.Klein: A new proof of localization in the Anderson tight binding model. *Commun. Math. Phys.* **124**, 285–299 (1989).
32. Faris, W.G. Random waves and localization. *Notices Amer. Math. Soc.* **42** (1995), no. 8, 848–853.
33. A.Figotin and A.Klein: Localization of classical waves I: Acoustic waves. *Commun. Math. Phys.* **180**, 439–482 (1996).
34. W.Fischer, H.Leschke and P.Müller: Spectral localization by Gaussian random potentials in multi-dimensional continuous space. *J. Stat. Phys.* **101**, 935–985 (2000).
35. J.Fröhlich: Les Houches lecture. In: *Critical Phenomena, Random Systems, Gauge Theories,* K. Osterwalder and R. Stora, eds. North-Holland, 1984
36. J.Fröhlich, F.Martinelli, E.Scoppola and T.Spencer: Constructive Proof of Localization in the Anderson Tight Binding Model. *Commun. Math. Phys.* **101**, 21–46 (1985).
37. J.Fröhlich and T.Spencer: Absence of Diffusion in the Anderson Tight Binding Model for Large Disorder or Low Energy. *Commun. Math. Phys.* **88**, 151–184 (1983).
38. F.Germinet: Dynamical Localization II with an Application to the Almost Mathieu Operator. *J. Statist. Phys.* **98** (2000), no. 5-6, 1135–1148.
39. F.Germinet and S.de Bievre: Dynamical Localization for Discrete and Continuous Random Schrödinger operators. *Commun. Math. Phys.* **194**, 323–341 (1998).

40. F.Germinet and S.Jitomirskaya: Strong dynamical localization for the almost Mathieu model. *Rev. Math. Phys.* **13** (2001), no. 6, 755–765
41. F.Germinet and A.Klein: Bootstrap multiscale analysis and localization in random media. *Comm. Math. Phys.* **222** (2001), no. 2, 415–448.
42. F.Germinet and H.Klein: A characterization of the Anderson metal-insulator transport transition, submitted.
43. I.Ya.Goldsheidt, S.A.Molchanov and L.A.Pastur: Typical one-dimensional Schrödinger operator has pure point spectrum, *Funktsional. Anal. i Prilozhen.* **11** (1), 1–10 (1977); Engl. transl. in *Functional Anal. Appl.* **11** (1977).
44. Goldstein, M. Schlag, W.: Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions. *Ann. of Math.* (2) **154** (2001), no. 1, 155–203.
45. G.M.Graf: Anderson localization and the space-time characteristic of continuum states. *J. Stat. Phys.* **75**, 337–343 (1994).
46. V. Grinshpun: Localization for random potentials supported on a subspace, *Lett. Math. Phys.* **34**(2), 103–117 (1995).
47. H.Holden and F.Martinelli: On the absence of diffusion for a Schrödinger operator on $L^2(R^\nu)$ with a random potential. *Commun. Math. Phys.* **93**, 197–217 (1984).
48. D.Hundertmark: On the time-dependent approach to Anderson localization. *Math. Nachrichten*, **214**, 25–38 (2000).
49. D. Hundertmark, W. Kirsch: Spectral theory of sparse potentials, in "Stochastic processes, physics and geometry: new interplays, I (Leipzig, 1999)," Amer. Math. Soc., Providence, RI, 2000, pp. 213–238.
50. V. Jakšić, Y. Last: Corrugated surfaces and a.c. spectrum, *Rev. Math. Phys.* **12**(11), 1465–1503 (2000).
51. V. Jakšić, Y. Last: Spectral structure of Anderson type hamiltonians, *Inv. Math.*(2000) **141** no. 3, 561–577
52. I. McGillivray, P. Stollmann and G. Stolz: Absence of absolutely continuous spectra for multidimensional Schrödinger operators with high barriers. *Bull. London Math. Soc.* **27** (1995), no. 2, 162–168.
53. V. Jakšić, S. Molchanov: On the surface spectrum in dimension two, *Helv. Phys. Acta* **71**(6), 629–657 (1998).
54. V. Jakšić, S. Molchanov: On the spectrum of the surface Maryland model, *Lett. Math. Phys.* **45**(3), 189–193 (1998).
55. V. Jakšić, S. Molchanov: Localization of surface spectra, *Comm. Math. Phys.* **208**(1), 153–172 (1999).
56. V. Jakšić, S. Molchanov, L. Pastur: On the propagation properties of surface waves, in "Wave propagation in complex media (Minneapolis, MN, 1994)," IMA Math. Appl., Vol. **96**, Springer, New York, 1998, pp. 143–154.
57. S. Jitomirskaya: Almost everything about the almost Mathieu operator II. In: *Proc. of the XIth International Congress of Mathematical Physics,* Paris 1994, Int. Press, 366–372 (1995).
58. S. Jitomirskaya and Y. Last: Anderson Localization for the Almost Mathieu Operator III. Semi-Uniform Localization, Continuity of Gaps and Measure of the Spectrum. *Commun. Math. Phys.* **195**, 1–14 (1998).
59. S. Jitomirskaya: Metal-Insulator transition for the Almost Mathieu operator. *Annals of Math.*, **150**, 1159–1175 (1999).
60. S. John: The Localization of Light and Other Classical Waves in Disordered Media. *Comm. Cond. Mat. Phys.* **14**(4), 193–230 (1988).

61. A. Klein: Extended states in the Anderson model on the Bethe lattice. *Adv. Math.* **133** (1998), no. 1, 163–184.
62. W. Kirsch: Wegner estimates and Anderson localization for alloy-type potentials. *Math. Z.* **221**, 507–512 (1996).
63. W. Kirsch: Scattering theory for sparse random potentials. *Random Oper. Stochastic Equations* **10** (2002), no. 4, 329–334.
64. W. Kirsch, M. Krishna and J. Obermeit: Anderson model with decaying randomness: Mobility edge, *Math. Z.*, **235**, 421–433 (2000).
65. W. Kirsch, P. Stollmann, G. Stolz: Localization for random perturbations of periodic Schrödinger operators, *Random Oper. Stochastic Equations* **6**(3), 241–268 (1998).
66. W. Kirsch, P. Stollmann, G. Stolz: Anderson localization for random Schrödinger operators with long range Interactions, *Comm. Math. Phys.* **195**(3), 495–507 (1998).
67. F. Kleespies and P. Stollmann: Localization and Lifshitz tails for random quantum waveguides, *Rev. Math. Phys.* **12**, No. 10, 1345–1365 (2000)
68. F. Klopp: Localization for semiclassical Schrödinger operators II: The random displacement model. *Helv. Phys. Acta* **66**, 810–841 (1993).
69. F. Klopp: Localization for some continuous random Schrödinger operators. *Commun. Math. Phys.* **167**, 553–569 (1995).
70. M. Krishna: Anderson model with decaying randomness: Existence of extended states. *Proc. Indian Acad. Sci. Math. Sci.* **100**, no. 3, 285–294.
71. M. Krishna: Absolutely continuous spectrum for sparse potentials. *Proc. Indian Acad. Sci. Math. Sci.* **103** (1993), no. 3, 333–339.
72. Krishna, M., Sinha, K. B. Spectra of Anderson type models with decaying randomness. *Proc. Indian Acad. Sci. Math. Sci.* **111** (2001), no. 2, 179–201
73. Y. Last: Almost everything about the almost Mathieu operator I. In: *Proc. of the XIth International Congress of Mathematical Physics,* Paris 1994, Int. Press, 373–382 (1995).
74. F. Martinelli and H. Holden: On absence of diffusion near the bottom of the spectrum for a random Schrödinger operator on $L^2(\mathbb{R}^\nu)$. *Commun. Math. Phys.* **93**, 197–217 (1984).
75. F. Martinelli and E. Scoppola: Introduction to the mathematical theory of Anderson localization. *Rivista Nuovo Cimento* **10**, 10 (1987).
76. F. Martinelli and E. Scoppola: Remark on the absence of the absolutely continuous spectrum for d-dimensional Schrödinger operator with random potential for large disorder or low energy. *Commun. Math. Phys.* **97**, 465–471 (1985), Erratum ibid. **98**, 579 (1985).
77. Molchanov, S. Multiscattering on sparse bumps. Advances in differential equations and mathematical physics (Atlanta, GA, 1997), 157–181, Contemp. Math., 217, Amer. Math. Soc., Providence, RI, 1998.
78. Molchanov, S., Vainberg, B.: Multiscattering by sparse scatterers. Mathematical and numerical aspects of wave propagation (Santiago de Compostela, 2000), 518–522, SIAM, Philadelphia, PA, 2000.
79. N.F. Mott and W.D. Twose: The theory of impurity conduction. *Adv. Phys.* **10**, 107–163 (1961).
80. L.A. Pastur and A. Figotin: *Spectra of Random and Almost-Periodic Operators.* Springer-Verlag, Berlin, 1992
81. D. Pearson: Singular Continuous Measures in Scattering Theory. *Commun. Math. Phys.* **60**, 13–36 (1976).

82. Schlag, W., Shubin, C., Wolff, T.: Frequency concentration and location lengths for the Anderson model at small disorders. Dedicated to the memory of Tom Wolff. *J. Anal. Math.* 88 (2002), 173–220.
83. C. Shubin, R. Vakilian and T. Wolff: Some harmonic analysis questions suggested by Anderson–Bernoulli models. *GAFA* **8**, 932–964 (1998).
84. B. Simon and T. Wolff: Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians. *Comm. Pure Appl. Math.* **39**, 75–90 (1986).
85. R. Sims and G. Stolz: Localization in one dimensional random media: a scattering theoretic approach. *Comm. Math. Phys.* **213**, 575–597 (2000).
86. T. Spencer: The Schrödinger equation with a random potential – a mathematical review. In: *Critical Phenomena, Random Systems, Gauge Theories,* K. Osterwalder and R. Stora, eds. North Holland 1984
87. T. Spencer: Localization for random and quasi-periodic potentials. *J. Stat. Phys.* **51**, 1009 (1988).
88. P. Stollmann: *Caught by disorder – bound states in random media*, Progress in Math. Phys., vol. 20, Birkhäuser, Boston (2001).
89. P. Stollmann: Wegner estimates and localization for continuum Anderson models with some singular distributions, *Arch. Math.* **75**(4), 307–311 (2000).
90. P. Stollmann and G. Stolz: Singular spectrum for multidimensional operators with potential barriers. *J. Operator Theory* **32**, 91–109 (1994).
91. G. Stolz: Localization for Schrödinger operators with effective barriers. *J. Funct. Anal.* **146** (1997), no. 2, 416–429.
92. G. Stolz: Localization for random Schrödinger operators with Poisson potential. *Ann. Inst. Henri Poincaré* **63**, 297–314 (1995).
93. Thouless, D. J.: Introduction to disordered systems. Phènoménes critiques, systémes alèatoires, thèories de jauge, Part I, II (Les Houches, 1984), 681–722, North-Holland, Amsterdam, 1986.
94. I. Veselic: Localisation for random perturbations of periodic Schrödinger operators with regular Floquet eigenvalues. *Ann. Henri Poincare* **3** (2002), no. 2, 389–409.
95. W.M. Wang: Microlocalization, Percolation, and Anderson Localization for the Magnetic Schrödinger operator with a Random Potential. *J. Funct. Anal.* **146**, 1–26 (1997).
96. H. Zenk: Anderson localization for a multidimensional model including long range potentials and displacements. *Rev. Math. Phys.* **14** (2002), no. 3, 273–302