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Scattering by Obstacles of Finite Capacity

PETER STOLLMANN

Fachbereich Mathematik, Johann Wolfgang Goethe-Universität, Robert-Mayer-Str. 10, 6000 Frankfurt am Main, Germany

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For very general generators of diffusion semigroups we show that the essential and absolutely continuous spectra do not change when one adds an extra Dirichlet boundary condition on a "small" set. This is done by proving that the corresponding semigroup differences are Hilbert–Schmidt or trace class, respectively. Our method consists in a factorization argument which is based on calculating the semigroup difference via the Feynman–Kac formula. We also derive trace class estimates for differences of resolvent powers, provided the underlying semigroup has finite local dimension.

1. INTRODUCTION

Consider a Hamiltonian H associated with a regular Dirichlet form whose semigroup maps $L_1(X)$ to $L_\infty(X)$. The purpose of this note is to prove trace class properties of the semigroup difference $\exp(-tH)-\exp(-tH_G)$, where H_G denotes the operator obtained by imposing Dirichlet boundary conditions on a closed set $B := X\setminus G$ (the obstacle) of finite capacity.

In [6] the reader can find various examples which illustrate the physical interest of such results.

Our Theorem 1 partly extends results from [7], showing that the difference is Hilbert–Schmidt if B satisfies a condition which is weaker than requiring that B has finite capacity. In particular, H and H_G have the same essential spectrum in this case.

In Theorem 2 we can even show that the semigroup difference is trace class, provided B satisfies a somewhat stronger condition. In case that H is the Laplacian on $\mathbb{R}^d$ there is an extensive literature on these so-called exterior domain problems for which we refer the reader to [1, 2, 6–8, 12, 14]. Even for this special case, however, our results are not included in the above-mentioned articles. What seems to be more important is that our method of proof is new. It is based on a simple idea: Using the Feynman–Kac formula, one obtains a suitable factorization for $\exp(-tH)-\exp(-tH_G)$,

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from which the result follows by standard operator ideal theory. This not only allows us to treat a quite general class of Hamiltonians but also gives explicit bounds on the trace norms which can then be exploited in the discussion of resolvent powers. More precisely, if H generates a semigroup which is of finite local dimension d and the obstacle $B = \mathsf{X}\backslash G$ is "small enough," then $(H+1)^{-s} - (H_G + 1)^{-s}$ is Hilbert–Schmidt for $s > d/4$ and nuclear for $s > d/2$.

2. The Results

Assume that H is a self adjoint non-negative operator on $L_2(\mathsf{X}, m)$, where X is a locally compact, second countable space equipped with a Radon measure m of full support. We are going to use the following assumptions:

(A.1) The form $\mathfrak{h}$ defined by $D(\mathfrak{h}) = D(H^{1/2})$, $\mathfrak{h}[u, v] := (H^{1/2}u \mid H^{1/2}v)$ is a regular Dirichlet form (see [10]).

(A.2) For any $t > 0$, $\exp(-tH)$ induces a bounded operator from L_1 to L_∞.

(Note that this is more general than the conditions used in [1, 6, 7].) Recall that under condition (A.1), $\exp(-tH)$ maps L_∞ to itself, so that (A.2) and interpolation imply that $\exp(-tH)$ is a bounded operator from L_2 to L_∞.

From the general theory developed in [10] we infer that there exists a Markov process $(\Omega, (\mathbb{P}^x; x \in \mathsf{X}), (X_t; t \geqslant 0))$ with state space $\mathsf{X} \cup \{\infty\}$ such that the semigroup of H can be calculated as

$$\exp(-tH) f(x) = \mathbb{E}^x(f \circ X_t)$$

for $t \geqslant 0$, $f \in L_2$, and a.e. $x \in \mathsf{X}$.

For any open set G the operator H_G with *Dirichlet boundary conditions* on $B = \mathsf{X}\backslash G$ is defined as the unique self adjoint operator in $L_2(G)$ associated with the closure of the form $\mathfrak{h} \mid D(\mathfrak{h}) \cap C_c(G)$.

Hence, H and H_G act in different spaces, $L_2(\mathsf{X})$ and $L_2(G)$. To consider the difference between the semigroups we use the restriction mapping $J: L_2(\mathsf{X}) \to L_2(G)$, whose adjoint J^* is the natural inclusion. We want to study

$$\exp(-tH) - J^* \exp(-tH_G) J.$$

In order to factorize this difference, we use the Feynman–Kac formula (in a simple form) which asserts that

$$\exp(-tH_G) f(x) = \mathbb{E}^x(\mathbb{1}_{\{\tau_B > t\}} \cdot f \circ X_t),$$

where

$$\tau_B := \inf\{s \geq 0;\ X_s \in B\}$$

denotes the *hitting time* of B; see [10, Sect. 4.1 and Theorem 4.4.2, p. 111].

Recall that if the capacity (see [10, Sect. 3.1, p. 61 ff.])

$$\operatorname{cap}(B) = \inf\{\mathfrak{h}[f, f] + \|f\|^2;\ f \geq \mathbb{1}_U,\ U \text{ open},\ B \subset U\}$$

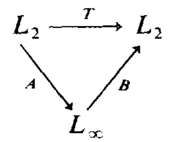
of the set B is finite,

$$e_B(x) := \mathbb{E}^x(\exp(-\tau_B))$$

defines an element $e_B \in D(\mathfrak{h})$, the so-called 1-*equilibrium potential* of B (see [10, p. 75]) which satisfies $\operatorname{cap}(B) = \mathfrak{h}[e_B, e_B] + \|e_B\|^2$.

As we already mentioned in the introduction, we are going to use techniques from the theory of operator ideals. More precisely, we will refer to the following fundamental facts:

An operator T which admits a factorization of the form



is Hilbert–Schmidt.

We will use this result for order preserving operators A, B only, in which case it is not very deep and one even has the inequality $\|T\|_{HS} \leq \|A\|\ \|B\|$ (see [5, 11.2, 11.6]). The general case is a consequence of what is sometimes called the "little Grothendieck theorem" ([5, 11.11] or [21, III.F, III.G]).

To understand this properly one has to consider the ideal $\mathfrak{P}_2$ of *absolutely* 2-*summing operators*. An operator T acting between Banach spaces E and F belongs to $\mathfrak{P}_2(E, F)$ if there exists a constant c such that

$$\left(\sum \|Tx_i\|^2\right)^{1/2} \leq c \cdot \sup_{x' \in B_{E'}} \left(\sum |\langle x', x_i\rangle|^2\right)^{1/2}$$

holds for all $x_1, ..., x_n \in E$. The standard reference for operator ideals is Pietsch's classic [13]. It is not hard to see that $\mathfrak{P}_2(H_1, H_2)$ equals $\mathfrak{HS}(H_1, H_2)$, by which we denote the *Hilbert–Schmidt operators*, provided, of course, that H_1 and H_2 are Hilbert spaces. A considerable strengthening of the corresponding well-known result on the product of Hilbert–Schmidt operators can be written as the inclusion $\mathfrak{P}_2 \circ \mathfrak{P}_2 \subset \mathfrak{N}$, which means that the composition of two absolutely 2-summing operators is nuclear. Recall that the nuclear operators between Hilbert spaces are exactly the trace operators. Denoting the norms in the above operator ideal by $\|\cdot\|_P$, $\|\cdot\|_{HS}$, and $\|\cdot\|_N$, respectively, the "little Grothendieck theorem" reads as follows:

If $E = L_1$ or L_∞, then $\mathfrak{L}(E, L_2) = \mathfrak{P}_2(E, L_2)$ and $\|T\|_P \leqslant K_G \|T\|$ for any $T \in \mathfrak{L}(E; L_2)$, where K_G is the Grothendieck constant.

In the proof of our Theorem 2 we use that every operator T which admits a factorization of the form

$$\begin{array}{ccc} L_2 & \xrightarrow{T} & L_2 \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle B} \\ L_\infty(\mu) & \xrightarrow[I]{} & L_1(\mu) \end{array}$$

where I is the canonical embedding and μ a finite measure, is nuclear. Its nuclear norm satisfies the inequality

$$\|T\|_N \leqslant \|A\| \cdot \|B\| \cdot \|I\|.$$

This follows, e.g., from [5, 10.6 and D.2] or from [9, p. 258 ff. and p. 248]. To explain this result we would have to introduce the ideal of operators which are integral in the sense of Grothendieck (introduced in [11]) and use the fact that Hilbert spaces enjoy the Radon–Nikodým property. Instead we stop our excursion here and return to our subject matter. We hope that those readers who are not (yet) familiar with operator ideals realize the usefulness of the above-mentioned facts.

Our first main result is:

THEOREM 1. *Assume that* $B = \mathsf{X} \backslash G$ *satisfies*

$$\int_{\mathsf{X}} (\mathbb{P}^x(\tau_B \leqslant t_0))^2 \, dm(x) < \infty \tag{1}$$

for some $t_0 > 0$. *Then*

$$D(t) := \exp(-tH) - J^* \exp(-tH_G) J \in \mathfrak{H}\mathfrak{S}(L_2, L_2)$$

for all $t > 0$. *More precisely,*

$$\|D(t)\|_{HS} \leqslant 2 \, \|\mathbb{P}^\cdot(\tau_B \leqslant t/2)\|_2 \, \|\exp(-t/2H) \colon L_2 \to L_\infty\|$$

for $t \leqslant 2t_0$.

Proof. Denote $U(t) := \exp(-tH)$, $U_G(t) := J^* \exp(-tH_G) J$. Since

$$D(2t) = D(t)\, U(t) + U_G(t)\, D(t),$$

it is clear that we may assume $t \leqslant t_0$. $U(t)$ as well as $U_G(t)$ map from L_2 to L_∞, so that it suffices to prove that $D(t)$ is bounded from L_∞ to L_2. In fact, the first of the above-mentioned factorization arguments will then imply that both $D(t)\, U(t)$ and $U_G(t)\, D(t) = (D(t)\, U_G(t))^*$ are in $\mathfrak{H}\mathfrak{S}$.

In order to check that $D(t) \in \mathfrak{L}(L_\infty, L_2)$ we calculate

$$\begin{aligned} D(t)\, 1(x) &= \mathbb{P}^x(1) - \mathbb{P}^x(\tau_B > t) \\ &= \mathbb{P}^x(\tau_B \leqslant t). \end{aligned}$$

Since $D(t)$ acts order preserving, $\|D(t) : L_\infty \to L_2\| \leqslant \|\mathbb{P}^{\cdot}(\tau_B \leqslant t)\|_2$. ∎

As a remarkable spectral theoretic consequence of Theorem 1 we obtain:

COROLLARY. *Under the assumptions of Theorem* 1, *the essential spectra of H and H_G coincide.*

Proof. We show that $(H+E)^{-1} - J^*(H_G+E)^{-1} J$ is compact for $E > 0$, from which the assertion follows by Weyl's theorem as stated for example in [15, Theorem XIII.14]. To this end we use an argument borrowed from [19]:

$$(H+E)^{-1} - J^*(H_G+E)^{-1} J = \int_0^\infty e^{-Et} D(t)\, dt.$$

Using the compactness of $D(t)$ and the dominated convergence theorem, we see that the integral maps weak null sequences to norm null sequences. ∎

Note that obstacles of finite capacity satisfy the condition in the last theorem; in fact

$$\begin{aligned} \mathbb{P}^x(\tau_B \leqslant s) &\leqslant e^s \mathbb{P}^x(e^{-\tau_B}) \\ &= e^s e_B(x) \end{aligned}$$

if B has finite capacity. It is interesting to note that e_B is even in L_1 which implies

$$\int_X \mathbb{P}^x(\tau_B \leqslant 1)\, dm(x) < \infty.$$

We think that this is the right condition in order to obtain that $\exp(-tH) - \exp(-tH_G)$ is trace class (in fact, this has been claimed in [1]; there is, however, a gap in the proof). We want to stress the fact that van Casteren isolated the right sort of condition. In this respect, [1] influenced the present note considerably, although we use totally different techniques.

THEOREM 2. *Assume that $B = \mathsf{X} \setminus G$ satisfies*

$$\int_{\mathsf{X}} (\mathbb{P}^x(\tau_B \leqslant t_0))^{1/2} \, dm(x) < \infty \tag{2}$$

for some $t_0 > 0$. Then

$$\exp(-tH) - J^* \exp(-tH_G) \, J \in \mathfrak{N}(L_2, L_2)$$

for all $t > 0$. More precisely,

$$\|D(t)\|_{\mathcal{N}} \leqslant 2 \, \|(\mathbb{P}^{\cdot}(\tau_B \leqslant t/2))^{1/2}\|_1 \, \|\exp(-t/2H) : L_1 \to L_\infty\|$$

for $t \leqslant 2t_0$.

Proof. Since $D(2t) = U(t) \, D(t) + D(t) \, U_G(t)$, we may assume that $t \leqslant 2t_0$. We check that $U(t) \, D(t)$ can be factorized as in the second diagram above: Let $f \in L_2$, $\|f\| \leqslant 1$. Then, for all $x \in \mathsf{X}$

$$\begin{aligned} |D(t) \, f(x)| &= |\mathbb{E}^x(\mathbb{1}_{\{\tau_B \leqslant t\}} \, f \circ X_t)| \\ &\leqslant (\mathbb{P}^x(\tau_B \leqslant t))^{1/2} \cdot (\mathbb{E}^x(|f|^2 \circ X_t))^{1/2} \\ &\leqslant (\mathbb{P}^x(\tau_B \leqslant t))^{1/2} \cdot \sup_y \, (\mathbb{E}^y(|f|^2 \circ X_t))^{1/2}. \end{aligned}$$

Since $U(t)$ maps L_1 to L_∞, we arrive at

$$|D(t) \, f(x)| \leqslant (\mathbb{P}^x(\tau_B \leqslant t))^{1/2} \cdot \|U(t) : L_1 \to L_\infty\|^{1/2}.$$

Write $h(x) := (\mathbb{P}^x(\tau_B \leqslant t))^{1/2}$ and denote by M the operator of multiplication by h^{-1}. Let $d\mu := h \, dm$. Then the last inequality may be restated in the following way:

$$A := MD(t) \in \mathfrak{L}(L_2(\mathsf{X}, m), L_\infty(\mathsf{X}, \mu)), \qquad \|A\| \leqslant \|U(t) : L_1 \to L_\infty\|^{1/2}.$$

Moreover, $\|I\| = \mu(\mathsf{X}) = \|h\|_1$ and

$$B := U(t) \, M^{-1} \in \mathfrak{L}(L_1(\mathsf{X}, \mu), L_2(\mathsf{X}, m))$$

with norm less than $\|U(t) : L_1 \to L_2\|$. This implies that

$$U(t) \, D(t) \in \mathfrak{N}(L_2, L_2),$$

with nuclear norm estimated by

$$\|(\mathbb{P}^{\cdot}(\tau_B \leqslant t))^{1/2}\|_1 \cdot \|U(t) : L_1 \to L_\infty\| \cdot \|U(t) : L_1 \to L_2\|.$$

By Riesz–Thorin, $\|U(t)\colon L_1 \to L_2\| \leqslant \|U(t)\colon L_1 \to L_\infty\|^{1/2}$. It remains to check that $D(t)\, U_G(t)$ is also nuclear, but one can use the arguments above to prove that its adjoint $U_G(t)\, D(t)$ is nuclear. ∎

We should remark that condition (2) is satisfied for any bounded set in case of the classical Dirichlet form on $\mathbb{R}^d$. This is a consequence of the exponential decay of the resolvent kernel. Moreover, $\|(\mathbb{P}^{\cdot}(\tau_{B(r)} \leqslant t))^{1/2}\|_1 \to 0$ for $r \to 0$ if $d \geqslant 2$ (where $B(r)$ is a ball with radius r). Hence it is possible to give examples of unbounded domains, which can be handled by successive application of Theorem 2:

EXAMPLE. Let $H = -\frac{1}{2}\Delta$ on $X := \mathbb{R}^d$, $d \geqslant 2$. Assume that $B = \bigcup B_n$, where B_n is a set with diameter less than r_n. If r_n is small enough, then

$$\exp(-tH) - J^* \exp(-tH_G)\, J \in \mathfrak{N}$$

for all $t > 0$.

Note that such a B may even be "dense at infinity" in the sense that $\operatorname{dist}(x, B) \to 0$ for $x \to \infty$. This is somewhat surprising, for (due to results in [4]) one might be tempted to think that H_G has discrete spectrum in that case. From the above corollary, however, we know that $\sigma_{\text{ess}}(H_G) = [0, \infty)$. Even more is true, namely, $\sigma_{\text{ac}}(H_G) = [0, \infty)$, which follows from our next result.

COROLLARY. *Under the assumptions of Theorem 2, the wave operators*

$$\Omega_{+/-}(H, H_G, J) := s\text{-}\lim_{t \to +/-\infty} e^{iH_G t} J e^{-iHt} P_{\text{ac}}(H)$$

exist and are complete. ($P_{\text{ac}}(H)$ denotes the projection onto the absolutely continuous subspace of H.) In particular, the absolutely continuous spectra of H and H_G coincide.

Proof. This follows from the invariance principle; see the discussion in [12, Sect. 2]. ∎

Next we are going to study differences of resolvent powers instead of the semigroups. The general method we use is the representation of the resolvent via the Laplace transform. Demuth has employed this technique in [6, 7] to obtain corresponding results for a perturbation of the Laplacian on $\mathbb{R}^d$ and less general obstacles. The idea is the following: Once one has inequalities like those in Theorem 1 and Theorem 2 one must only control the singularity of the Hilbert–Schmidt or the trace norm of $D(t)$ as $t \to 0$. Since the latter is essentially governed by the singularity of $\|\exp(-tH)\colon L_1 \to L_\infty\|$ for the nuclear and $\|\exp(-tH)\colon L_1 \to L_2\|$ for the Hilbert–Schmidt norm, it is clear that the following notion is most useful.

We say that a semigroup $(U(t);\, t \geqslant 0)$ is of *local dimension* d, provided

$$\|U(t)\colon L_1 \to L_\infty\| \leqslant \text{const} \cdot t^{-d/2} \qquad (0 < t < 1);$$

see [3, 18] for background information and related conditions. Clearly, any semigroup of finite local dimension satisfies our general assumption (A.2). With this key notion at hand, we can state:

THEOREM 3. *Assume that* $(\exp(-tH);\, t \geqslant 0)$ *is of local dimensional* d.

(a) *Let* $B = X \setminus G$ *satisfy* (1). *Then, for* $s > d/4$,

$$(H+1)^{-s} - J^*(H_G+1)^{-s} J \in \mathfrak{H}\mathfrak{S}.$$

(b) *Let* B *satisfy* (2). *Then, for* $s > d/2$,

$$(H+1)^{-s} - J^*(H_G+1)^{-s} J \in \mathfrak{N}.$$

Proof. We only give the proof of (a), since (b) can be verified analogously. Let us first recall that the Riesz–Thorin convexity theorem implies that

$$\|\exp(-tH)\colon L_1 \to L_2\| \leqslant \text{const} \cdot t^{-d/4}.$$

From

$$(H+1)^{-s} = \Gamma(s)^{-1} \int_0^\infty t^{s-1} e^{-t} \exp(-tH)\, dt$$

and the corresponding formula for H_G, it follows that

$$(H+1)^{-s} - J^*(H_G+1)^{-s} J = \Gamma(s)^{-1} \int_0^\infty t^{s-1} e^{-t} D(t)\, dt,$$

so that it suffices to estimate

$$\Gamma(s)^{-1} \int_0^\infty t^{s-1} e^{-t}\, \|D(t)\|_{HS}\, dt.$$

Since

$$D(2t) = D(t)\, U(t) + U_G(t)\, D(t)$$

for all t, only the singularity at $t = 0$ plays a role. But

$$\|D(t)\|_{HS} \leqslant \text{const} \cdot \|\exp(-tH)\colon L_1 \to L_2\| \leqslant \text{const} \cdot t^{-d/4},$$

which implies, for $s > d/4$,

$$\int_0^1 t^{s-1} e^{-t} \|D(t)\|_{HS}\, dt < \infty. \quad \blacksquare$$

As a final remark let us mention that the above methods are also applicable, if H is replaced by a perturbed operator $H+V$ for suitable potentials. Such perturbations have been treated in [1, 6, 7] (in a different setting with different methods). Using the results from [17] we can even allow perturbations by measures $H+\mu$ where the negative part of the measure satisfies a Kato condition. We do not go into more details concerning such results here and instead refer the reader to [16].

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