Mathematical Physics, Analysis and Geometry **2:** 279–289, 1999.

Lifshitz Asymptotics via Linear Coupling of Disorder

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(Received: 22 December 1998)

Abstract. We present a simple method for proving Lifshitz asymptotics for random Schrödinger operators and apply it to the Anderson and Poisson model.

Mathematics Subject Classifications (1991): 82B44, 47B80, 60H25.

Key words: Lifshitz tails, integrated density of states.

1. Introduction

In this note, we present a very simple approach to proving Lifshitz asymptotics for random operators and apply it to Schrödinger operators with Anderson and Poisson potentials. Let us first briefly recall what Lifshitz asymptotics is about: consider $H_0 = -\Delta$ on $\mathbf{R}^d$ and a random potential term V_ω. The operator $H_0 + V_\omega$ is to be thought of as the effective Hamiltonian of fixed a realization of a random solid. If we assume that the V_ω are bounded below uniformly in ω, $V_\omega \geqslant 0$ say, the restriction $(H_0 + V_\omega)_\Lambda$ of $H_0 + V_\omega$ to an open cube Λ with Neumann boundary conditions has compact resolvent. Therefore, the spectral counting function

$$n(E, (H_0 + V_\omega)_\Lambda) := \mathrm{tr}\big[\chi_{[0,E]}(H_0 + V_\omega)_\Lambda\big]$$

which gives the number of eigenvalues below E, counted with multiplicity, is finite. This function bears important information about the random potential under consideration. In fact, its limit as Λ exhausts the whole space has an asymptotic behaviour characteristic of disorder. To see that, let us first recall that by the celebrated Weyl asymptotic formula, in absence of disorder, i.e. for $V_\omega = 0$, we have

$$n(E, (H_0)_\Lambda) = C_d E^{d/2}(|\Lambda| + o(|\Lambda|)) \quad E \geqslant 0$$

(where we use $|\Lambda|$ for the volume of the cube) which means that

$$\lim_{\Lambda \nearrow \mathbf{R}^d} \frac{1}{|\Lambda|} n(E, (H_0)_\Lambda) =: N_0(E) = C_d E^{d/2}.$$

A submultiplicative ergodic theorem implies that the respective limit

$$N(E) := \lim_{\Lambda \nearrow \mathbf{R}^d} \frac{1}{|\Lambda|} n(E, (H_0 + V_\omega)_\Lambda)$$

exists for a.e. ω under some mild and very natural ergodicity assumption on V_ω (see [4]). Moreover, this limit is independent of the choice of ω outside some set of measure zero and equals

$$N(E) = \inf_\Lambda \frac{1}{|\Lambda|} \mathbf{E}\{n(E, (H_0 + V_\omega)_\Lambda)\}.$$

Now the right-hand side above is readily interpreted as the expected number of energy levels per unit volume below E. Clearly, this quantity is of importance both mathematically and from the physicists point of view. In his landmark work, Lifshitz predicted an asymptotic behaviour of $N(E)$ which differs drastically from the dimension-dependent power law decay of $N_0(E)$. Namely, he claimed that for nontrivial V_ω which obeys some spatial independence (this will be explained below),

$$N(E) \sim \exp(-\gamma E^{-d/2}) \quad \text{as } E \searrow 0.$$

(Here we assume that 0 is the inf of the spectrum of $H_0 + V_\omega$ a.e. for notational convenience.) His reasoning is as follows:

first of all $n(E, (H_0 + V_\omega)_\Lambda) \leqslant n(E, (H_0)_\Lambda)$ as the nonnegative potential term shifts the eigenvalues to the right. Therefore, with $E_1(\ldots)$ denoting the bottom eigenvalue of the operator in question, we have

$$\begin{aligned} N(E) &\leqslant \frac{1}{|\Lambda|} \int n(E, (H_0 + V_\omega)_\Lambda) \chi_{\{E_1((H_0+V_\omega)_\Lambda) \leqslant E\}} \,\mathrm{d}\mathbf{P}(\omega) \\ &\leqslant CE^{d/2} \mathbf{P}\{\omega : E_1((H_0 + V_\omega)_\Lambda) \leqslant E\}. \end{aligned}$$

Now we want to estimate the probability of having small eigenvalues. If ϕ is a normalized eigenfunction of $(H_0+V_\omega)_\Lambda$ with eigenvalue $E \sim 0$ it must be localized to a region where $V_\omega = 0$, as $E = (-\Delta\phi|\phi) + (V_\omega\phi|\phi)$. As the kinetic energy of a function localized to a set of diameter R is at least of order R^{-2}, there must be a ball of radius $E^{-1/2}$ on which V_ω vanishes essentially. The spatial independence referred to above means that we assume that the restrictions of V_ω to disjoint subsets are independent of each other. In that case, the probability that V_ω vanishes on a ball of radius R goes to zero exponentially in the volume R^d of the ball as R goes to infinity. Inserting the length $R = E^{-1/2}$ found above, we get that

$$\mathbf{P}\{\omega : E_1((H_0 + V_\omega)_\Lambda) \leqslant E\} \leqslant \text{const} \exp(-\gamma E^{-d/2}).$$

Of course, this is not a mathematically rigorous proof. The point which certainly has to be made precise is the existence of a large enough region where $V_\omega = 0$.

For due to tunneling effects, ϕ might still live on parts of space where $V_\omega \gg 0$. Of course, V_ω may not increase the potential energy too much.

Our way around that difficulty goes as follows: Let $H(\omega) = H_0 + V_\omega$ be a random Schrödinger operator with $V_\omega \geqslant 0$. By what we said above, Lifshitz behaviour for the integrated density of states can be deduced from an estimate of the following form, where $H_\Lambda(\omega)$ denotes $H_0 + V_\omega$ in $L^2(\Lambda)$, with Neumann boundary condition (b.c.), $\Lambda = \Lambda_l(0)$ an open cube with sidelength l in $\mathbf{R}^d$ and $E_1(\cdot)$ the first eigenvalue.

$$\mathbf{P}\{E_1(H_\Lambda(\omega)) \leqslant E_1(H_0) + bl^{-2}\} \leqslant 4\exp(-l^d\gamma). \tag{1.1}$$

This latter inequality states in precise terms that it is very unlikely to find really small eigenvalues of $H_\Lambda(\omega)$. In order to prove such an inequality, one has to overcome the following main problem: for a simple-minded lower bound on the first eigenvalue one would need a uniform lower bound on the perturbation V_ω. Such uniform lower bounds only hold with small probability. On the other hand, what one knows by standard probabilistic tools are lower bounds for the mean of V_ω for typical ω. So what we need is a relation between the mean of V_ω and the first eigenvalue. In our approach we choose a derivative related with $E_1(\omega)$ as such a link. As you will see, that provides a conceptually simple proof of inequality (1.1).

More precisely, let us consider

$$H_\Lambda(\omega, t) = H_0 + tV_\omega \quad \text{on } L^2(\Lambda) \text{ with Neumann b.c.}$$

Then the first eigenvalue $E_1(\omega, t)$ of this operator behaves like

$$E_1(H_\Lambda(\omega)) \geqslant E_1(\omega, t) \approx E_1(H_0) + tE_1'(\omega, 0) \quad \text{for small } t, \tag{1.2}$$

with

$$E_1'(\omega, 0) = (V_\omega\phi_0|\phi_0),$$

where ϕ_0 is the normalized ground state of H_0.

Now, let us take a closer look at (1.2). We have to find out just how large we may take t. Analytic perturbation theory suggests $t \approx l^{-2}$, as this is the distance of $E_1(H_0)$ to $E_2(H_0)$ for typical Schrödinger operators. With this choice and b small enough, from $E_1(\omega) \leqslant E_1(H_0) + bl^{-2}$ it follows that $E_1'(\omega, 0)$ has to be small. But, in the Anderson case,

$$E_1'(\omega, 0) = (V_\omega\phi_0|\phi_0) = \left(\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\omega(i)\right)\text{const.} \tag{1.3}$$

is essentially the mean of a sum of $|\Lambda| = l^d$ i.i.d. variables. The probability that this mean differs from the expectation by some fixed constant goes to zero exponentially in the number of independent copies, i.e. $\exp(-\gamma|\Lambda|)$ which is exactly the decay we need. For Poisson potentials, we provide a rather elementary large deviation estimate, reducing it to something like (1.3). The above considerations constitute already the main idea of our method which we call ‘linear coupling of disorder’ for obvious reasons. The rest of the paper is devoted to carrying out the details needed to turn the above heuristics into a rigorous proof.

In principle, this requires three steps: firstly, the standard procedure to deduce Lifshitz tails from (1.1) above. To prove (1.1) we need, of course, large deviation results for $(V_\omega\phi_0|\phi_0)$. As a last ingredient, a remainder estimate for the first-order approximation to $E_1(\omega, t)$ is needed, which, again, is quite standard.

As all three steps are rather straightforward, the expert reader could stop at this point. (But please, read on.) To appreciate the simplicity of our approach, the reader should compare it with the proofs available so far; see [1, 10], where detailed references to the literature can be found. Usually, there is some tricky part when it comes down to showing the main point: small $E_1(\omega)$ come from large deviation from the typical ω.

For the Anderson model, Temple’s or Thirring’s inequality is used at that point.

The Poisson model was treated using the celebrated work of Donsker and Varadhan [2] on the asymptotics of the Wiener sausage. A beautiful introduction to this circle of ideas can be found in [8].

In our approach we single out a very convenient link, namely $E_1'(\omega, 0)$.

This enabled us to apply our method to a random quantum waveguide model [5], quite reminiscent of the Anderson model. For this model, however, determining the derivative $E_1'(\omega, 0)$ is harder (and more interesting), and the methods using Temple’s or Thirring’s inequality fail. Despite of all the advertisement for our method, we should stress that, so far, we haven’t achieved ‘the right constant’ in the Poisson case.

2. Lifshitz Tails for Anderson and Poisson Models

Let us first fix the notation and the basic assumptions. Throughout the following, $\Lambda = \Lambda_l(x)$ denotes an open cube of sidelength l centered at x. Moreover, p is an exponent such that $p \geqslant d/2$ for $d \geqslant 4$ and $p = 2$ for $d \leqslant 3$, and $f \in L^p(\mathbf{R}^d)$, $f \geqslant 0$ and supp $f \subset \Lambda_1(0)$. We will consider the following random potentials:

(A) THE ANDERSON MODEL. Let $S > 0$ and μ be a probability measure on $[0, S]$ with $0 \in$ supp μ and $M = \int x \,\mathrm{d}\mu > 0$, i.e. μ is not just δ. Denote its variance by $v = \int x^2 \,\mathrm{d}\mu$. Let $\Omega = [0, S]^{\mathbf{Z}^d}$, $\mathbf{P} = \mu^{\mathbf{Z}^d}$ and define $V_\omega^A(x) = \sum_{i\in\mathbf{Z}^d} \omega(i) f(x - i)$.

(P) THE POISSON MODEL. Let Ω denote the point measures on $\mathbf{R}^d$ and $\mathbf{P}$ the Poisson measure on Ω, which is concentrated on $\{\sum_i \delta_{X_i};$ for some discrete sequence $(X_i)\}$. Define

$$V_\omega^P(x) = f * \omega(x) = \sum_i f(x - X_i(\omega)).$$

These two random measures correspond to quite different types of disorder: while V_ω^A is used to model solids with defects, with some periodicity still present in the random potential, V_ω^P describes an amorphous medium in which the nuclei (at $X_i(\omega)$) are distributed erratically in space.

As many of the following considerations apply to both the Anderson and the Poisson model, we will often write V_ω to denote either of them, and use a superscripts A, P to distinguish between them. With this convention, denote $H(\omega) = -\Delta + V_\omega$ in $L^2(\mathbf{R}^d)$ and by $H_\Lambda(\omega)$ the restriction of this operator to $L^2(\Lambda)$ with Neumann boundary conditions.

The *integrated density of states* for $H(\omega)$ is given by

$$\begin{aligned} N(t) &= \inf_\Lambda \frac{1}{|\Lambda|}\mathbf{E}\left\{\mathrm{tr}\left[\chi_{[0,t]}(H_\Lambda(\omega))\right]\right\} \\ &= \lim_{\Lambda\nearrow\mathbf{R}^d} \frac{1}{|\Lambda|}\mathrm{tr}\left[\chi_{[0,t]}(H_\Lambda(\omega))\right] \quad \mathbf{P}\text{-a.s.} \end{aligned}$$

We refer to [1, 4, 10] for a discussion of this very important quantity. Note that the trace appearing above simply counts the number of eigenvalues below t, so that $N(t)$ is interpreted as the number of energy levels per unit volume of $H(\omega)$. The fact that

$$\frac{1}{|\Lambda|}\mathbf{E}\left\{\mathrm{tr}\left[\chi_{[0,t]}(H_\Lambda(\omega))\right]\right\}$$

decreases as Λ increases is due to our choice of the boundary condition. Since we are working with Neumann boundary conditions the spectral counting function is subadditive on disjoint open sets. It is also possible to work with Dirichlet boundary conditions instead, in which case the spectral counting function is superadditive. For reasonably well defined V_ω the limits are in fact the same. See [4] for a thorough discussion of this point.

The estimate given in the next theorem is usually referred to as Lifshitz tail behaviour and is one of the central topics of disordered systems ever since Lifshitz' seminal contribution [9]:

THEOREM 2.1. *The integrated density of states $N(t)$ satisfies*

$$\limsup_{t\searrow 0} \frac{\log N(t)}{t^{-d/2}} \leqslant -\gamma \tag{2.1}$$

for some $\gamma > 0$. For the Anderson model, $\gamma = \gamma_A$ depends upon f, M, S and for the Poisson model, $\gamma = \gamma_P$ depends upon f.

The inequality (2.1) will easily follow from the next result, as we will show at the end of this section. In the supplement given there one can see the dependence quite clearly. Note that we write $E_1(\cdot)$ for the first eigenvalue of the operator in question.

PROPOSITION 2.2. (A) *There exist universal constants $c, K > 0$ such that with $c_A = c \cdot S \cdot \|f\|_p/\|f\|_1$, for every*

$$b \leqslant \min\left\{\frac{\pi^2}{4}, \frac{M^2}{c_A^2}\right\},$$

we have

$$\begin{aligned}&\mathbf{P}\{E_1(H_\Lambda^A(\omega)) \leqslant b \cdot l^{-2}\}\\ &\quad\leqslant K \exp\left[-l^d \frac{M - c_A\sqrt{b}}{K \cdot S} \log\left(1 + \frac{S \cdot (M - c_A\sqrt{b})}{v}\right)\right].\end{aligned} \tag{2.2}$$

(P) *There exists a universal constant* c' *such that, for* $M = (e-1)/e = v$,

$$c_P = c' \cdot \frac{\|f\|_p}{\|f\|_1},$$

and every

$$b \leqslant \min\left\{\frac{\pi^2}{4}, \frac{M^2}{c_P^2}\right\},$$

we have

$$\begin{aligned}&\mathbf{P}\{E_1(H_\Lambda^P(\omega)) \leqslant b \cdot l^{-2}\}\\ &\quad\leqslant K \exp\left[-(l-2)^d \frac{M - c_P\frac{l^d}{(l-2)^d}\sqrt{b}}{K} \log\left(1 + \frac{M - c_P\frac{l^d}{(l-2)^d}\sqrt{b}}{v}\right)\right].\end{aligned} \tag{2.3}$$

Let us first single out an important step in the proof of Proposition 2.2. To this end, fix $V \in L^p_{\mathrm{loc,unif}}(\Lambda)$, let $H(t) = -\Delta + t \cdot V$ in $L^2(\Lambda)$ with Neumann b.c. and denote its first eigenvalue by $E_1(t)$. Note that $E_1(0) = 0$.

LEMMA 2.3. *There exists a universal constant* C *such that for* $\tau = C \cdot \|V\|^{-1}_{p,\mathrm{loc,unif}}$ *and* $0 \leqslant t \leqslant \tau l^{-2}$ *we have*

$$|E_1(t) - tE_1'(0)| \leqslant \frac{\pi^2}{4\tau^2} \cdot l^2 \cdot t^2.$$

Proof. To estimate the remainder term in the Taylor expansion we want to use [3], formula II(3.6). The isolation distance ϑ defined as the distance of $E_1(0)$ to the rest of the spectrum of $H(0)$ is given by $\vartheta = \pi^2/l^2$.

As Γ we choose a circle around E_0 with radius $\vartheta/2$. We need an estimate for the r_0 appearing in [3], II(3.3), which means that we have to consider

$$r(\zeta) = \|V(H(0) - \zeta)^{-1}\|^{-1} \quad \text{for } \zeta \in \Gamma.$$

As $(H(0) - \zeta)^{-1}$ maps L^2 to the Sobolev space $W^{2,2}$ with norm controlled by $\mathrm{dist}(\zeta, \sigma(H(0))) \geqslant \vartheta/2$, we have by Sobolev's inequality that

$$\|V(H(0) - \zeta)^{-1}\| \leqslant c' \cdot \frac{2}{\vartheta} \cdot \|V\|_{p,\mathrm{loc,unif}},$$

so that

$$r_0 = \min_{\zeta \in \Gamma} r(\zeta) \geqslant C'' \frac{\vartheta}{\|V\|_{p,\mathrm{loc,unif}}},$$

and an appeal to [3], estimate II(3.6) finishes the proof. □

We are now ready to present the proof of Proposition 2.2.

Proof of Proposition 2.2. Denote $H(\omega, t) = -\Delta + t \cdot V_\omega$ in $L^2(\Lambda)$ with Neumann b.c. and denote by $E_1(\omega, t)$ its first eigenvalue. From the remainder estimate in Lemma 2.3 we have

$$|E_1(\omega, t) - t \cdot E_1'(\omega, 0)| \leqslant \frac{\pi^2}{4\tau^2} \cdot l^2 \cdot t^2 \quad (0 \leqslant t \leqslant \tau l^{-2}),$$

where

$$\tau = C \cdot \frac{1}{\|V_\omega\|_{p,\mathrm{loc,unif}}} \geqslant C \cdot \frac{1}{S \cdot \|f\|_p}$$

is bounded away from 0, independently of ω. Assume that $E_1(\omega) \leqslant b \cdot l^{-2}$ for $b \leqslant \pi^2/4$. Then the above inequality yields

$$t \cdot E_1'(\omega, 0) \leqslant \frac{\pi^2}{4\tau^2} \cdot l^2 \cdot t^2 + b \cdot l^{-2} \quad \text{for all } 0 \leqslant t \leqslant \tau l^{-2}.$$

Inserting $t = s\tau l^{-2}$ we get

$$E_1'(\omega, 0) \leqslant \frac{\pi^2 s}{4\tau} + \frac{b}{\tau s} \quad \text{for all } 0 \leqslant s \leqslant 1.$$

Optimizing w.r.t. s we get $s = \frac{2}{\pi}\sqrt{b}$ and

$$E_1'(\omega, 0) \leqslant \frac{\pi}{\tau}\sqrt{b},$$

which implies

$$(V_\omega \phi_0 | \phi_0) \leqslant \frac{\pi}{\tau}\sqrt{b},$$

where

$$\phi_0 = \frac{1}{|\Lambda|^{1/2}} \chi_\Lambda.$$

We now specialize to the case (A):

Then

$$(V_\omega \phi_0 | \phi_0) = \|f\|_1 \left(\frac{1}{|\Lambda|} \sum_{i \in \Lambda} \omega(i) \right).$$

Define

$$c_A = \frac{\pi \cdot S \cdot \|f\|_p}{C \cdot \|f\|_1},$$

so that

$$\frac{\pi}{\|f\|_1 \cdot \tau} \leqslant c_A$$

Now, if $0 \leqslant b \leqslant M^2/c_A^2$ it follows that

$$\begin{aligned}\mathbf{P}\left\{E_1(\omega, 1) \leqslant b \cdot l^{-2}\right\} &\leqslant \mathbf{P}\left\{\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\omega(i) \leqslant c_A\sqrt{b}\right\}\\ &\leqslant \mathbf{P}\left\{\left|\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\omega(i) - M\right| \geqslant M - c_A\sqrt{b}\right\}.\end{aligned}$$

By [11], Thm. 1.4, this latter probability can be estimated by

$$K \exp\left[-l^d \frac{M - c_A\sqrt{b}}{K \cdot S} \log\left(1 + \frac{S \cdot (M - c_A\sqrt{b})}{v}\right)\right],$$

the assertion.

To treat case (P), we want to use a similar calculation and subdivide Λ into the unit cubes $\Lambda_1(m)$, where m runs through $\Lambda_l(0) \cap \mathbf{Z}^d$.

We introduce the random variables

$$Y_m(\omega) := \begin{cases} 1, & \text{if there is an } X_i(\omega) \in \Lambda_1(m),\\ 0, & \text{else,}\end{cases}$$

for $m \in \Lambda_{l-2}(0)$. By the properties of the Poisson process, these r.v. are i.i.d. with expectation and variance equal to $M = (e-1)/e = v$. We define an auxiliary random potential by

$$W_\omega(x) := \sum_{m\in\Lambda_{l-2}(0)} Y_m(\omega) \cdot f(x - X_{i(m)}(\omega)),$$

where $X_{i(m)}(\omega)$ is one of the Poisson points in $\Lambda_1(m)$, if $Y_m(\omega) = 1$, and zero else. Clearly,

$$W_\omega(x) \leqslant V_\omega(x) \quad \text{for all } \omega \in \Omega, x \in \mathbf{R}^d,$$

and, hence,

$$\mathbf{P}\{E_1(H_\Lambda(\omega)) \leqslant b \cdot l^{-2}\} \leqslant \mathbf{P}\{E_1(-\Delta + W_\omega) \leqslant b \cdot l^{-2}\}.$$

Now the latter probability can be estimated by the same calculation as in the case (A) above, since

$$(W_\omega \phi_0|\phi_0) \geqslant \|f\|_1 \frac{(l-2)^d}{l^d}\left(\frac{1}{(l-2)^d}\sum_m Y_m\right),$$

and

$$\|W_\omega\|_{p,\mathrm{loc,unif}} \leqslant 3^d \cdot \|f\|_p,$$

where for the last inequality we counted the neighbouring boxes and thus the maximal number of nontrivial terms in the sum which defines W_ω. Thus, we are again left with applying a large deviation result for sums of i.i.d. variables. We get, with

$$c_P = \frac{\pi}{C} \cdot \frac{\|f\|_p}{\|f\|_1},$$

that

$$\begin{aligned}
&\mathbf{P}\big\{E_1(\omega, 1) \leqslant b \cdot l^{-2}\big\} \\
&\quad \leqslant \mathbf{P}\left\{\frac{1}{(l-2)^d} \sum_{m \in \Lambda_{l-2}} Y_m \leqslant c_P \frac{l^d}{(l-2)^d}\sqrt{b}\right\} \\
&\quad \leqslant \mathbf{P}\left\{\left|\frac{1}{|\Lambda|} \sum_{m \in \Lambda_{l-2}} Y_m - M\right| \geqslant M - c_P \frac{l^d}{(l-2)^d}\sqrt{b}\right\} \\
&\quad \leqslant K \exp\left[-(l-2)^d \frac{M - c_P \frac{l^d}{(l-2)^d}\sqrt{b}}{K} \log\left(1 + \frac{M - c_P \frac{l^d}{(l-2)^d}\sqrt{b}}{v}\right)\right],
\end{aligned}$$

by [11], Thm. 1.4. □

It remains to prove the theorem. In order to give more precise information on the exponent, let us introduce some notation. Denote

$$\gamma_*(b) = b^{d/2} \frac{M - c_*\sqrt{b}}{K \cdot S} \log\left(1 + \frac{S \cdot (M - c_*\sqrt{b})}{v}\right)$$

for $* = A, P$, where M, v, S are defined in **(A)** for the Anderson case and $M = v = (e-1)/e$, $S = 1$ in the Poisson case.

SUPPLEMENT TO THEOREM 2.1. *Inequality* (2.1) *holds for*

$$\gamma_* = \max\left\{\gamma_*(b); 0 \leqslant b \leqslant \min\left\{\frac{\pi^2}{4}, \frac{M^2}{c_*}\right\}\right\}.$$

Proof. We first deduce inequality (1.1) with a b-dependent exponent $\gamma(b)$, where b is as in Proposition 2.2. The result above will then follow by optimizing with respect to b.

First note that

$$\begin{aligned}
N(t) &= \inf_\Lambda \frac{1}{|\Lambda|} \mathbf{E}\left\{\mathrm{tr}\left[\chi_{[0,t]}(H_\Lambda(\omega))\right]\right\} \\
&\leqslant \inf_\Lambda \frac{1}{|\Lambda|} \mathbf{P}\left\{E_1(H_\Lambda(\omega)) \leqslant t\right\} \cdot c \cdot |\Lambda|
\end{aligned}$$

by Weyl's law, referred to in the introduction; choosing $t = b \cdot l^{-2}$ with

$$0 \leqslant b \leqslant \min\left\{\frac{\pi^2}{4}, \frac{M^2}{c_A^2}\right\}, \quad \text{and} \quad \Lambda = \Lambda_l(0),$$

we get the assertion. □

3. Concluding Remarks

Of course, one could shorten the above proof if one isn't interested in the exponent.

There are different quite easy perturbation theoretic proofs for Lifshitz tail asymptotics which use Temple's or Thirring's inequality. See [4] for a detailed explanation and references. The method presented here has the advantage that the link between spectral and probability theory provided by the derivative allows for a conceptually more transparent proof, at least in our opinion. Moreover, the derivative is in many cases easy to calculate or at least easy to guess, which provides a road map for the rigorous proof. An example is the application of the above method in [5], where we didn't see how to use the methods previously available.

So far we haven't been able to strengthen our arguments so as to obtain the correct value of the exponent which is known to be $C \cdot \gamma_d^{d/2}$, where C is a known constant and γ_d is the lowest eigenvalue of the Dirichlet Laplacian on a ball of unit volume in $\mathbf{R}^d$; see [10] for an extensive discussion. This correct value is related with isoperimetric inequalities and is obtained by using the celebrated results of Donsker and Varadhan; see [2, 8, 4, 10].

To date there are more detailed results available for the bottom (i.e. principal) eigenvalue of a Schrödinger operator with Poissonian obstacles; we refer the reader to [12] and the literature cited there.

A recent thorough investigation of the attractive Poissonian case (i.e. the case where f is nonpositive) is given in [7].

Let us further mention recent deep work of Klopp, [6], which deals with what is called internal Lifshitz tails. Consider $H_0 = -\Delta + V_0$, where V_0 is a $\mathbf{Z}^d$-periodic potential, and $H(\omega) = H_0 + V_\omega^A$. The spectrum of H_0 consists of a number of closed intervals, called bands, separated by open intervals called gaps. The same is true for $H(\omega)$, where the bands are usually shifted and somewhat enlarged, depending on sign and size of V_ω^A. Lifshitz predicted that the behaviour of the integrated density of states N_0 for H_0 near band edges E_0 should have the same power law decay as in the case $V_0 = 0$ at energy 0, i.e.

$$N_0(E_0 + \varepsilon) \sim N_0(E_0) + \varepsilon^{d/2} \quad \text{as } \varepsilon \searrow 0$$

if E_0 is the left endpoint of one of the bands. Moreover, for the randomized operator $H(\omega)$ he claimed that the integrated density of states N should exhibit the exponential decay discussed above for the inf of the spectrum. Interestingly enough, both claims are still not proved nor disproved in general. Klopps work establishes an equivalence between them, saying that at band edges at which N_0 behaves as predicted so does N. Presumably, this can be proven by our methods above (work in progress, jointly with G. Stolz). However, Klopp's article gives more information: expanding the projection of the periodic Hamiltonian onto a band into Wannier functions he establishes a certain equivalence with a discrete Schrödinger type operator. This equivalence is used to reduce the proof of Lifshitz asymptotics near band edges to the proof of Lifshitz asymptotics of an associated discrete operator at the bottom of the spectrum.

Acknowledgement

Heartfelt thanks go to F. Kleespies and R. Lang for most useful discussions.

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