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RESEARCH ARTICLE

Schrödinger Operators with Singular Complex Potentials as Generators: Existence and Stability

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Abstract

For Hamiltonians H associated with Dirichlet forms and complex locally integrable V with negative real part in the Kato class we construct an extension of $H+V$ which generates a C_0-semigroup on L^p for $p \in [1,\infty)$ and prove the continuity of the semigroup with respect to V.

1. Introduction

In [2] Brezis and Kato constructed m-accretive realizations of Schrödinger operators with complex potentials generalizing earlier results by Nelson [10] and Kato [7]. More precisely they showed that the maximal operator in L^2 is maximal accretive; similar results were proven by Devinatz in [4] using probabilistic methods. We show how to define the corresponding generator of a strongly continuous semigroup in L^p for all $p \in [1,\infty)$ in a more general context: the Laplacian is replaced by a selfadjoint operator associated with a Dirichlet form, thus giving a much wider range of applications. We use the common approach of approximating the singular complex potential by bounded ones. This gives, at the same time, a Feynman-Kac representation for the semigroup. Two key ideas used in this procedure are of independent interest: the "local test" and the fact that L^1_{loc}-convergence of potentials implies strong resolvent convergence of the generators, provided a uniform Kato condition holds.

Let us now introduce the framework of regular Dirichlet forms very briefly, referring to [6] as a standard reference for the properties of Dirichlet forms used in the text. We work on a locally compact, second countable space X as in [6] endowed with a Radon measure m of full support. Let $\mathfrak{h}$ with domain D be a regular, closed Dirichlet form in $L^2 = L^2(X,m)$ whose associated selfadjoint operator is denoted by H_2. We write $-H_p$ for the generator of the contraction semigroup which is induced on L^p by $exp(-tH)$. As a standard example take $X = \Omega$ an open subset of $\mathbb{R}^d$ and $\mathfrak{h}[u,v] := \int \nabla u \nabla \bar{v} dx$ with domain $H^1_0(\Omega)$ for which the associated self adjoint operator is the Dirichlet Laplacian $-\Delta$ and the semigroup is the heat semigroup with absorption at the boundary, which is defined and contractive on all L^p-spaces.

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2. The results

First of all let us recall the appropriate equivalent of the Kato condition in our abstract setting. Consider

$$F_a(V) : L^1(X,m) \to \mathbb{C}, f \mapsto \int_X (H+a)^{-1} f(x)V(x)\, dm(x)\,,$$

$$\hat{K} := \{V \in L^1_{loc,+} : \exists\, a > 0 \text{ s.t. } F_a(V) \text{ is bounded}\}$$

and for such a potential V the Kato constant

$$c_a(V) := \|F_a(V)\|\,.$$

It was shown in [15] that for the case of the classical Dirichlet form associated with the Laplacian on $\mathbb{R}^d$ the potentials $V \in \hat{K}$ satisfying $c_a(V) = 0$ form the usual Kato class K_d. Therefore we call the $\hat{K}$ above the extended Kato class. For $V \in L^1_{loc}$ real valued with negative part $V^- \in \hat{K}$ such that $c_a(V) < 1$ for some a it was shown in [15] that $-H_p - V$ is a generator in L^p for all $p \in [1,\infty)$. This semigroup allows a very convenient probabilistic representation, which we use in the sequel. To this end recall that there is a Markov process with state space $X \cup \{\infty\}$ such that

$$e^{-tH_p} f(x) = \mathbb{E}^x [f \circ X_t]\,.$$

See [6] for this fundamental result in the theory of Dirichlet forms. Using the Trotter-Kato theorem it is quite easy to deduce the following representation for perturbed semigroups, for a potential V as above (see [11, 14] for details in the classical case):

$$e^{-t(H_p+V)} f(x) = \mathbb{E}^x [e^{-\int_0^t V\circ X_s ds} f \circ X_t] \tag{1}$$

This is the celebrated *Feynman-Kac formula* which also holds for more general potentials and can be used as a starting point for defining the corresponding semigroups by means of additive functionals. Generally speaking, there are two ways of defining semigroups associated with singular perturbations: Showing that the Feynman-Kac formula indeed defines a semigroup with the desired properties (see [1] and the references given there), or approximating the singular perturbation by well-behaved ones and proving that the limit exists in an appropriate sense (see [15, 14]). In the next theorem we show that these ways lead to the same result for the complex singular perturbation. From what we said above and the Trotter-Kato theorem it is easily deduced that for a *complex* V with bounded imaginary part $\Im V$ we have equation (1).

Theorem 1. *Let $p \in [1,\infty)$. Let $V \in L^1_{loc}(X,\mathbb{C})$ such that $c_a((\Re V)^-) < 1$ for some $a > 0$. Assume that V_n is a sequence of bounded functions converging to V in $L^1_{loc}(X,\mathbb{C})$ such that $(\Re V_n)^- \le (\Re V)^-$ for all $n \in \mathbb{N}$. Then $H_p + V_n$ converges in strong resolvent sense to an operator denoted by $H_p + V$ such that $-H_p - V$ generates a C_0-semigroup on L^p, which satisfies the Feynman-Kac formula* (1). ■

At this point we want to caution the reader that $H_p + V$ does not stand for the operator sum. It is quite easy to see that it is an extension of the latter. In the case when the imaginary part of V is zero, the generator constructed in Theorem 1 corresponds to the form sum and the convergence result can be deduced from the dominated convergence theorem in [16]. In the case of the Laplacian on $\mathbb{R}^d$, Devinatz [4] has constructed a maximal realization satisfying the Feynman-Kac formula. In this context our result can be considered as an approximation theorem for this generator.

In the proof of the theorem we need to introduce *Dirichlet boundary conditions*, by which we mean the following: For an open subset G of X we can restrict the form $\mathfrak{h}$ to $D \cap C_c(G)$ and obtain a closable form whose domain is dense in $L^2(G)$, the closure $\mathfrak{h}_G$ again being a regular Dirichlet form. We denote the associated generator in $L^p(G)$ by $(-H_p)_G$. Using the hitting time τ_B for a subset B of X it follows (see [6], § 4.4) that

$$e^{-t(H_p+V)_G} f(x) = \mathbb{E}^x[f \circ X_t \cdot e^{-\int_0^t V \circ X_s ds} \cdot \mathbb{1}_{\tau_{G^c}>t}] . \tag{2}$$

We extend these semigroups to $L^p(X)$ in the obvious manner.

Proof of Theorem 1. We write $U = \Re V$, $W = \Im V$ for the real and imaginary part of V. From (1) for bounded W it is clear that

$$|e^{-t(H_p+U_n+iW_n)} f| \le e^{-t(H_p-U^-)}|f| . \tag{3}$$

By the Trotter-Kato-Neveu theorem we have to show that for fixed $f \in L^p$

$$\sup_{t\in[0,1]} \|e^{-t(H_p+U_n+iW_n)} f - e^{-t(H_p+U_m+iW_m)} f\|_p \to 0 \text{ for } n, m \to \infty . \tag{4}$$

By the uniform bound in (3) it suffices to check (4) for $f \in L^p \cap L^\infty$. In order to use the L^1_{loc}-convergence we introduce Dirichlet boundary conditions. Abbreviate $T_n := H_p + U_n + iW_n$ and write $\mathfrak{G}$ for the relatively compact open subsets of X, ordered by inclusion. For $G \in \mathfrak{G}$ we have

$$e^{-tT_n} - e^{-tT_m} = e^{-tT_n} - e^{-t(T_n)_G} + e^{-t(T_n)_G} - e^{-t(T_m)_G} + e^{-t(T_m)_G} - e^{-tT_m} . \tag{5}$$

We first show that

$$\sup_{t\in[0,1]} \|e^{-tT_m} f - e^{-t(T_m)_G} f\|_p \to 0 \tag{6}$$

for $G \nearrow X$ uniformly in m. To this end recall that

$$(H_p - U^-)_G \xrightarrow{srs} H_p - U^- \text{ as } G \nearrow X . \tag{7}$$

In fact, for $p = 2$ the convergence (7) follows from monotone convergence for forms (see [13] for details). For arbitrary p it follows from the Feynman-Kac formula that the semigroups are monotonically increasing as $G \nearrow X$ and hence converge strongly. The limit is as asserted, since it coincides with $e^{-t(H_p-U^-)}$ on $L^p \cap L^2$.

To prove (6) we use the Feynman-Kac formula (2) to calculate

$$\begin{aligned} |\left(e^{-tT_n} - e^{-t(T_n)_G}\right) f(x)| &= |\mathbb{E}^x[e^{-\int_0^t V_n \circ X_s ds} f \circ X_t, \tau_{G^c} \le t]| && (8)\\ &\le \mathbb{E}^x[e^{\int_0^t U^- \circ X_s ds} |f| \circ X_t, \tau_{G^c} \le t] && (9)\\ &= \left(e^{-t(H_p - U^-)} - e^{-t(H_p - U^-)_G}\right) |f|(x). && (10) \end{aligned}$$

Together with (7) this proves (6). Note that (6) enables us to estimate the first and last term on the RHS of (5) by choosing G large enough. We now show how to estimate the middle term for large n, m using the L^1_{loc}-convergence of V_n. By Duhamel's formula

$$\begin{aligned} \left(e^{-t(T_n)_G} - e^{-t(T_m)_G}\right) f &= \int_0^t e^{-(t-s)(T_n)_G} (T_m - T_n) e^{-s(T_m)_G} f ds && (11)\\ &= \int_0^t e^{-(t-s)(T_n)_G} \chi_G (V_m - V_n) e^{-s(T_m)_G} f ds && (12) \end{aligned}$$

so that we have the estimate

$$\| \left(e^{-t(T_n)_G} - e^{-t(T_m)_G}\right) f \|_1 \le c \|\chi_G (V_m - V_n)\|_1 \|f\|_\infty \tag{13}$$

where we have used the fact, that the semigroups are bounded on L^1 and L^∞ independently of n and G. By assumption on the V_n the RHS of (13) converges to zero for n, m tending to infinity. To estimate the L^p-norm of

$$\left(e^{-t(T_n)_G} - e^{-t(T_m)_G}\right) f = g$$

we only have to note that

$$\|g\|_p \le \|g\|_1^{\frac{1}{p}} \|g\|_\infty^{\frac{p-1}{p}} . \tag{14}$$

To finish the proof of (4) which in turn gives the convergence asserted in the Theorem, fix $\varepsilon > 0$. According to (6) we find a $G \in \mathfrak{G}$ such that

$$\| \left(e^{-tT_n} - e^{-t(T_n)_G}\right) f \|_p \le \varepsilon$$

for all $n \in \mathbb{N}$. With this G fixed use (13) and (14) to find $N \in \mathbb{N}$ such that

$$\| \left(e^{-t(T_n)_G} - e^{-t(T_m)_G}\right) f \|_p \le \varepsilon \quad (n, m \ge N).$$

Using (5) gives the desired result.

Recall that we know already that

$$e^{-t(H_p + V_n)} f(x) = \mathbb{E}_x[e^{-\int_0^t V_n \circ X_s ds} f \circ X_t]$$

and that the LHS converges in L^p to $e^{-t(H_p+V)} f$. Take V_n to be a truncation of V. Once we can assure that for a.e. x

$$\int_0^t |W \circ X_s| ds < \infty \quad \text{for} \quad \mathbb{P}^x - a.e.\omega \tag{15}$$

the Feynman-Kac formula follows from Lebesgue's theorem. However, (15) follows in the same way as the implication $(i) \Rightarrow (ii)$ of Proposition 6.1(d) in [15]. ■

We are now heading towards somewhat strengthened versions of the two basic steps used in the above proof which constitute results of independent interest. To simplify their statement we say that a sequence U_n in $L^1_{loc,+}$ *satisfies a uniform Kato condition*, if there is a $\gamma < 1$ and an $a > 0$ such that $c_a(U_n) \le \gamma$ for all n.

Lemma 2. **(Local test)** *Let $p \in [1, \infty)$. Let $V_n \in L^1_{loc}(X, \mathbb{C})$ and assume that the $(\Re V_n)^-$ satisfy a uniform Kato condition. If $(H_p + V_n)_G \xrightarrow{srs} (H_p + V)_G$ for all relatively compact $G \subset X$, then*

$$H_p + V_n \xrightarrow{srs} H_p + V .$$

Proof. Write $U_n = \Re V_n$, $T_n = H_p + V_n$. Using the analogue of (5) from the above proof

$$e^{-tT_n} - e^{-tT} = e^{-tT_n} - e^{-t(T_n)_G} + e^{-t(T_n)_G} - e^{-t(T)_G} + e^{-t(T)_G} - e^{-tT}, \tag{16}$$

it remains to check the equivalent of (6), since the convergence of the middle term in the RHS of (16) is just our assumption. We may restrict ourselves to considering $f \in L^1 \cap L^\infty$ by uniform boundedness, which in turn follows from the uniform Kato condition. To prove (6) we want to estimate

$$|\left(e^{-t(H_p+V_n)} - e^{-t(H_p+V_n)_G}\right) f(x)| \le \mathbb{E}^x[e^{\int_0^t U_n^- \circ X_s ds} |f| \circ X_t, \tau_{G^c} \le t] \tag{17}$$

in L^p-norm, uniformly in n. Choose $\alpha > 1$ such that $\alpha\gamma < 1$, where $\gamma < 1$ is chosen according to the uniform Kato condition. With this choice αU_n still satisfy a uniform Kato condition, and therefore the semigroups $e^{-t(H_p - \alpha U_n^-)}$ admit a uniform bound as operators from L^∞ to L^∞. Denote the dual exponent of α by β and apply Hölders inequality to the RHS of (17) which gives

$$\ldots \le \left(\mathbb{E}^x[e^{\int_0^t \alpha U_n^- \circ X_s ds} |f| \circ X_t]\right)^{\frac{1}{\alpha}} \left(\mathbb{E}^x[|f| \circ X_t, \tau_{G^c} \le t]\right)^{\frac{1}{\beta}} . \tag{18}$$

Note that we omitted the negative and imaginary exponentials which can be estimated by 1 in absolute value. The first term on the RHS of (18) is uniformly bounded in x and n by what we said above. Using the trick involving (14) as in the proof of Theorem 1 it suffices to show that the second factor on the right hand side of (18) converges to zero in L^1. This follows easily from (7), since

$$(\mathbb{E}^x[|f| \circ X_t, \tau_{G^c} \le t]) = \left(e^{-tH} - e^{-t(H)_G}\right) |f(x)| .$$

The same reasoning as in the proof of Theorem 1 yields the conclusion. ∎

A particularly nice consequence of the "local test lemma" is the following

Theorem 3. *Let $V_n, V \in L^1_{loc}(X, \mathbb{C})$ satisfy a uniform Kato condition. Then*

$$V_n \to V \text{ in } L^1_{loc}(X, \mathbb{C}) \Longrightarrow H_p + V_n \xrightarrow{srs} H_p + V .$$ ∎

Note that several special cases of Theorem 3 in the Hilbert space setting and for real-valued potentials follow from abstract considerations:

- if $V_n \geq V$ one can apply [5], Proposition 7.9 with $C_c(X) \cap D$ as a common form core of the $H_2 + V_n$.
- if there is a common core for the operators $H_2 + V_n$ consisting of C_c-functions, one can apply [5], Proposition 7.6.

Moreover, in the case of the Laplacian and for non-negative potentials V_n the assertion of Theorem 3 follows from Lemma 3.1 in [3].

Proof of Theorem 3. With the help of the local test lemma we are almost done. If it were not for domain questions we could use the same arguments involving Duhamel's formula as in the proof of Theorem 1 leading to (13). This minor problem is circumvented as follows: Cut off V_n and V and denote the result by $V_n^{(k)}$ and $V^{(k)}$. As in the proof of Theorem 1 we find

$$\|e^{-t(H_p+V_n^{(k)})_G}f - e^{-t(H_p+V^{(k)})_G}f\|_1 \leq c\|\chi_G(V_n - V)\|_1\|f\|_\infty$$

using that the difference of the cut-off functions is dominated by the difference of the functions themselves. From this the assertion follows as in the proof of Theorem 1. ■

We finish by several remarks:

- If the negative real part of the potential is in the Kato class, the Feynman-Kac formula and [12], Theorem B.7.1 immediately imply a Gaussian estimate for the semigroup constructed in Theorem 1.
- For some of the above results it is clear and for some it is quite plausible that one can treat measures instead of potentials.
- Another direction of research concerns dominated semigroups which include as special case Hamiltonians with magnetic field; see [8].
- The results above remain valid for quasi-regular forms on topological spaces. This is an easy consequence of the transfer method described in [9], Chapter VI. As a consequence, our results may also be applied in the infinite dimensional setting; see [9] for more information.

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