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Essential self-adjointness, generalized eigenforms, and spectra for the $\bar{\partial}$-Neumann problem on G-manifolds

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Abstract

Let M be a complex manifold with boundary, satisfying a subelliptic estimate, which is also the total space of a principal G-bundle with G a Lie group and compact orbit space $\overline{M}/G$. Here we investigate the $\bar{\partial}$-Neumann Laplacian $\square$ on M. We show that it is essentially self-adjoint on its restriction to compactly supported smooth forms. Moreover we relate its spectrum to the existence of generalized eigenforms: an energy belongs to $\sigma(\square)$ if there is a subexponentially bounded generalized eigenform for this energy. Vice versa, there is an expansion in terms of these well-behaved eigenforms so that, spectrally, almost every energy comes with such a generalized eigenform.

Keywords: Essential self-adjointness; Eigenfunctions; Spectra; $\bar{\partial}$-Neumann problem

Contents

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1. Introduction

The approach to the theory of several complex variables via partial differential equations involves the analysis of a self-adjoint boundary value problem for an operator $\Box$ similar to the Hodge Laplacian. This problem, called the $\bar\partial$-Neumann problem, is the subject of this article and we will give a brief description here.

We will assume that M is a complex manifold, $n = \dim_{\mathbb{C}} M$, with smooth boundary bM such that $\overline{M} = M \cup bM$. Assume further that $\overline{M}$ is strictly contained in a slightly larger complex manifold $\widetilde{M}$ of the same dimension. For any integers p, q with $0 \leqslant p, q \leqslant n$ denote by $C^\infty(M, \Lambda^{p,q})$ the space of all C^∞ forms of type (p, q) on M, *i.e.* the forms which can be written in local complex coordinates $(z^1, z^2, \dots, z^n)$ as

$$\phi = \sum_{|I|=p,|J|=q} \phi_{I,J}\, dz^I \wedge d\bar z^J \tag{1}$$

where $dz^I = dz^{i_1} \wedge \cdots \wedge dz^{i_p}$, $d\bar z^J = d\bar z^{j_1} \wedge \cdots \wedge d\bar z^{j_q}$, $I = (i_1, \dots, i_p)$, $J = (j_1, \dots, j_q)$, $i_1 < \cdots < i_p$, $j_1 < \cdots < j_q$, with the $\phi_{I,J}$ smooth functions in local coordinates. For such a form ϕ, the value of the antiholomorphic exterior derivative $\bar\partial\phi$ is

$$\bar\partial\phi = \sum_{|I|=p,|J|=q} \sum_{k=1}^{n} \frac{\partial\phi_{I,J}}{\partial\bar z^k}\, d\bar z^k \wedge dz^I \wedge d\bar z^J$$

so $\bar\partial = \bar\partial|_{p,q}$ defines a linear map $\bar\partial : C^\infty(M, \Lambda^{p,q}) \to C^\infty(M, \Lambda^{p,q+1})$. With respect to a smoothly varying Hermitian structure in the fibers of the tangent bundle, and a corresponding volume form, define the spaces $L^2(M, \Lambda^{p,q})$. Let us consider $\bar\partial$ as the maximal operator in L^2 and let $\bar\partial^*$ be its Hilbert space adjoint operator (this involves the introduction of boundary conditions). Define the nonnegative form

$$Q(\phi, \psi) = \langle \bar\partial\phi, \bar\partial\psi\rangle_{L^2(M,\Lambda^{p,q+1})} + \langle \bar\partial^*\phi, \bar\partial^*\psi\rangle_{L^2(M,\Lambda^{p,q-1})}, \tag{2}$$

with domain $\mathrm{dom}(Q) = \mathrm{dom}(Q^{p,q}) \subset L^2(M, \Lambda^{p,q})$ and denote the associated self-adjoint operator in $L^2(M, \Lambda^{p,q})$ by

$$\Box = \Box_{p,q} = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*,$$

using $+$ for the *form sum* of two self-adjoint operators; see [12]. The Laplacian is elliptic but its natural boundary conditions are not coercive, thus, in the interior of M, the operator gains two degrees in the Sobolev scale, as a second-order operator, while in neighborhoods of the boundary, it gains less. The gain at the boundary depends on the geometry of the boundary, and the best such situation is that in which the boundary is *strongly pseudoconvex*. In that case, the operator gains one degree on the Sobolev scale in neighborhoods of bM and so global estimates including both interior and boundary neighborhoods gain only one degree.

More generally, one says that the Laplacian satisfies a *pseudolocal estimate with gain* $\epsilon > 0$ *in* $L^2(M, \Lambda^{p,q})$ in the following situation.

If $U \subset \overline{M}$ is a neighborhood with compact closure, $\zeta, \zeta' \in C_c^\infty(U)$ for which $\zeta'|_{\mathrm{supp}(\zeta)} = 1$, and $\alpha|_U \in H^s(U, \Lambda^{p,q})$, then $\zeta(\Box + \mathbf{1})^{-1}\alpha \in H^{s+\epsilon}(\overline{M}, \Lambda^{p,q})$ and there exists a constant $C_{\zeta,\zeta'} > 0$ such that

$$\left\|\zeta(\Box + \mathbf{1})^{-1}\alpha\right\|_{H^{s+\epsilon}(M,\Lambda^{p,q})} \leqslant C_{\zeta,\zeta'}\left(\left\|\zeta'\alpha\right\|_{H^s(M,\Lambda^{p,q})} + \|\alpha\|_{L^2(M,\Lambda^{p,q})}\right) \tag{3}$$

uniformly for all α satisfying the assumption. See [20–22,13,11] for these results.

Mostly for the simplicity that a group symmetry implies, let us assume in this paper that the manifolds in consideration satisfy the following requirements.

Definition 1.1. We will say that M satisfies *assumption* (A) if the following hold. First, assume that M is a complex manifold which is also the total space of a principal G-bundle with G a Lie group acting by holomorphic transformations and with compact orbit space $\overline{M}/G$:

$$G \longrightarrow M \longrightarrow X.$$

Assume also that M has a smooth pseudoconvex boundary and that $\Box = \Box_{p,q}$ satisfies a pseudolocal estimate with gain $\epsilon > 0$ in $L^2(M, \Lambda^{p,q})$.

Though our results hold in substantially greater generality, which we will indicate where we feel necessary, we keep our setting as above, with exact invariances. We note that in the case in which G is unimodular, there is a good generalized Fredholm theory for the $\Box$ as well as generalized Paley–Wiener theorems for G-bundles which together provide an effective framework for understanding the solvability of equations involving $\Box$. These are worked out and applied in [30,31,9]. In [32], the unimodularity condition is dropped, as in our setting here.

We will in this article be concerned with the following fundamental properties of the operator $\Box$, whose domain is denoted $\mathrm{dom}(\Box)$.

Theorem 1. *Assume* (A) *from* 1.1. *Then* $\Box$ *is essentially self-adjoint on* $C_c^\infty(\overline{M}, \Lambda^{p,q}) \cap \mathrm{dom}\,\Box$.

This type of result is very common for many natural partial differential operators on manifolds without boundary. It is important because it provides that there is only one way to extend the operator from a domain consisting of smooth, compactly supported forms to a self-adjoint operator. The case at hand is more complicated due to the boundary, which moreover plays an important role and comes with noncoercive boundary conditions. We prove Theorem 1 in Section 7 by a cutoff procedure that requires taking the boundary condition into account. We borrow from [3] and from discussions of the first-named author with E. Straube.

The reader will notice that we state this theorem first and prove it last. The reason for this is that we base the following two results on quadratic form methods for which we do not need the more precise description of the domain of the operator.

Theorem 2 *(Schnol-type theorem). Assume* (A) *from* 1.1. *The existence of a generalized eigenform for $\Box$ with eigenvalue λ satisfying certain growth conditions implies that $\lambda \in \sigma(\Box)$.*

This type of result is often called Schnol's theorem in the literature. The precise statements are Theorem 5.2 and Corollary 5.4 below. Actually, the original result of Schnol's paper [38] is an equivalence, so Theorem 2 and the following Theorem 3 together give results reminiscent of Schnol's theorem from Schrödinger operator theory; see [38] and the discussion in [5,25,26] for a list of references and recent results in the Dirichlet form context. See also [39,40] for results on general elliptic operators on sections of vector bundles over complete manifolds.

Theorem 3 *(Eigenfunction expansion). Assume* (A) *from* 1.1. *Let $\omega \in L^2(M, \mathbb{R})$ with $\omega^{-1} \geqslant 1$. Then, for spectrally a.e. $\lambda \in \sigma(\Box)$ there is a generalized eigenform ε_λ for $\Box$ with eigenvalue λ so that $\omega\varepsilon_\lambda \in L^2(M, \Lambda^{p,q})$.*

For the proof of Theorem 2 we follow the strategy from [5], see also [25]: starting from the well-behaved generalized eigenform u we construct a singular sequence $u_k = \eta_k u$ for the form Q of $\Box$. The cutoff functions have to be such that the product $\eta_k u$ belongs to the domain of the form Q. That is achieved by using the intrinsic metric of $\Box$ to define η_k. The intrinsic metric for $\Box$ was introduced in our previous work [33] and turned out to be useful in estimating the heat kernel of $\Box$. Here we provide some more results and a useful characterization of the intrinsic metric in Section 3. That the cutoff does provide a singular sequence is a consequence of a Caccioppoli type inequality, which is the subject of Section 4. In Section 5 we prove two variants of Theorem 2, making precise what "certain growth conditions" means.

Expansion in generalized eigenelements is typically based on strong compactness properties. Here we use the method developed in [4] for Dirichlet forms, based on an abstract result from [34]. The main input is from [33], where we showed ultracontractivity of the heat semigroup corresponding to $\Box$; we also refer to this paper for more pointers to related literature.

2. Preliminaries and examples

2.1. Invariant structures

We will have to describe smoothness of functions, forms, and sections of vector bundles using G-invariant Sobolev spaces which we define here.

We denote by $C^\infty(M, \Lambda^{p,q})$ the space of smooth (p,q)-forms on M, by $C^\infty(\overline{M}, \Lambda^{p,q})$ the subspace of those forms that can be smoothly extended to $\overline{M}$ and by $C_c^\infty(\overline{M}, \Lambda^{p,q})$ the subspace of the latter, consisting of those smooth forms with compact support. Given any G-invariant, pointwise Hermitian structure

$$C^\infty\big(\overline{M}, \Lambda^{p,q}\big) \ni u, v \longmapsto \big\langle u(x), v(x)\big\rangle_{\Lambda_x^{p,q}} \in \mathbb{C} \quad (x \in \overline{M}),$$

and its volume form μ, we define the L^p-spaces $L^p(M, \Lambda^{q,r})$ of forms, for $1 \leqslant p \leqslant \infty$, as those forms u, for which the norm

$$\|u\|_{L^p(M,\Lambda^{q,r})} = \left[\int_M \langle u, u\rangle_{\Lambda^{q,r}}^{p/2}\, d\mu \right]^{1/p}$$

is finite, with the obvious modification for $p = \infty$. We will sometimes abbreviate $\langle u, u\rangle_{\Lambda^{q,r}}$ by writing $|u|^2_{\Lambda^{q,r}}$ instead. Also, we will write $\langle\cdot,\cdot\rangle_{\Lambda^p}$ to mean the Hermitian structure on $\mathbb{C} \otimes \Lambda^p = \bigoplus_S \Lambda^{q,r}$ with $S = \{(q,r) \mid q + r = p\}$ as well as the Riemannian metric on Λ^1 associated, see [33, §3.2].

As we have a manifold with bounded geometry, there exist partitions of unity with bounded multiplicity and derivatives, [15,16,23,24,37,40] and, by differentiating componentwise with respect to local geodesic coordinates, we may assemble G-invariant integer Sobolev spaces $H^s(M, \Lambda^{p,q})$, for $s = 0, 1, 2, \dots$.

Because $\overline{M}/G$ is compact, the spaces $H^s(M, \Lambda^{p,q})$ do not depend on the choice of an invariant Hermitian structure on $\Lambda^{p,q}$. The usual duality relations for L^p spaces hold (polarizing the above norm) as well as the Sobolev lemma, *etc.* Background on this is provided in [14]. We will also need the L^p-Sobolev spaces

$$W^{s,p}\big(M, \Lambda^{q,r}\big) := \big\{ u \in L^p\big(M, \Lambda^{q,r}\big) \bigm| D^\alpha u \in L^p \text{ for } |\alpha| \leqslant s \big\},$$

for $1 \leqslant p \leqslant \infty$, $s \in \mathbb{N}$, where the differentiation D is understood componentwise, with respect to local geodesic coordinates, and in the distributional sense.

As mentioned above, the group invariance and the compactness of the quotient provide us with a number of useful uniformities. This applies, *e.g.* to the pseudolocal estimates required in assumption (A) from 1.1 above in that all we will ever need will be derivable from the estimate for a single neighborhood U and a fixed pair of cutoffs ζ, ζ', yielding a universal $\epsilon > 0$ and constant $C_{\zeta,\zeta'}$, as in [33]. We refer the reader to [7,6,11] for a discussion of this type of estimates as well as sufficient geometric properties.

2.2. *Examples*

Complex manifolds satisfying our assumptions fall into two major categories. Manifolds in the first category, treated in [17] and corresponding to zero-dimensional structure groups, are most naturally obtained as follows. Let X be a strongly pseudoconvex, complex manifold with compact closure $\bar{X} = X \cup bX$. Assume also that the fundamental group $\pi_1(X)$ is infinite. It follows that $\pi_1(X)$ acts on the universal cover $\widetilde{X} = M$ of X by deck transformations, and estimates involving the boundary are uniform as they are determined on the compact $\bar{X}$. Covers of X corresponding to subgroups of π_1 will share this uniformity property.

For the second, in [18] a large class of manifolds was constructed which are also amenable to our treatment. These are obtained as follows. Suppose that a Lie group G acts properly by C^ω transformations on a C^ω manifold Y. The most natural example to take here is Y as the underlying manifold of G itself. It turns out that the action of G on Y can be extended to a complexification $Y^{\mathbb{C}}$ of Y in such a way that the action of G on $Y^{\mathbb{C}}$ is by holomorphic transformations. In addition, the authors construct a strictly plurisubharmonic function φ in a neighborhood of Y in $Y^{\mathbb{C}}$ such that φ is constant on the orbits of G. It follows that for $\epsilon > 0$ sufficiently small, the tube $M = \{\varphi < \epsilon\}$ is a strongly pseudoconvex complex G-manifold and if Y/G is compact, then $\overline{M}/G$ is too. Defining invariant structures as above, we obtain uniform estimates for $\Box$, as required by our techniques.

Whenever the group in the setting of [18] contains a cocompact lattice, of course the present situation reduces to that of [17]. However, even for the restricted class of unimodular Lie groups, it is generically not the case that a Lie group G possess such a subgroup, [29].

Concrete examples of tubes of matrix groups can be found in [9,27]; one is given as follows, constructed by the technique of [18]. For $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, define the three-dimensional Heisenberg group

$$\mathbb{H}_3(\mathbb{K})=\left\{\begin{pmatrix}1 & z_1 & z_3\\ 0 & 1 & z_2\\ 0 & 0 & 1\end{pmatrix}\,\middle|\, z_k\in\mathbb{K}\right\}.$$

The function $\varphi:\mathbb{H}_3(\mathbb{C})\to\mathbb{R}$ given by

$$\varphi(Z)=(\mathfrak{Im}\,z_1)^2+(\mathfrak{Im}\,z_2)^2+(\mathfrak{Im}\,z_3-\mathfrak{Re}\,z_2\,\mathfrak{Im}\,z_1)^2$$

is invariant under right multiplication by matrices in $\mathbb{H}_3(\mathbb{R})$. An easy calculation shows that $M_\epsilon=\{\varphi<\epsilon\}\subset\mathbb{C}^3$ is strongly pseudoconvex as long as $\epsilon<1$, and it is true that M_1 satisfies a pseudolocal estimate though it is not strongly pseudoconvex. Since $\mathbb{H}_3(\mathbb{R})$ contains lattices, the manifolds M_ϵ are examples of the setting of the discrete structure group as well as that of a bundle.

Finally, [17, §3] contains a remarkable example of a G manifold in $\mathbb{C}^2$ which is not a tube but satisfies all of our requirements. This manifold has a trivial Bergman space though the $\bar\partial$-Neumann problem is somewhat tractable, as shown in [32]. Our treatment is valid there as well.

Let us end this section with a final word on forms and forms: Unfortunately we need to use these completely different concepts that bear the same name in this paper. From Hilbert space theory we need *sesquilinear forms* that are *bounded below*, *e.g.*, the $Q=Q^{p,q}$ above. See Kato's [19] and Reed and Simon's [35] classics and Faris' excellent lecture notes [12] for background. These forms are defined on L^2-spaces of *differential forms*, as we already mentioned. The standard reference for the relevant notions of differential forms related to the $\bar\partial$-Neumann problem is [13].

3. The intrinsic metric

In [33] we used the intrinsic metric to bound the heat kernel of the $\bar\partial$-Neumann Laplacian. Here it will again turn out to be extremely useful. In this section we give a characterization and prepare the ground for a cutoff procedure that is well suited to forms in the domain of the form $Q=Q^{p,q}$. We rely on assumption (A) from 1.1, as usual.

Definition 3.1. We define the G-invariant pseudo-metric on M by

$$d_\square(x,y)=\sup\bigl\{w(y)-w(x)\,\big|\,w\in L^\infty\cap C^\infty(\overline{M},\mathbb{R}),\ \langle\bar\partial w,\bar\partial w\rangle_{\Lambda^{0,1}}\leqslant 1\bigr\}.$$

We define the distance between sets accordingly,

$$d_\square(A;B):=\sup\Bigl\{\inf_B w-\sup_A w\,\Big|\,w\in L^\infty\cap C^\infty(\overline{M},\mathbb{R}),\ \langle\bar\partial w,\bar\partial w\rangle_{\Lambda^{0,1}}\leqslant 1\Bigr\}$$

for arbitrary $A,B\subset\overline{M}$.

Compared to the above definition, we extend the family of functions over which we take the supremum as follows.

Lemma 3.2. *Let* $\mathcal{A}^1 = \{w \in C(\overline{M}, \mathbb{R}) \mid |\bar\partial w|_{\Lambda^{0,1}} \leqslant 1,\ \mu\text{-a.e.}\}$ *where the derivative is understood in the distributional sense. It follows that any* $w \in \mathcal{A}^1$ *is a limit, locally uniformly, of smooth functions* w_k *with* $|\bar\partial w_k|_{\Lambda^{0,1}} \leqslant 1$.

Proof. Apply Friedrichs mollifiers. □

Corollary 3.3. $d_\square(x, y) = \sup\{w(y) - w(x) \mid w \in \mathcal{A}^1\}$.

Definition 3.4. For $E \subset \overline{M}$, put

$$\rho_E(x) = \inf\bigl\{d_\square(x, y) \bigm| y \in E\bigr\}.$$

Lemma 3.5. *The function* $\rho_E \in \mathcal{A}^1$ *and* $d_\square(\{x\}, E) = \rho_E(x)$.

We deduce the preceding lemma from the following description of the saturation properties of $\mathcal{A}^1$.

Proposition 3.6. *(See [5, Propositions A.1, A.2].) For* $\mathcal{A}^1$ *as above, we have the following properties*:

1. $\mathcal{A}^1$ *is* balanced, *i.e. it is convex and closed under multiplication by* -1.
2. $\mathcal{A}^1$ *is closed under the operations* min *and* max.
3. $\mathcal{A}^1$ *is closed under the operation of adding constants.*
4. $\mathcal{A}^1$ *is closed under pointwise convergence of functions uniformly bounded on compacts.*
5. *Let* $\mathcal{F} \subset \mathcal{A}^1 \cap C(\overline{M}, \mathbb{R})$ *be stable under* max (*resp.* min). *If* $u = \sup\{v \mid v \in \mathcal{F}\}$ (*resp.* $u = \inf\{v \mid v \in \mathcal{F}\}$) *then* $u \in \mathcal{A}^1$.

Proof. Note that the form $\mathcal{E}(u, v) = \langle\bar\partial u, \bar\partial v\rangle_{\Lambda^{0,1}}$ from [33] on $\mathcal{D} = \operatorname{dom}(Q^{0,0})$ is a strongly local Dirichlet form with energy measure

$$\Gamma(u, v) = \bigl\langle\bar\partial u(x), \bar\partial v(x)\bigr\rangle_{\Lambda^{0,1}}\, d\mu(x),$$

so the formalism of [5, Appendix] applies. □

Now let us turn to an alternative description of the intrinsic metric. As calculated in [33, §3.2], if the Hermitian metric on $\Lambda^{0,1}$ is associated to the Riemannian metric on Λ^1, then, pointwise

$$2\langle\bar\partial w, \bar\partial w\rangle_{\Lambda^{0,1}} = \langle dw, dw\rangle_{\Lambda^1}, \quad w \in C^\infty(\overline{M}, \mathbb{R}).$$

By [36,45], the form on the right induces the Laplace–Beltrami operator Δ_{LB} on functions. Recall the usual Riemannian distance on M:

Definition 3.7. Put

$$L(\gamma) = \int_a^b \left|\dot\gamma(t)\right|_{T^1}\,dt$$

for γ a piecewise smooth curve $\gamma : [a,b] \to \overline{M}$, and where the length of $\dot\gamma$ is measured with the Riemannian metric on T^1M. Let

$$\rho(x,y) = \inf\bigl\{L(\gamma) \bigm| \gamma \text{ is a piecewise smooth curve joining } x \text{ and } y\bigr\}.$$

Corollary 3.8. *In the situation above, we have*

$$\begin{aligned} d_\Box(x,y) &= \sqrt{2}\sup\bigl\{w(y) - w(x) \bigm| \langle dw, dw\rangle_{\Lambda^1} \leqslant 1\bigr\} \\ &= \sqrt{2}\rho(x,y). \end{aligned}$$

Proof. Fix a $w \in \mathcal{A}^1 \cap C^\infty(\overline{M},\mathbb{R})$ and let $\gamma : [a,b] \to \overline{M}$ be a curve with

$$\rho(x,y) \geqslant L(\gamma) + \epsilon.$$

We get

$$\begin{aligned} w(y) - w(x) = w\bigl(\gamma(1)\bigr) - w\bigl(\gamma(0)\bigr) &= \int_0^1 \frac{d}{dt} w \circ \gamma(t)\,dt \\ &= \int_0^1 \bigl\langle dw\bigl(\gamma(t)\bigr), \dot\gamma(t)\bigr\rangle\,dt \\ &\leqslant \int_0^1 \bigl|dw\bigl(\gamma(t)\bigr)\bigr|_{\Lambda^1}\left|\dot\gamma(t)\right|_{T^1}\,dt \leqslant \sqrt{2}\int_0^1 \left|\dot\gamma(t)\right|_{T^1}\,dt \\ &\leqslant \sqrt{2}\bigl(\rho(x,y) + \epsilon\bigr). \end{aligned}$$

Thus $d_\Box(x,y) \leqslant \sqrt{2}\rho(x,y)$. To show the reverse inequality, fix $y \in \overline{M}$ and note that it is enough to prove that for $w(x) = \rho(x,y)$ we have weak differentiability and $|dw(x)|_{\Lambda^1} \leqslant 1$, μ-almost everywhere in M. By Rademacher's theorem, this amounts to showing that

$$\bigl|w(x) - w\bigl(x'\bigr)\bigr| \leqslant \rho\bigl(x,x'\bigr)$$

which follows from the triangle inequality for ρ. □

Remark 3.9. The existence of minimizing geodesics in the case at hand is demonstrated in [1]. In the general Dirichlet form setting, the intrinsic metric gives at least a length space, as shown in [42]. From now on we will simply write $d(\cdot,\cdot)$ instead of $d_\Box(\cdot,\cdot)$.

We now use the intrinsic metric to define cutoff functions. Let $b > 0$ and $\zeta \in C^1(\mathbb{R}, [0, 1])$ so that $\zeta|_{(-\infty,0]} \equiv 0$, $\zeta|_{[b,\infty)} \equiv 1$, and $\sup|\zeta'(t)| < 2/b$. It follows that

- $\zeta \circ \rho_E \in W^{1,\infty}(\overline{M}, \mathbb{R})$,
- $|\bar\partial(\zeta \circ \rho_E)| \leqslant 2/b$,
- $\zeta \circ \rho_E|_{\bar E} \equiv 1$,
- $\zeta \circ \rho_E|_{B_b(E)^c} \equiv 0$,

where $B_b(E) = \{y \in \overline{M} \mid d(y, E) \leqslant b\}$ is the b-neighborhood of E.

A word on notation: For two quantities A and B, we write $A \lesssim B$ to mean that there exists a constant $C > 0$ such that $|A(\phi)| \leqslant C|B(\phi)|$ uniformly for ϕ in whatever set relevant to the context.

We have the following elementary:

Lemma 3.10. *For $u \in L^\infty(\overline{M}, \Lambda^k)$ with support in E and $v \in L^2(M, \Lambda^l)$, we have*

$$\|u \wedge v\|^2_{L^2(M,\Lambda^{k+l})} \lesssim \|u\|^2_{L^\infty(M,\Lambda^k)} \int_E \left|v(x)\right|^2_{\Lambda^l}.$$

Proof. Pointwise, we have $|u \wedge v|_{\Lambda^{k+l}} \lesssim |u|_{\Lambda^k}|v|_{\Lambda^l}$, and by Section 2, $\|u\|_{L^\infty} = \operatorname{esssup}_M |u|_{\Lambda^k}$, from which the result follows on integration. □

Proposition 3.11. *Let $\phi \in W^{1,\infty}(\overline{M}, \mathbb{R})$ and $u \in \operatorname{dom} Q$. Then $\phi u \in \operatorname{dom} Q$ and*

$$Q(\phi u) \lesssim \|\phi\|^2_{W^{1,\infty}}\big[Q(u) + \|u\|^2_{L^2}\big].$$

Proof. First assume that ϕ and u are smooth. Then

$$\bar\partial(\phi u) = \phi\bar\partial u + \bar\partial\phi \wedge u, \qquad \bar\partial^*(\phi u) = \phi\bar\partial^* u - \star[\partial\phi \wedge \star u],$$

cf. [33, Lemma 3.10]. The fact that the Hodge $\star$ is an isometry gives

$$\begin{aligned} Q(\phi u) &= \left\|\bar\partial(\phi u)\right\|^2_{L^2} + \left\|\bar\partial^*(\phi u)\right\|^2_{L^2} \\ &= \|\phi\bar\partial u\|^2_{L^2} + \left\|\phi\bar\partial^* u\right\|^2_{L^2} + \|\bar\partial\phi \wedge u\|^2_{L^2} + \|\partial\phi \wedge \star u\|^2_{L^2} \\ &\quad + 2\,\mathfrak{Re}\langle\phi\bar\partial u, \bar\partial\phi \wedge u\rangle_{L^2} - 2\,\mathfrak{Re}\big\langle\phi\bar\partial^* u, \star[\partial\phi \wedge \star u]\big\rangle_{L^2}. \end{aligned}$$

With the previous lemma, Cauchy–Schwarz, and again the fact that $\star$ is an isometry, we obtain

$$\begin{aligned} \cdots &\lesssim \|\phi\|^2_{L^\infty} Q(u) + \|\bar\partial\phi\|^2_{L^\infty}\|u\|^2_{L^2} + \|\partial\phi\|^2_{L^\infty}\|u\|^2_{L^2} \\ &\quad + \|\phi\bar\partial u\|^2_{L^2} + \|\bar\partial\phi \wedge u\|^2_{L^2} + \left\|\phi\bar\partial^* u\right\|^2_{L^2} + \|\partial\phi \wedge \star u\|^2_{L^2}, \end{aligned}$$

and each of these terms is bounded by a constant multiple of the right-hand side in the assertion. Now drop the assumption of smoothness and choose $(\phi_k)_k \subset W^{1,\infty} \cap C^1$ so that $\phi_k \to \phi$, pointwise a.e. and with $\|\phi_k\|_{W^{1,\infty}} \leqslant \|\phi\|_{W^{1,\infty}}$, and similarly $(u_k)_k \subset C^\infty_c(\overline{M}, \Lambda^{p,q})$ so that

$$Q(u_k - u) \longrightarrow 0 \quad \text{and} \quad \|u_k - u\|_{L^2} \longrightarrow 0.$$

It follows that $\phi_k u_k \to \phi u$ in L^2 and $\sup Q(\phi_k u_k) < \infty$. Standard Fatou-type arguments [28] give that $\phi u \in \operatorname{dom} Q$ and

$$Q(\phi u) \leqslant \liminf Q(\phi_k u_k),$$

giving the assertion. □

4. The Caccioppoli inequality

As usual, we work under the assumption (A) from 1.1 above. Let us first introduce the notion of a generalized eigenform.

Definition 4.1. A form $u \in L^2_{\mathrm{loc}}(M, \Lambda^{p,q})$ is said to be a *generalized eigenform* for $\Box = \Box_{p,q}$ if:

1) $u \in \operatorname{dom}_{\mathrm{loc}} Q$. That is, for any compact $K \subset \overline{M}$ there is a $v \in \operatorname{dom} Q$ such that $v|_K = u|_K$.
2) There exists a $\lambda \in \mathbb{R}$ such that $Q(u, \phi) = \lambda \langle u, \phi \rangle$ for all $\phi \in C_c^\infty(\overline{M}, \Lambda^{p,q})$.

Remark 4.2. Note that $\operatorname{dom}_{\mathrm{loc}} Q$ is in $H^{\epsilon/2}_{\mathrm{loc}}$; see [17, Proposition 1.2]. Note also that the identity in 2) is a weak form of the equation $\Box u = \lambda u$.

By locality of the energy we can define $Q(u, \phi) = Q(v, \phi)$ provided $\operatorname{supp} \phi \subset K$ and v is as in the definition. Alternatively, we can write

$$Q(u, \phi) = \int \big\langle \bar\partial u(x), \bar\partial \phi(x) \big\rangle_{\Lambda^{p,q+1}} + \big\langle \bar\partial^* u(x), \bar\partial^* \phi(x) \big\rangle_{\Lambda^{p,q-1}} \, d\mu(x)$$

noting that the integral is convergent. Moreover, we have that

$$u \in \operatorname{dom}_{\mathrm{loc}} Q \quad \Leftrightarrow \quad \bar\partial u \in L^2_{\mathrm{loc}}\big(M, \Lambda^{p,q+1}\big) \quad \text{and} \quad \bar\partial^* u \in L^2_{\mathrm{loc}}\big(M, \Lambda^{p,q-1}\big).$$

The Caccioppoli inequality states that for any generalized eigenform u, the energy

$$M \ni x \longmapsto \big\langle \bar\partial u(x), \bar\partial u(x) \big\rangle_{\Lambda^{p,q+1}} + \big\langle \bar\partial^* u(x), \bar\partial^* u(x) \big\rangle_{\Lambda^{p,q-1}}$$

is locally bounded by the L^2-norm of u. We follow the strategy of [5] in what follows; see also [2, Proposition 3], for a similar result. The authors of the former paper had not been aware of the latter at the time their paper appeared. The reader should note one important difference between the result in [2] and in the following result: u is not supposed to be in the domain of the operator, or even locally in the Sobolev space H^2.

Theorem 4.3 *(Caccioppoli inequality). For any generalized eigenform u of $\Box_{p,q}$ associated to an eigenvalue $\lambda \geqslant 0$, every compact set $E \subset \overline{M}$, and every $b \in (0,1]$ we have*

$$\int_E |\bar\partial u|^2_{\Lambda^{p,q+1}} + \big|\bar\partial^* u\big|^2_{\Lambda^{p,q-1}} \leqslant 2\lambda \int_E |u|^2_{\Lambda^{p,q}} + \frac{4}{b^2} \int_{B_b(E)} |u|^2_{\Lambda^{p,q}}.$$

Remark 4.4. Note that in order to control the energy on E we need to take the L^2-norm on a slightly larger set $B_b(E)$, the b-neighborhood of E.

Proof. Pick a cutoff function $\eta = \zeta \circ \rho_E$ as constructed in Section 3.1, so that $|\bar{\partial}\eta| \leqslant 2/b$, $\eta|_{\bar{E}} \equiv 1$, and $\eta|_{B_b(E)^c} \equiv 0$. The eigenvalue equation

$$Q(u,\phi) = \lambda\langle u,\phi\rangle_{L^2(M,\Lambda^{p,q})} \quad \bigl(\phi \in C_c^\infty\bigl(\overline{M},\Lambda^{p,q}\bigr)\bigr),$$

extends to arbitrary $\phi \in \operatorname{dom} Q$ by approximation. Therefore we may calculate

$$\begin{aligned}
\lambda\bigl\langle u,\eta^2 u\bigr\rangle &= Q\bigl(u,\eta^2 u\bigr) \\
&= \int\limits_{B_b(E)} \bigl\langle \bar{\partial}u, \bar{\partial}\bigl(\eta^2 u\bigr)\bigr\rangle + \bigl\langle \bar{\partial}^* u, \bar{\partial}^*\bigl(\eta^2 u\bigr)\bigr\rangle \\
&= \int\limits_{B_b(E)} \eta^2\bigl[|\bar{\partial}u|^2 + \bigl|\bar{\partial}^* u\bigr|^2\bigr] + \bigl\langle \bar{\partial}u, \bar{\partial}\eta^2 \wedge u\bigr\rangle - \bigl\langle \bar{\partial}^* u, \star\bigl[\partial\eta^2 \wedge \star u\bigr]\bigr\rangle.
\end{aligned}$$

Leibniz' rule gives

$$\cdots = \int\limits_{B_b(E)} \eta^2\bigl[|\bar{\partial}u|^2 + \bigl|\bar{\partial}^* u\bigr|^2\bigr] + 2\langle \eta\bar{\partial}u, \bar{\partial}\eta \wedge u\rangle - 2\bigl\langle \eta\bar{\partial}^* u, \star[\partial\eta \wedge \star u]\bigr\rangle.$$

Now rearrange terms, apply Cauchy–Schwarz, and Lemma 3.10 to get

$$\begin{aligned}
\int\limits_{B_b(E)} \eta^2\bigl[|\bar{\partial}u|^2 + \bigl|\bar{\partial}^* u\bigr|^2\bigr] &\leqslant \lambda\bigl\langle u,\eta^2 u\bigr\rangle + 2\|\eta\bar{\partial}u\|\|\bar{\partial}\eta \wedge u\| + 2\bigl\|\eta\bar{\partial}^* u\bigr\|\|\partial\eta \wedge \star u\| \\
&\leqslant \lambda \int\limits_{B_b(E)} \eta^2|u|^2 + \frac{1}{2}\bigl[\bigl\|\eta\bar{\partial}^* u\bigr\|^2 + 4\|\partial\eta \wedge \star u\|^2\bigr].
\end{aligned}$$

The second term on the right-hand side is $1/2$ the left-hand side so we have

$$\frac{1}{2}\int\limits_{B_b(E)} \eta^2\bigl[|\bar{\partial}u|^2 + \bigl|\bar{\partial}^* u\bigr|^2\bigr] \lesssim \lambda \int\limits_{B_b(E)} |u|^2_{\Lambda^{p,q}} + \frac{2}{b}\int\limits_{B_b(E)\setminus E} |u|^2_{\Lambda^{p,q}},$$

which yields the assertion since $\eta \equiv 1$ on E. $\square$

5. Subexponentially bounded eigenforms induce spectrum

Here we closely follow [5], see also [25,26]. The treatment here rests on two main observations. The first is a criterion for $\lambda \in \sigma(H)$ in terms of the quadratic form h associated with H. *Singular sequences* or *Weyl sequences* for H and λ are sequences $(f_n)_{n\in\mathbb{N}} \subset \operatorname{dom}(H)$ that satisfy $\|f_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\|Hf_n - \lambda f_n\| \longrightarrow 0 \quad \text{for } n \to \infty.$$

Clearly, the existence of such a sequence implies that $\lambda \in \sigma(H)$. However, due to the requirement $f_n \in \mathrm{dom}(H)$, such singular sequences may be hard to construct. The next proposition gives a quadratic form version, which clearly is easier to find. Note that the terms "singular sequence" and "Weyl sequence" are most commonly used in a stricter sense; namely, it is required that, additionally, $f_n \xrightarrow{w} 0$. In this case one even gets that λ lies in the essential spectrum of H. Our criterion below does not require this. E.g., for an eigenvalue λ one could simply take $f_n = f$, where f is a normalized eigenelement of H with eigenvalue λ.

Throughout this section we assume (A) from 1.1.

Proposition 5.1 *(Weyl type criterion). Let h be a closed semibounded form and let H be the associated self-adjoint operator. Then the following are equivalent*:

1) $\lambda \in \sigma(H)$.
2) *There exists a sequence $(u_k)_k$ in* $\mathrm{dom}\, h$ *with $\|u_k\| \to 1$ and*

$$\sup\bigl\{\bigl|h(u_k, v) - \lambda\langle u_k, v\rangle_{L^2}\bigr| \bigm| v \in \mathrm{dom}\, h,\ \|v\|_h \leqslant 1\bigr\} \longrightarrow 0$$

for $k \to \infty$.

For the proof see [43, Lemma 1.4.4] and [10].

As a last ingredient for the main result, let us introduce the *inner b-collar* of a set $E \subset M$, given by

$$\mathcal{C}_b(E) = \bigl\{x \in E \bigm| d\bigl(x, E^c\bigr) \leqslant b\bigr\}.$$

Theorem 5.2 (1/2*-Schnol). Assume that $\lambda \in \mathbb{R}$ admits a generalized eigenform u so that there exists a sequence E_k of compact subsets of $\overline{M}$ and $b > 0$ with*

$$\frac{\|u\mathbf{1}_{\mathcal{C}_b(E_k)}\|_{L^2}}{\|u\mathbf{1}_{E_k}\|_{L^2}} \longrightarrow 0 \quad \textit{as } k \to \infty.$$

Then $\lambda \in \sigma(H)$.

Proof. Let us first calculate, for $\eta \in W_c^{1,\infty}(\overline{M}, \mathbb{R})$ and $u, v \in \mathrm{dom}_{\mathrm{loc}}\, Q^{p,q}$,

$$\begin{aligned} Q(\eta u, v) - Q(u, \eta v) &= \int_M \bigl\langle\bar\partial(\eta u), \bar\partial v\bigr\rangle - \bigl\langle\bar\partial u, \bar\partial(\eta v)\bigr\rangle + \bigl\langle\bar\partial^*(\eta u), \bar\partial^* v\bigr\rangle - \bigl\langle\bar\partial^* u, \bar\partial^*(\eta v)\bigr\rangle \\ &= \int_M \langle\bar\partial\eta \wedge u, \bar\partial v\rangle - \langle\bar\partial u, \bar\partial\eta \wedge v\rangle + \cdots \\ &\quad + \bigl\langle\star[\partial\eta \wedge \star u], \bar\partial^* v\bigr\rangle - \bigl\langle\bar\partial^* u, \star[\partial\eta \wedge \star v]\bigr\rangle. \end{aligned} \tag{4}$$

Now choose a sequence E_k as in the assumptions and define

$$F_k = \bigl\{x \in E_k \bigm| d\bigl(x, E_k^c\bigr) \geqslant b/2\bigr\},$$

with which we will define suitable cutoff functions. So pick $\zeta \in C^1(\mathbb{R})$ with $0 \leqslant \zeta \leqslant 1$, $\zeta|_{(-\infty,0]} \equiv 1$, $\zeta|_{[b/4,\infty)} \equiv 0$, and $\sup|\zeta'| \leqslant 8/b$. Note that $\eta_k := \zeta \circ \rho_{F_k} \in W_c^{1,\infty}(\overline{M}, \mathbb{R})$ satisfies $0 \leqslant \eta_k \leqslant \mathbf{1}_{B_{b/4}(F_k)}$ and $\operatorname{supp}|\partial\eta_k| \subset B_{b/4}(F_k) \setminus F_k =: G_k$. Moreover, note that $B_{b/4}(G_k) \subset C_b(E_k)$.

We now show that

$$u_k = \frac{\eta_k u}{\|\eta_k u\|_{L^2}}$$

gives an approximate eigensequence as required by the Weyl criterion above.

Let $v_k = \eta_k u$. For $v \in \operatorname{dom} Q^{p,q}$ with $\|v\|_Q \leqslant 1$, we estimate

$$\begin{aligned} Q(u_k, v) - \lambda\langle u_k, v\rangle_{L^2} &= \frac{1}{\|v_k\|_{L^2}}\big[Q(v_k, v) - \lambda\langle v_k, v\rangle_{L^2}\big] \\ &= \frac{1}{\|v_k\|_{L^2}}\big[Q(\eta_k u, v) - \lambda\langle u, \eta_k v\rangle_{L^2}\big] \\ &= \frac{1}{\|v_k\|_{L^2}}\big[Q(\eta_k u, v) - Q(u, \eta_k v)\big], \end{aligned}$$

since η_k is real-valued. We have used that u is a generalized eigenform with eigenvalue λ and the fact, discussed in the proof of Caccioppoli's inequality, that $\eta_k v$ can be taken to be a test function. Now, as in (4),

$$\begin{aligned} Q(u_k, v) - \lambda\langle u_k, v\rangle_{L^2} = \frac{1}{\|v_k\|_{L^2}} \int_M &\langle\bar\partial\eta_k \wedge u, \bar\partial v\rangle - \langle\bar\partial u, \bar\partial\eta_k \wedge v\rangle \\ &- \big\langle \star[\partial\eta_k \wedge \star u], \bar\partial^* v\big\rangle + \big\langle\bar\partial^* u, \star[\partial\eta_k \wedge \star v]\big\rangle. \end{aligned}$$

Due to the support properties of the η_k, we know that

$$\begin{aligned} \cdots &\lesssim \frac{1}{\|v_k\|_{L^2}}\|\bar\partial\eta_k\|_\infty\big[\|u\|_{L^2(G_k)}\|\bar\partial v\|_{L^2} + \|u\|_{L^2(G_k)}\big\|\bar\partial^* v\big\|_{L^2}\big] \\ &\quad + \frac{1}{\|v_k\|_{L^2}}\|\bar\partial\eta_k\|_\infty\big[\|\bar\partial u\|_{L^2(G_k)}\|v\|_{L^2} + \big\|\bar\partial^* u\big\|_{L^2(G_k)}\|v\|_{L^2}\big] \\ &\lesssim \frac{1}{\|v_k\|_{L^2}}\big[\|u\|_{L^2(G_k)}\|v\|_Q + \|\bar\partial u\|_{L^2(G_k)}\|v\|_{L^2} + \big\|\bar\partial^* u\big\|_{L^2(G_k)}\|v\|_{L^2}\big]. \end{aligned}$$

Now apply Caccioppoli with $E = G_k$ and $b/4$ to get

$$\begin{aligned} \cdots &\lesssim \frac{1}{\|v_k\|_{L^2}}\big[\|u\mathbf{1}_{G_k}\|_{L^2} + \|u\|_{L^2(B_{b/4}(G_k))}\big] \\ &\lesssim \frac{\|u\mathbf{1}_{C_b(E_k)}\|_{L^2}}{\|u\mathbf{1}_{E_k}\|_{L^2}} \end{aligned}$$

since $0 \leqslant \eta_k \leqslant \mathbf{1}_{E_k}$ and $B_{b/4}(G_k) \subset C_b(E_k)$. □

Generalizing the notion from statistical mechanics, let us call a sequence (E_k) a *van Hove sequence* if it has the property that

$$\frac{\operatorname{vol} C_b(E_k)}{\operatorname{vol} E_k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty,$$

for some $b > 0$.

Corollary 5.3. *Assume that $\overline{M}$ admits a van Hove sequence. It follows that $0 \in \sigma(\Box_{0,0})$ and 1 is a generalized eigenfunction.*

Proof. Clearly, 1 is a generalized eigenfunction for the eigenvalue 0. By the preceding remark, it satisfies the requirement for the theorem above. □

Apart from certain uniformities, the G-invariance which we assume throughout this treatment is certainly too strong a condition to impose. Note that a suitable generalization of the theorem above allows for manifolds with very different geometries in different "directions to infinity." One such direction which supports a van Hove sequence is sufficient for 0 to be in the spectrum of $\Box_{0,0}$.

We now add some sufficient conditions for the assumptions in the theorem. They rest on the following notions: A function $J : [0,\infty) \to [0,\infty)$ is said to be *subexponentially bounded* if for any $\alpha > 0$ there exists a $C_\alpha > 0$ such that

$$J(r) \leqslant C_\alpha e^{\alpha r} \quad (r \geqslant 0).$$

Similarly, a form $u \in L^2_{\mathrm{loc}}(M, \Lambda^{0,1})$ will also be called subexponentially bounded if for some $z_0 \in \overline{M}$,

$$e^{-\alpha w} u \in L^2\big(M, \Lambda^{p,q}\big)$$

for any $\alpha > 0$, where $w(z) = d(z, z_0)$.

As in Lemmata 4.2, 4.3, and Theorem 4.4 of [5], we obtain:

Corollary 5.4. *Assume that $\lambda \in \mathbb{R}$ admits a subexponentially bounded eigenform for $\Box$. It follows that $\lambda \in \sigma(\Box)$.*

This or the previous corollary has as a special case the following.

Corollary 5.5. *Assume that there is a $z_0 \in \overline{M}$ such that $r \mapsto \operatorname{vol} B_r(z_0)$ is subexponentially bounded. It follows that $0 \in \sigma(\Box)$.*

Remark 5.6. See the example in [9, §5].

During the writing of [5,25] we were not aware of M. Shubin's papers [39,40], where strongly related results are presented. The main difference is that our approach is based on the underlying forms, making it applicable in cases where nothing is known about the domain of the operator. On the other hand, the latter papers contain results about higher order elliptic operators.

6. Expansion in generalized eigenforms

Here we prove Theorem 3, in fact the stronger result Proposition 6.3 below, where assumption (A) from 1.1 is required, as usual.

Some explanations are in order: *spectrally a.e.* means a.e. with respect to a spectral measure; in turn, a spectral measure ρ is a measure with the property that $\rho(I) = 0$ if and only if $E_I(\Box) = 0$, where $E.(\Box)$ denotes the spectral projection of the operator $\Box$.

The strategy of proof is sufficiently parallel to the one in [4] so that we do not carry out all the details but rather point at differences; we fix integers $p \geqslant 0, q > 0$ so that the pseudolocal estimate holds true. This latter condition is important in that we use ultracontractivity established in [33], *i.e.*, $e^{-t\Box} : L^2(M, \Lambda^{p,q}) \to L^\infty(M, \Lambda^{p,q})$ for $t > 0$. The compactness property referred to above is contained in the following:

Lemma 6.1. *In the situation of the theorem above let $\gamma(x) := e^{-tx}$ and $T := M_{\omega^{-1}}$ the multiplication operator. Then $\gamma(\Box)T^{-1}$ is Hilbert–Schmidt.*

Proof. This follows from the factorization principle based on Grothendieck's theorem. See [8] for the abstract background and [4] for an application in a situation similar to ours.

Indeed, for bounded operators, from

$$A : L^2 \longrightarrow L^\infty, \qquad B : L^\infty \longrightarrow L^2,$$

it follows that $BA : L^2 \to L^2$ is a Hilbert–Schmidt operator. We can apply this to deduce that

$$\big(\gamma(\Box)T^{-1}\big)^* = \big(T^{-1}\big)^*\gamma(\Box)^*$$

is Hilbert–Schmidt: $\gamma(\Box) : L^2 \to L^\infty$ is the above mentioned ultracontractivity and $T^{-1} = M_\omega : L^\infty \to L^2$, since ω is an L^2 function. Since the adjoint of a Hilbert–Schmidt operator is likewise Hilbert–Schmidt, we have the result. □

Suppressing the indices p, q, let

$$\mathcal{H}_+ := \big\{\alpha \in L^2\big(M, \Lambda^{p,q}\big) \bigm| \alpha \in \operatorname{dom}(T)\big\},$$

and $\mathcal{H}_-$, the completion of $\mathcal{H} := L^2(M, \Lambda^{p,q})$ with respect to the inner product $\langle\alpha, \beta\rangle_- := \langle T^{-1}\alpha, T^{-1}\beta\rangle_{\mathcal{H}}$. We have a special case of a Gelfand triple here, considering on $\mathcal{H}_+$ the inner product $\langle\alpha, \beta\rangle_+ := \langle T\alpha, T\beta\rangle_{\mathcal{H}}$.

Remark 6.2. We have that

$$C_c^\infty\big(M, \Lambda^{p,q}\big) \subset \big\{\alpha \in \operatorname{dom}(\Box) \cap \operatorname{dom}(T) \bigm| \Box\alpha \in \operatorname{dom}(T)\big\}$$

is dense in $\mathcal{H}$. In the next section we will prove much more, namely that $C_c^\infty(\overline{M}, \Lambda^{p,q})$ is a core for $\Box$. Note the important difference between M and $\overline{M}$ here.

In the following result we see a much stronger though more technical version of the theorem above. It uses the notion of an *ordered spectral representation*, that goes as follows: Given is a self-adjoint operator H in some Hilbert space $\mathcal{H}$, a spectral measure ρ of H, $N \in \mathbb{N} \cup \{\infty\}$ a sequence $(M_j)_{j<\infty}$ of measurable subsets $M_j \subset \mathbb{R}$ so that $M_j \supset M_{j+1}$ and a unitary

$$U = (U_j)_{j<\infty} : \mathcal{H} \longrightarrow \bigoplus_{j<N} L^2(M_j, \rho)$$

so that

$$U\varphi(\Box) = M_\varphi U$$

for every bounded measurable function φ on $\mathbb{R}$.

Proposition 6.3. *Let ρ be a spectral measure for $\Box$ and $U = (U(j))_{j<N}$, $N \in \mathbb{N} \cup \{\infty\}$, an ordered spectral representation for $\Box$. Also let ω, T, $\mathcal{H}_+$ and $\mathcal{H}_-$ be as above. Then there are measurable functions $M_j \to \mathcal{H}_-$, $\lambda \mapsto \varepsilon_{j,\lambda}$ for $j \in \mathbb{N}$, $j < N$ such that*:

(1) *$U_j\alpha(\lambda) = \langle \alpha, \varepsilon_{j,\lambda}\rangle$ for $\alpha \in \mathcal{H}_+$ and ρ-a.e. $\lambda \in M_j$.*
(2) *For every $g = (g_j)_{j<N} \in \bigoplus_{j<N} L^2(M_j, \rho)$ we have*

$$U^{-1}g = \lim_{m\to N, R\to\infty} \sum_{j=1}^{m} \int_{M_j\cap[-R,R]} g_j(\lambda)\varepsilon_{j,\lambda}\, d\rho(\lambda),$$

and therefore, for every $\alpha \in \mathcal{H}$,

$$\alpha = \lim_{m\to N, R\to\infty} \sum_{j=1}^{m} \int_{M_j\cap[-R,R]} U_j\alpha(\lambda)\varepsilon_{j,\lambda}\, d\rho(\lambda).$$

(3) *If $\alpha \in \operatorname{dom}(\Box) \cap \mathcal{H}_+$ with $\Box\alpha \in \mathcal{H}_+$, then*

$$\langle \Box\alpha, \varepsilon_{j,\lambda}\rangle = \lambda\langle \alpha, \varepsilon_{j,\lambda}\rangle \quad \text{for } \rho\text{-a.e. } \lambda \in M_j.$$

For details on ordered spectral representations, see [34]; this reference is the basis for our proof of the eigenform expansion.

Part (3) of the above proposition ensures that

$$\Box\varepsilon_{j,\lambda} = \lambda\varepsilon_{j,\lambda}$$

in the weak sense. This is why we speak of a *generalized eigenform*.

Remark 6.4. Due to the interior ellipticity of $\Box$, [13, Theorem 2.2.9] we obtain that the eigenforms constructed above are in $C^\infty(M, \Lambda^{p,q})$ for $q > 0$.

7. Essential self-adjointness of $\Box$

As we explained in the introduction, $\Box$ is defined via its sesquilinear form, so its domain $\mathrm{dom}(\Box)$ is only given implicitly. In the previous sections we have seen that even without explicit knowledge of its domain we can analyze important properties of $\Box$.

On the other hand it is known for manifolds without boundary that elliptic operators are typically essentially self-adjoint on smooth compactly supported forms, see *e.g.* [39–41] and the literature cited there. Thus it is a natural question whether the same holds true in the situation at hand with two important differences: there is a boundary, and we do not have ellipticity but only subellipticity.

Essential self-adjointness means that there is a unique self-adjoint extension of $\Box|_{\mathrm{dom}_c}$ and this is in turn equivalent to the fact that $\mathrm{dom}_c := \mathrm{dom}_c \Box := \mathrm{dom}(\Box) \cap C_c^\infty(\overline{M}, \Lambda^{p,q})$ is a *core for* $\Box$, *i.e.*, $\overline{\Box|_{\mathrm{dom}_c}} = \Box$, where $\bar{T}$ denotes, as usual, the closure of the operator T. We want to point out that there is a big difference due to the boundary: in the usual complete case without boundary, the so-called *minimal operator*, defined on $C_c^\infty(\overline{M}, \Lambda^{p,q})$ is essentially self-adjoint. This fails in our situation. There are various different self-adjoint extensions. E.g., the operator $\Box_{p,q}$ we consider is obviously different from the Friedrichs extension of $\Box_{p,q}|_{C_c^\infty(M,\Lambda^{p,q})}$ which would usually be called the $\Box$ with Dirichlet boundary conditions, and which has a smaller form domain.

A first step in showing the asserted essential self-adjointness is the following result from [17]. As usual, assumption (A) from 1.1 is in force. Here and for what follows we fix ρ to be the (positive) distance to bM as given by a G-invariant Riemannian metric on M.

Proposition 7.1. *Let ϑ be the formal adjoint operator to $\bar\partial$, and denote by $\sigma = \sigma(\vartheta, \cdot)$ its principal symbol. Assume also that $q > 0$ and let $\Box = \Box_{p,q}$. Then*

$$\mathrm{dom}_0 \Box := \bigl\{u \in C^\infty\bigl(M, \Lambda^{p,q}\bigr) \bigm| u, \bar\partial u, \vartheta u \in L^2, \\ \sigma(\vartheta, d\rho)u|_{bM} = 0,\ \sigma(\vartheta, d\rho)\bar\partial u|_{bM} = 0\bigr\}$$

is a core for $\Box$.

Proof. Let $u \in \mathrm{dom}\,\Box_{p,q}$. Then $\Box u + u = \alpha \in L^2$. Now let $(\alpha_k)_k \subset C_c^\infty(\overline{M}, \Lambda^{p,q})$, so that $\alpha_k \to \alpha$ in L^2 and put $u_k = (\Box + \mathbf{1})^{-1}\alpha_k$. Since $(\Box + \mathbf{1})^{-1}$ is defined everywhere and is bounded, we have that $(u_k)_k$ is Cauchy in L^2 with limit u. Applying the pseudolocal estimate [11], [33, Theorem 2.4] we have

$$\|\zeta u_k\|_{H^{s+\varepsilon}} = \bigl\|\zeta(\Box + \mathbf{1})^{-1}\alpha_k\bigr\|_{H^{s+\varepsilon}} \lesssim \bigl\|\zeta'\alpha_k\bigr\|_{H^s} + \|\alpha_k\|_{L^2},$$

thus $(u_k)_k \subset C^\infty(\overline{M}, \Lambda^{p,q})$ and we have shown that the assertion is true. $\Box$

For the proof of essential self-adjointness we need some geometrical tools. First recall the following:

Definition 7.2. (See [13, p. 33], [44, §2.2].) A *special boundary chart U* is a chart intersecting bM having the following properties:

(1) With ρ the function defining bM as above, the functions $t := \{t_1, \dots, t_{2n-1}\}$, together with ρ form a coordinate system on U.
(2) The functions $\{t, \rho = 0\}$ form a coordinate system on $bM \cap U$.
(3) With respect to the Riemannian structure in the cotangent bundle, choose a local orthonormal basis $\omega_1, \dots, \omega_n$ for $C^\infty(U, \Lambda^{1,0})$ such that $\omega_n = \sqrt{2}\,\partial\rho$ on U.

Let us describe $\operatorname{dom}\Box$ by restating the boundary conditions as in [13, §5.2]. In terms of the Hermitian structure $\langle\ ,\ \rangle_\Lambda$ in $\Lambda^{p,q}$, the above conditions on the symbol $\sigma(\vartheta, d\rho)$ translate to the following criteria. Members of $\operatorname{dom}_0\Box$ are those forms $\phi \in C^\infty(\overline{M}, \Lambda^{p,q})$ satisfying the following $\bar\partial$-Neumann boundary conditions:

(1) $\langle\phi, \bar\partial\rho \wedge \psi\rangle_\Lambda|_{bM} = 0$ ($\psi \in \Lambda^{p,q-1}$), and
(2) $\langle\bar\partial\phi, \bar\partial\rho \wedge \psi\rangle_\Lambda|_{bM} = 0$ ($\psi \in \Lambda^{p,q}$).

The first condition (equivalent to $\phi \in \operatorname{dom}_0\vartheta$) is obviously preserved by introduction of a cutoff function $\phi \to \chi\phi$ since the condition is algebraic.

The second "free boundary" condition becomes

$$\big\langle\bar\partial(\chi\phi), \bar\partial\rho \wedge \psi\big\rangle_\Lambda|_{bM} = \big\langle(\bar\partial\chi) \wedge \phi, \bar\partial\rho \wedge \psi\big\rangle_\Lambda|_{bM} + \langle\chi\bar\partial\phi, \bar\partial\rho \wedge \psi\rangle_\Lambda|_{bM} = 0.$$

Upon restriction to the boundary, the second term is zero by assumption that $\phi \in \operatorname{dom}\Box$, which assumes that $\bar\partial\phi \in \operatorname{dom}\bar\partial^*$. Thus we are interested in the condition

$$\big\langle(\bar\partial\chi) \wedge \phi, \bar\partial\rho \wedge \psi\big\rangle_\Lambda|_{bM} = 0, \quad \forall\psi \in \Lambda^{p,q}.$$

In terms of the forms defined in the special boundary chart, we have the formulas

$$\bar\partial\rho = \frac{1}{\sqrt{2}}\bar\omega^n, \qquad \bar\partial\chi = \sum_k (\bar L_k\chi)\bar\omega^k$$

so cutoff functions χ satisfying

$$\bar L_n\chi|_{bM} = 0 \tag{5}$$

preserve $\operatorname{dom}\Box$. Notice that there are no other restrictions on $\chi \in C^\infty(\overline{M})$ beyond this one at the boundary, so χ satisfying (5) may be extended smoothly to the interior of M in an arbitrary way.

We may write the relation (5) in such a way that manifestly separates the tangential and normal derivatives of χ, as indicated in [3, p. 86]. First note that since L_n is dual to $\omega^n = \sqrt{2}\partial\rho$, we have

$$L_n\rho = d\rho(L_n) = \big\langle(\partial + \bar\partial)\rho, \omega^n\big\rangle_\Lambda = \sqrt{2}\langle\partial\rho, \partial\rho\rangle_\Lambda,$$

and similarly $\bar L_n\rho = \sqrt{2}\langle\bar\partial\rho, \bar\partial\rho\rangle_\Lambda$. It follows that $(\bar L_n - L_n)\rho = 0$ and thus $\bar L_n - L_n$ is a vector field tangential to bM. If J is the complex structure, then L_n and $\bar L_n$ lie in the i and $-i$ eigenspaces of J, respectively and

$$J(\bar L_n - L_n) = -i(\bar L_n + L_n)$$

must not be tangential; indeed, $(\bar{L}_n + L_n)\rho = 2\sqrt{2}\langle\bar{\partial}\rho, \bar{\partial}\rho\rangle \neq 0$. The same calculations provide that the equation

$$-iJ(\bar{L}_n - L_n)\chi = (\bar{L}_n - L_n)\chi \tag{6}$$

(in bM) is equivalent to the property $\bar{L}_n\chi|_{bM} = 0$. Since only the normal derivative is prescribed at the boundary, it follows that given any smooth function χ in bM, there exists an extension to a collar of bM which fulfills the requirement (5), *cf.* Lemma 7.5 below.

Definition 7.3. A sequence of functions $(\chi_k)_k$ in $C_c^\infty(\overline{M}, \mathbb{R})$ is called a *good cutoff-exhaustion of* $\overline{M}$ if

(C1) $\chi_k \to 1$ as $k \to \infty$,
(C2) $\bar{L}_n\chi_k|_{bM} = 0$ for all $k \in \mathbb{N}$, and
(C3) $\sup\{\|\partial^\alpha\chi_k\|_\infty, |\alpha| \leqslant m\} < \infty$, for any $m \in \mathbb{N}$,

where the derivatives in the last condition are with respect to geodesic coordinates. Note that $\bar{L}_n$ is globally defined in a collar of the boundary of M.

Our goal here will be to demonstrate the existence of good cutoff-exhaustions of $\overline{M}$ and to use such a sequence to show that dom_c is a core for $\Box$. We start with:

Proposition 7.4. *Let U be a special boundary chart and $\chi \in C_c^\infty(U, \mathbb{R})$ with $\bar{L}_n\chi|_{bM} = 0$. Then, for any $u \in \mathrm{dom}\,\Box_{p,q}$ with $q > 0$ it follows that*

$$\chi u \in \mathrm{dom}\,\Box$$

and

$$\big\|\Box(\chi u)\big\|_{L^2} \lesssim \underbrace{\sup\big\{\big\|\partial^\alpha\chi\big\|_\infty, |\alpha| \leqslant 2\big\}}_{=\|\chi\|_{W^{2,\infty}}} \cdot \big(\|\Box u\|_{L^2}^2 + \|u\|_{L^2}^2\big)^{\frac{1}{2}}. \tag{7}$$

Proof. The factor $(\|\Box u\|_{L^2}^2 + \|u\|_{L^2}^2)^{\frac{1}{2}}$ appearing above is called the operator norm $\|u\|_\Box$ of u. It dominates the form norm $\|u\|_Q$ in the sense that

$$\|u\|_Q := \big(Q(u,u) + \|u\|_{L^2}^2\big)^{\frac{1}{2}} \lesssim \|u\|_\Box.$$

Let us first consider the case that $u \in \mathrm{dom}_0\,\Box_{p,q}$. By the calculation above, $\chi u \in \mathrm{dom}\,\Box$. For the proof of the estimate (7), we use the following straightforward calculation:

$$\begin{aligned}\bar{\partial}(\chi u) &= (\bar{\partial}\chi) \wedge u + \chi\bar{\partial}u,\\ \bar{\partial}^*(\chi u) &= (-\star\partial\star)(\chi u)\\ &= -\star\big[(\partial\chi) \wedge \star u\big] + \chi\bar{\partial}^* u,\end{aligned}$$

from which we get that

$$\begin{aligned}
&\bigl|\langle\Box(\chi u), v\rangle - \langle\chi\Box u, v\rangle\bigr| \\
&\quad = \bigl|Q(\chi u, v) - Q(u, \chi v)\bigr| \\
&\quad = \bigl|\langle\bar\partial(\chi u), \bar\partial v\rangle + \langle\bar\partial^*(\chi u), \bar\partial^* v\rangle - \langle\bar\partial u, \bar\partial(\chi v)\rangle - \langle\bar\partial^* u, \bar\partial^*(\chi v)\rangle\bigr| \\
&\quad = \bigl|\langle\bar\partial\chi \wedge u, \bar\partial v\rangle - \langle\bar\partial u, \bar\partial\chi \wedge v\rangle - \langle\star(\partial\chi \wedge \star u), \bar\partial^* v\rangle + \langle\bar\partial^* u, \star(\partial\chi \wedge \star v)\rangle\bigr|.
\end{aligned}$$

The first term can be estimated as

$$\bigl|\langle\bar\partial^*(\bar\partial\chi \wedge u), v\rangle\bigr| \lesssim \|\chi\|_{W^{2,\infty}}\|u\|_Q\|v\|_{L^2}$$

and similarly we can bound the third term. The second and fourth terms are easily bounded and we get

$$\bigl|\langle\Box(\chi u), v\rangle - \langle\chi\Box u, v\rangle\bigr| \lesssim \|\chi\|_{W^{2,\infty}}\|u\|_Q\|v\|_{L^2}$$

for arbitrary $v \in \operatorname{dom} Q$. Since the latter is dense in L^2, we obtain the estimate

$$\bigl\|\Box(\chi u) - \chi\Box u\bigr\| \lesssim \|\chi\|_{W^{2,\infty}}\|u\|_\Box.$$

Since

$$\|\chi\Box u\| \lesssim \|\chi\|_{W^{2,\infty}}\|u\|_\Box$$

is obvious, we arrive at the desired estimate. Since $\operatorname{dom}_0\Box$ is a core for $\Box$ the assertion carries over to arbitrary $u \in \operatorname{dom}\Box$. □

Before going on, let us note that due to the invariance under the group action and the compact quotient our manifold has bounded geometry. We rely on [37] for the definition and a number of nice technical properties that come with bounded geometry. The first is the existence of $r_c > 0$ so that the geodesic collar

$$j : N = [0, r_c) \times bM \longrightarrow M, \quad (\tau, x) \longmapsto \exp_x(\tau\nu_x)$$

is a diffeomorphism onto its image, with ν_x denoting the unit inward normal vector at x; so τ refers to the distance ρ to the boundary mentioned previously. Denote $j([0, \frac{1}{3}r_c) \times bM) =: N_{\frac{1}{3}}$ and define $N_{\frac{2}{3}}$ accordingly.

Lemma 7.5. *Let $U \subset N_{\frac{2}{3}}$ be a special boundary chart and $\varphi \in C_c^\infty(U, \mathbb{R})$. Then there exists a $\psi \in C_c^\infty(U, \mathbb{R})$ so that*

$$\psi|_{bM} = \varphi|_{bM}, \qquad \bar L_n\psi|_{bM} = 0. \tag{8}$$

Moreover, if $\varphi|_V = 1$ on a set of the form $V = j([0, r) \times R) \subset U$ with R a relatively open subset of bM, then $\psi|_V = 1$.

Proof. We set $\zeta = \psi - \varphi$, so we want the derivatives of ζ to satisfy

$$-iJ(\bar{L}_n - L_n)[\varphi + \zeta]|_{bM} = (\bar{L}_n - L_n)[\varphi + \zeta]|_{bM}.$$

We should have $\zeta|_{bM} = 0$, so we obtain $(\bar{L}_n - L_n)\zeta|_{bM} = 0$ since the vector field is tangent to bM, thus

$$-iJ(\bar{L}_n - L_n)[\varphi + \zeta]|_{bM} = (\bar{L}_n - L_n)\varphi|_{bM}$$

and so

$$-iJ(\bar{L}_n - L_n)\zeta|_{bM} = (\bar{L}_n - L_n)\varphi|_{bM} + iJ(\bar{L}_n - L_n)\varphi|_{bM} = 2\bar{L}_n\varphi|_{bM}.$$

Following the computations in [33, §3.2], one can derive that $L_n\zeta = d\zeta(L_n) = \langle d\zeta, \omega^n\rangle_\Lambda$ and likewise $\bar{L}_n\zeta = \langle d\zeta, \bar{\omega}^n\rangle_\Lambda$, so that

$$iJ(\bar{L}_n - L_n)\zeta = (L_n + \bar{L}_n)\zeta = \sqrt{2}\langle d\zeta, d\rho\rangle_{\Lambda^1} = \sqrt{2}\frac{\partial\zeta}{\partial\rho}.$$

In the special boundary chart U, we are left with solving the equations

$$\begin{cases} \left.\dfrac{\partial\zeta}{\partial\rho}(t,\rho)\right|_{\rho=0} = -\sqrt{2}\bar{L}_n\varphi(t,0), \\ \zeta(t,0) = 0. \end{cases}$$

Define now, for $r < \frac{2}{3}r_c$,

$$\zeta(t,r) := -\sqrt{2}\int_0^r d\rho\, \bar{L}_n\varphi(t,\rho).$$

It follows that a solution ψ to Eq. (8) exists. Clearly, it satisfies the required bound on the derivatives as well as the assertion on the level sets. □

Proposition 7.6. *There exists a good cutoff-exhaustion of M.*

Proof. We begin by constructing a sequence of functions with bounded derivatives that converges to 1. To this end, let $(\varphi_i)_{i\in\mathbb{Z}}$ be a partition of unity as in [37, Lemma 3.22]. Without loss of generality, we may choose the supports of these functions to have diameter smaller than $\frac{1}{3}r_c$ and to have a uniform bound on the number of j's for which the support of φ_j meet a given point. We fix $x_0 \in M$ and let

$$I_k^{(0)} := \left\{ i \in \mathbb{Z} \;\middle|\; \operatorname{supp}\varphi_i \cap B_{k+r_c}(x_0) \cap N_{\frac{1}{3}} \neq \emptyset \right\}.$$

Note that $\varphi_k^{(0)} := \sum_{i\in I_k^{(0)}} \varphi_i$ satisfies

$$1 \geqslant \varphi_k^{(0)} \geqslant 1_{B_{k+r_c}(x_0)\cap N_{\frac{1}{3}}}.$$

Due to the uniform bounds for the partition of unity,

$$\sup_k \left\| \varphi_k^{(0)} \right\|_{W^{m,\infty}} < \infty.$$

Note that $\varphi_k^{(0)} \in C_c^\infty(U, \mathbb{R})$, where U is the interior (in $\overline{M}$) of $B_{k+r_c+1}(x_0) \cap N_{\frac{2}{3}}$. The functions $\varphi_k^{(0)}$ build the "boundary part" of a smooth exhaustion we want to construct. We will now modify them in a way to make sure that the product with any function in dom_0 is in the domain $\mathrm{dom}(\Box)$.

To this end we use Lemma 7.5 to find $\psi_k^{(0)}$ for $\varphi_k^{(0)}$ so that $\psi_k^{(0)}$ satisfies the requirement from (8), *mutatis mutandis*. Moreover,

$$\psi_k^{(0)}\big|_{[0,\frac{1}{3}r_c)\times(B_k(x_0)\cap bM)} = 1,$$

since $\varphi_k^{(0)}$ is 1 on the respective set by definition and the triangle inequality.

Denote $I_k := \{i \in \mathbb{Z} \mid \operatorname{supp}\varphi_i \subset B_{k+r_c}(x_0)\}$ and

$$\psi_k^{(1)} := \big(1 - \psi_k^{(0)}\big) \cdot \sum_{i\in I_k} \varphi_i.$$

By the assumption on the support of the φ_i, the sum is 1 on $B_k(x_0)$ and $\operatorname{supp}\psi_k^{(1)} \subset B_{k+r_c}(x_0)$. In particular, $\psi_k^{(1)} \in C_c^\infty(M, \mathbb{R})$ and

$$\chi_k := \psi_k^{(0)} + \psi_k^{(1)}$$

is 1 on $B_k(x_0)$. Thus (C1) from the definition of a good cutoff-exhaustion above is satisfied. The function $\psi_k^{(0)}$ was constructed so that the required condition (5) holds and since $\psi_k^{(1)}$ is supported away from the boundary, χ_k satisfies (C2). The uniform bound on the derivatives is evident from the definition and the properties of the partition of unity. $\Box$

Proof of Theorem 3. We have to show that any $u \in \mathrm{dom}\,\Box$ can be approximated by a sequence (u_k) in $C_c^\infty(\overline{M}, \Lambda^{p,q})$ in the Hilbert space $(\mathrm{dom}\,\Box, \|\cdot\|_\Box)$. Since $\mathrm{dom}_0\,\Box$ is dense by Proposition 7.1 we can restrict to the case in which $u \in \mathrm{dom}_0\,\Box$. Since $C_c^\infty(\overline{M}, \Lambda^{p,q})$ is convex, its weak and norm closures in $(\mathrm{dom}\,\Box, \|\cdot\|_\Box)$ coincide, so we are left with finding (u_k) that converges weakly in the latter space. We take a good cutoff-exhaustion (χ_k) and claim that $u_k := \chi_k u$ does the job. By (C1) and (C3) we know that $u_k \to u$ in $L^2(M, \Lambda^{p,q})$ as $k \to \infty$. Moreover, by (7) in Proposition 7.4 it follows that (u_k) is bounded in $(\mathrm{dom}\,\Box, \|\cdot\|_\Box)$. It thus has a weakly convergent subsequence that has to converge to u by uniqueness of the limit and the L^2-convergence we already established. $\Box$

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