THE ALLEGRETTO-PIEPENBRINK THEOREM FOR STRONGLY LOCAL FORMS

DANIEL LENZ[1], PETER STOLLMANN[2], AND IVAN VESELIĆ[3]

Dedicated to Jürgen Voigt in celebration of his 65th birthday

Abstract. The existence of positive weak solutions is related to spectral information on the corresponding partial differential operator.

Introduction

The Allegretto-Piepenbrink theorem relates solutions and spectra of 2nd order partial differential operators H and has quite some history, cf [1, 4, 5, 6, 24, 25, 26, 27, 28].

One way to phrase it is that the supremum of those real E for which a nontrivial positive solution of $H\Phi = E\Phi$ exists coincides with the infimum of the spectrum of H. In noncompact cases this can be sharpened in the sense that nontrivial positive solutions of the above equation exist for all $E \leq \inf \sigma(H)$.

In the present paper we consider the Allegretto-Piepenbrink theorem in a general setting in the sense that the coefficients that are allowed may be very singular. In fact, we regard $H = H_0 + \nu$, where H_0 is the generator of a strongly local Dirichlet form and ν is a suitable measure perturbation. Let us stress, however, that one main motivation for the present work is the conceptual simplicity that goes along with the generalisation.

The Allegretto-Piepenbrink theorem as stated above consists of two statements: the first one is the fact that positive solutions can only exist for E below the spectrum. Turned around this means that the existence of a nontrivial positive solution of $H\Phi = E\Phi$ implies that $H \geq E$. For a strong enough notion of positivity, this comes from a "ground state transformation". We present this simple extension of known classical results in Section 2, after introducing the necessary set-up in Section 1. For the ground state transformation not much structure is needed.

For the converse statement, the existence of positive solutions below $\sigma(H)$, we need more properties of H and the underlying space: noncompactness, irreducibility and what we call a Harnack principle. All these analytic properties are well established in the classical case. Given these tools, we prove this part of the Allegretto-Piepenbrink theorem in Section 3 with arguments reminiscent of the corresponding discussion in [12]. For somewhat complementary results we refer to [14] where it is shown that existence of a nontrivial subexponentially bounded solution of $H\Phi = E\Phi$ yields that $E \in \sigma(H)$. This implies, in particular, that the positive solutions we construct for energies below the spectrum cannot behave to well near infinity.

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We dedicate this paper to Jürgen Voigt - teacher, collaborator and friend - in deep gratitude and wish him many more years of fun in analysis.

1. Basics and notation concerning strongly local Dirichlet forms and measure perturbations

Dirichlet forms. We will now describe the set-up; we refer to [15] as the classical standard reference as well as [10, 13, 16, 22] for literature on Dirichlet forms. Let us emphasize that in contrast to most of the work done on Dirichlet forms, we treat real and complex function spaces at the same time and write $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

Throughout we will work with a locally compact, separable metric space X endowed with a positive Radon measure m with supp$m = X$.

The central object of our studies is a regular Dirichlet form $\mathcal{E}$ with domain $\mathcal{D}$ in $L^2(X)$ and the selfadjoint operator H_0 associated with $\mathcal{E}$. This means that $\mathcal{D} \subset L^2(X, m)$ is a dense subspace, $\mathcal{E}\colon \mathcal{D} \times \mathcal{D} \to \mathbb{K}$ is sesquilinear and $\mathcal{D}$ is closed with respect to the energy norm $\| \cdot \|_{\mathcal{E}}$, given by

$$\|u\|_{\mathcal{E}}^2 = \mathcal{E}[u, u] + \|u\|_{L^2(X,m)}^2,$$

in which case one speaks of a *closed form* in $L^2(X, m)$. In the sequel we will write

$$\mathcal{E}[u] := \mathcal{E}[u, u].$$

The selfadjoint operator H_0 associated with $\mathcal{E}$ is then characterized by

$$D(H_0) \subset \mathcal{D} \text{ and } \mathcal{E}[f, v] = (H_0 f \mid v) \quad (f \in D(H_0), v \in \mathcal{D}).$$

Such a closed form is said to be a *Dirichlet form* if $\mathcal{D}$ is stable under certain pointwise operations; more precisely, $T : \mathbb{K} \to \mathbb{K}$ is called a *normal contraction* if $T(0) = 0$ and $|T(\xi) - T(\zeta)| \leq |\xi - \zeta|$ for any $\xi, \zeta \in \mathbb{K}$ and we require that for any $u \in \mathcal{D}$ also

$$T \circ u \in \mathcal{D} \text{ and } \mathcal{E}[T \circ u] \leq \mathcal{E}[u].$$

Here we used the original condition from [8] that applies in the real and the complex case at the same time. Today, particularly in the real case, it is mostly expressed in an equivalent but formally weaker statement involving $u \vee 0$ and $u \wedge 1$, see [15], Thm. 1.4.1 and [22], Section I.4.

A Dirichlet form is called *regular* if $\mathcal{D} \cap C_c(X)$ is large enough so that it is dense both in $(\mathcal{D}, \| \cdot \|_{\mathcal{E}})$ and $(C_c(X), \| \cdot \|_\infty)$, where $C_c(X)$ denotes the space of continuous functions with compact support.

Capacity. Due to regularity, we find a set function, the *capacity* that allows to measure the size of sets in a way that is adapted to the form $\mathcal{E}$: For $U \subset X$, U open,

$$\text{cap}(U) := \inf\{\|v\|_{\mathcal{E}}^2 \mid v \in \mathcal{D}, \chi_U \leq v\}, (\inf \emptyset = \infty),$$

and

$$\text{cap}(A) := \inf\{\text{cap}(U) \mid A \subset U\}$$

(see [15], p. 61f.). We say that a property holds *quasi-everywhere*, short *q.e.*, if it holds outside a set of capacity 0. A function $f : X \to \mathbb{K}$ is said to be *quasi-continuous*, *q.c.* for short, if, for any $\varepsilon > 0$ there is an open set $U \subset X$ with $\text{cap}(U) \leq \varepsilon$ so that the restriction of f to $X \setminus U$ is continuous.

A fundamental result in the theory of Dirichlet forms says that every $u \in \mathcal{D}$ admits a q.c. representative $\tilde{u} \in u$ (recall that $u \in L^2(X, m)$ is an equivalence class of functions) and that two such q.c. representatives agree q.e. Moreover, for every Cauchy sequence (u_n) in $(\mathcal{D}, \|\cdot\|_{\mathcal{E}})$ there is a subsequence (u_{n_k}) such that the $(\tilde{u}_{n_k})$ converge q.e. (see [15], p.64f).

Measure perturbations. We will be dealing with Schrödinger type operators, i.e., perturbations $H = H_0 + V$ for suitable potentials V. In fact, we can even include measures as potentials. Here, we follow the approach from [33, 34]. Measure perturbations have been regarded by a number of authors in different contexts, see e.g. [3, 17, 36] and the references there.

We denote by $\mathcal{M}_R(U)$ the signed Radon measures on the open subset U of X and by $\mathcal{M}_{R,0}(U)$ the subset of measures ν that do not charge sets of capacity 0, i.e., those measures with $\nu(B) = 0$ for every Borel set B with $\mathrm{cap}(B) = 0$. In case that $\nu = \nu_+ - \nu_- \in \mathcal{M}_{R,0}(X)$ we can define

$$\nu[u, v] = \int_X \tilde{u}\overline{\tilde{v}} d\nu \text{ for } u, v \in \mathcal{D} \text{ with } \tilde{u}, \tilde{v} \in L^2(X, \nu_+ + \nu_-).$$

We have to rely upon more restrictive assumptions concerning the negative part ν_- of our measure perturbation. We write $\mathcal{M}_{R,1}$ for those measures $\mu \in \mathcal{M}_R(X)$ that are $\mathcal{E}$-bounded with bound less than one; i.e. measures for which there is a $\kappa < 1$ and a c_κ such that

$$\mu[u, u] \leq \kappa \mathcal{E}[u] + c_\kappa \|u\|^2.$$

The set $\mathcal{M}_{R,1}$ can easily be seen to be a subset of $\mathcal{M}_{R,0}$.

By the KLMN theorem (see [29], p. 167), the sum $\mathcal{E} + \nu$ given by $D(\mathcal{E} + \nu) = \{u \in \mathcal{D} \mid \tilde{u} \in L^2(X, \nu_+)\}$ is closed and densely defined (in fact $\mathcal{D} \cap C_c(X) \subset D(\mathcal{E} + \nu)$) for $\nu_+ \in \mathcal{M}_{R,0}(X)$ and $\nu_- \in \mathcal{M}_{R,1}$. We denote the associated selfadjoint operator by $H_0 + \nu$. An important special case is given by $\nu = V dm$ with $V \in L^1_{loc}(X)$. As done in various papers, one can allow for more singular measures, a direction we are not going to explore here due to the technicalities involved.

Strong locality and the energy measure. $\mathcal{E}$ is called *strongly local* if

$$\mathcal{E}[u, v] = 0$$

whenever u is constant a.s. on the support of v.

The typical example one should keep in mind is the Laplacian

$$H_0 = -\Delta \text{ on } L^2(\Omega), \quad \Omega \subset \mathbb{R}^d \text{ open},$$

in which case

$$\mathcal{D} = W_0^{1,2}(\Omega) \text{ and } \mathcal{E}[u, v] = \int_\Omega (\nabla u | \nabla v) dx.$$

Now we turn to an important notion generalizing the measure $(\nabla u|\nabla v)dx$ appearing above.

In fact, every strongly local, regular Dirichlet form $\mathcal{E}$ can be represented in the form

$$\mathcal{E}[u, v] = \int_X d\Gamma(u, v)$$

where Γ is a nonnegative sesquilinear mapping from $\mathcal{D} \times \mathcal{D}$ to the set of $\mathbb{K}$-valued Radon measures on X. It is determined by

$$\int_X \phi \, d\Gamma(u,u) = \mathcal{E}[u, \phi u] - \frac{1}{2}\mathcal{E}[u^2, \phi]$$

for realvalued $u \in \mathcal{D}$, $\phi \in \mathcal{D} \cap C_c(X)$ and called *energy measure*; see also [10]. The energy measure satisfies the Leibniz rule,

$$d\Gamma(u \cdot v, w) = u d\Gamma(v,w) + v d\Gamma(u,w),$$

as well as the chain rule

$$d\Gamma(\eta(u), w) = \eta'(u) d\Gamma(u,w).$$

One can even insert functions from $\mathcal{D}_{\text{loc}}$ into $d\Gamma$, where

$$\mathcal{D}_{\text{loc}} := \{u \in L^2_{\text{loc}} \text{ such that } \phi u \in \mathcal{D} \text{ for all } \phi \in \mathcal{D} \cap C_c(X)\},$$

as is readily seen from the following important property of the energy measure, **strong locality**:

Let U be an open set in X on which the function $\eta \in \mathcal{D}_{\text{loc}}$ is constant, then

$$\chi_U d\Gamma(\eta, u) = 0,$$

for any $u \in \mathcal{D}$. This, in turn, is a consequence of the strong locality of $\mathcal{E}$ and in fact equivalent to the validity of the Leibniz rule.

We write $d\Gamma(u) := d\Gamma(u,u)$ and note that the energy measure satisfies the **Cauchy-Schwarz inequality**:

$$\begin{aligned} \int_X |fg| d|\Gamma(u,v)| &\leq \left(\int_X |f|^2 d\Gamma(u)\right)^{\frac{1}{2}} \left(\int_X |g|^2 d\Gamma(v)\right)^{\frac{1}{2}} \\ &\leq \frac{1}{2}\int_X |f|^2 d\Gamma(u) + \frac{1}{2}\int_X |g|^2 d\Gamma(v). \end{aligned}$$

In order to introduce weak solutions on open subsets of X, we extend $\mathcal{E}$ and $\nu[\cdot,\cdot]$ to $\mathcal{D}_{\text{loc}}(U) \times \mathcal{D}_c(U)$: where,

$$\mathcal{D}_{\text{loc}}(U) := \{u \in L^2_{\text{loc}}(U) \text{ such that } \phi u \in \mathcal{D} \text{ for all } \phi \in \mathcal{D} \cap C_c(U)\},$$

$$\mathcal{D}_c(U) := \{\varphi \in \mathcal{D} | \text{supp}\varphi \text{ compact in } U\}.$$

For $u \in \mathcal{D}_{\text{loc}}(U), \varphi \in \mathcal{D}_c(U)$ we define

$$\mathcal{E}[u, \varphi] := \mathcal{E}[\eta u, \varphi],$$

where $\eta \in \mathcal{D} \cap C_c(U)$ is arbitrary with constant value 1 on the support of φ. This makes sense as the RHS does not depend on the particular choice of η by strong locality. In the same way, we can extend $\nu[\cdot,\cdot]$, using that every $u \in \mathcal{D}_{\text{loc}}(U)$ admits a quasi continuous version $\tilde{u}$. Moreover, also Γ extends to a mapping $\Gamma : \mathcal{D}_{\text{loc}}(U) \times \mathcal{D}_{\text{loc}}(U) \to \mathcal{M}_R(U)$.

The intrinsic metric. Using the energy measure one can define the *intrinsic metric* ρ by

$$\rho(x,y) = \sup\{|u(x) - u(y)| \ |u \in \mathcal{D}_{\text{loc}} \cap C(X) \text{ and } d\Gamma(u) \leq dm\}$$

where the latter condition signifies that $\Gamma(u)$ is absolutely continuous with respect to m and the Radon-Nikodym derivative is bounded by 1 on X. Note that, in general, ρ need not be a metric. We say that $\mathcal{E}$ is *strictly local* if ρ is a metric that induces the original topology on X. Note that this implies that X is connected, since otherwise points in x, y in different connected components would give $\rho(x,y) = \infty$, as characteristic functions of connected components are continuous and have vanishing energy measure. We denote the intrinsic balls by

$$B(x,r) := \{y \in X | \rho(x,y) \leq r\}.$$

An important consequence of the latter assumption is that the distance function $\rho_x(\cdot) := \rho(x,\cdot)$ itself is a function in $\mathcal{D}_{\text{loc}}$ with $d\Gamma(\rho_x) \leq dm$, see [35]. This easily extends to the fact that for every closed $E \subset X$ the function $\rho_E(x) := \inf\{\rho(x,y)|y \in E\}$ enjoys the same properties (see the Appendix of [14]). This has a very important consequence. Whenever $\zeta : \mathbb{R} \longrightarrow \mathbb{R}$ is continuously differentiable, and $\eta := \zeta \circ \rho_E$, then η belongs to $\mathcal{D}_{\text{loc}}$ and satisfies

$$d\Gamma(\eta) = (\zeta' \circ \rho_E)^2 d\Gamma(\rho_E) \leq (\zeta' \circ \rho_E)^2 dm. \tag{1}$$

Irreducibility. We will now discuss a notion that will be crucial in the proof of the existence of positive weak solutions below the spectrum. In what follows, $\mathfrak{h}$ will denote a densely defined, closed semibounded form in $L^2(X)$ with domain $D(\mathfrak{h})$ and positivity preserving semigroup $(T_t; t \geq 0)$. We denote by H the associated operator. Actually, the cases of interest in this paper are $\mathfrak{h} = \mathcal{E}$ or $\mathfrak{h} = \mathcal{E} + \nu$ with $\nu \in \mathcal{M}_{R,0} - \mathcal{M}_{R,1}$. We refer to [30], XIII.12 and a forthcoming paper [21] for details. We say that $\mathfrak{h}$ is *reducible*, if there is a measurable set $M \subset X$ such that M and its complement M^c are nontrivial (have positive measure) and $L^2(M)$ is a reducing subspace for M, i.e., $\mathbb{1}_M D(\mathfrak{h}) \subset D(\mathfrak{h})$ and $\mathfrak{h}$ restricted to $\mathbb{1}_M D(\mathfrak{h})$ is a closed form. If there is no such decomposition of $\mathfrak{h}$, the latter form is called *irreducible*. Note that reducibility can be rephrased in terms of the semigroup and the resolvent:

Theorem 1.1. *Let $\mathfrak{h}$ be as above. Then the following conditions are equivalent:*

- *$\mathfrak{h}$ is irreducible.*
- *T_t is positivity improving, for every $t > 0$, i.e. $f \geq 0$ and $f \neq 0$ implies that $T_t f > 0$ a.e.*
- *$(H + E)^{-1}$ is positivity improving for every $E < \inf \sigma(H)$.*

In [21] we will show that for a strictly local Dirichlet form $\mathcal{E}$ as above and a measure perturbation $\nu \in \mathcal{M}_{R,0} - \mathcal{M}_{R,1}$, irreducibility of $\mathcal{E}$ implies irreducibility of $\mathcal{E} + \nu$.

2. Positive weak solutions and the associated transformation

Throughout this section we consider a strongly local, regular Dirichlet form, $(\mathcal{E}, \mathcal{D})$ on X and denote by $\Gamma : \mathcal{D}_{loc} \times \mathcal{D}_{loc} \to \mathcal{M}(X)$ the associated energy measure. We will be concerned with weak solutions Φ of the equation

$$(H_0 + V)\Phi = E \cdot \Phi, \tag{2}$$

where H_0 is the operator associated with $\mathcal{E}$ and V is a realvalued, locally integrable potential. In fact, we will consider a somewhat more general framework, allowingfor measures instead of functions, as presented in the previous section. Moreover, we stress the fact that (2) is formal in the sense that Φ is not assumed to be in the operator domain of neither H_0 nor V. Here are the details.

Definition 2.1. *Let $\nu \in \mathcal{M}_{R,0}(U)$ be a signed Radon measure on U that charges no set of capacity zero. Let $U \subset X$ be open, $E \in \mathbb{R}$ and $\Phi \in L^2_{loc}(U)$. We say that Φ is a* weak supersolution *of $(H_0 + \nu)\Phi = E \cdot \Phi$ in U if:*

(i) $\Phi \in \mathcal{D}_{loc}(U)$,
(ii) $\tilde{\Phi} d\nu \in \mathcal{M}_R(U)$,
(iii) $\forall \varphi \in \mathcal{D} \cap C_c(U), \varphi \geq 0$:

$$\mathcal{E}[\Phi, \varphi] + \int_U \varphi \tilde{\Phi} d\nu \geq (\Phi|\varphi).$$

We call Φ a weak solution *of $(H_0 + \nu)\Phi = E \cdot \Phi$ in U if equality holds in (iii) above (which extends to all $\varphi \in \mathcal{D} \cap C_c(U)$). If $V \in L^1_{loc}(U)$ we say that Φ is a* weak (super-)solution *of $(H_0 + V)\Phi = E \cdot \Phi$ in U if it is a weak (super-) solution of $(H_0 + \nu)\Phi = E \cdot \Phi$ for $\nu = V dm$.*

Remark 2.2. (1) *If $\nu = V dm$ and $V \in L^2_{loc}(U)$, then property (ii) of the Definition above is satisfied.*
(2) *If $\Phi \in L^\infty_{loc}(U)$ and $\nu \in \mathcal{M}_R(U)$ then (ii) of the Definition above is satisfied.*
(3) *If $\nu \in \mathcal{M}_R(U)$ satisfies (ii) above then $\nu - E dm \in \mathcal{M}_R(U)$ satisfies (ii) as well and any weak solution of $(H_0 + \nu)\Phi = E \cdot \Phi$ in U is a weak solution of $(H_0 + \nu - E dm)\Phi = 0$ in U. Thus it suffices to consider the case $E = 0$.*

Here comes the first half of the Allegretto-Piepenbrink Theorem in a general form.

Theorem 2.3. *Let $(\mathcal{E}, \mathcal{D})$ be a regular, strictly local Dirichlet form, H_0 be the associated operator and $\nu \in \mathcal{M}_{R,0}(U)$. Suppose that Φ is a weak solution of $(H_0 + \nu)\Phi = E \cdot \Phi$ in U with $\Phi > 0$ m-a.e. and $\Phi^{-1} \in \mathcal{D}_{loc}(U)$. Then, for all $\varphi, \psi \in \mathcal{D} \cap C_c(U)$:*

$$\mathcal{E}[\varphi, \psi] + \nu[\varphi, \psi] = \int_X \Phi^2 d\Gamma(\varphi\Phi^{-1}, \psi\Phi^{-1}) + E \cdot (\varphi|\psi).$$

In particular, $\mathcal{E} + \nu \geq E$ on $\mathcal{D} \cap C_c(U)$.

Proof. The "in particular" is clear.

For the rest of the proof we may assume $E = 0$ without restriction, in view of the preceding Remark. We evaluate the RHS of the above equation, using the following identity. The chain rule implies that for arbitrary $w \in \mathcal{D}_{loc}(U)$:

$$0 = d\Gamma(w,1) = d\Gamma(w,\Phi\Phi^{-1}) = \Phi^{-1}d\Gamma(w,\Phi) + \Phi d\Gamma(w,\Phi^{-1}) \qquad (\bigstar)$$

Therefore, for $\varphi,\psi \in \mathcal{D}\cap C_c(X)$:

$$\begin{aligned}\int_X \Phi^2 d\Gamma(\varphi\Phi^{-1},\psi\Phi^{-1}) &= \int_X \Phi d\Gamma(\varphi,\psi\Phi^{-1}) + \int_X \Phi^2\varphi d\Gamma(\Phi^{-1},\psi\Phi^{-1})\\ &= \int_X d\Gamma(\varphi,\psi) + \int_X \Phi\psi d\Gamma(\varphi,\Phi^{-1}) + \int_X \Phi^2\varphi d\Gamma(\psi\Phi^{-1},\Phi^{-1})\end{aligned}$$

by symmetry,

$$\begin{aligned}\dots &= \mathcal{E}[\varphi,\psi] + \int_X \Phi^2 d\Gamma(\varphi\psi\Phi^{-1},\Phi^{-1})\\ &= \mathcal{E}[\varphi,\psi] - \int_X d\Gamma(\varphi\psi\Phi^{-1},\Phi)\end{aligned}$$

by $(\bigstar)$

$$\begin{aligned}\dots &= \mathcal{E}[\varphi,\psi] - \mathcal{E}[\varphi\psi\Phi^{-1},\Phi]\\ &= \mathcal{E}[\varphi,\psi] - \big(-\nu[\varphi\psi\Phi^{-1},\Phi]\big)\end{aligned}$$

since Φ is a weak solution

$$\dots = \mathcal{E}[\varphi,\psi] + \nu[\varphi,\psi].$$

□

We note a number of consequences of the preceding Theorem. The first is rather a consequence of the proof, however:

Corollary 2.4. *Assume that there is a weak supersolution Φ of $(H_0+\nu)\Phi = E\cdot\Phi$ on X with $\Phi > 0$ m-a.e. and $\Phi^{-1} \in \mathcal{D}_{loc}$. Then $\mathcal{E}+\nu \geq E$.*

For the *Proof* we can use the same calculation as in the proof of the Theorem with $\varphi = \psi$ and use the inequality instead of the equality at the end.

Remark 2.5.
(1) *We can allow for complex measures ν without problems, as long as the corresponding forms are accretive. In the context of PT–symmetric operators there is recent interest in this type of Schrödinger operators, see* [7]
(2) *Instead of measures also certain distributions could be included. Cf* [18] *for such singular perturbations.*

3. The existence of positive weak solutions below the spectrum

As noted in the preceding section, we find that $H_0+\nu \geq E$ whenever $\mathcal{E}+\nu$ is closable and admits a positive weak solution of $(H_0+\nu)\Phi = E\Phi$. In this section we prove the converse under suitable conditions. We use an idea from [12] where the corresponding statement for ordinary Schrödinger operators on $\mathbb{R}^d$ can be found. A key property is related to the celebrated *Harnack inequality.*

Definition 3.1. (1) We say that $H_0 + \nu$ *satisfies a Harnack inequality* for $E \in \mathbb{R}$ if, for every relatively compact, connected open $X_0 \subset X$ there is a constant C such that all positive weak solutions Φ of $(H_0 + \nu)\Phi = E\Phi$ on X_0 are locally bounded and satisfy

$$\mathrm{esssup}_{B(x,r)} u \leq C \mathrm{essinf}_{B(x,r)} u,$$

for every $B(x,r) \subset X_0$ where esssup and essinf denote the essential supremum and infimum.

(2) We say that $H_0 + \nu$ satisfies the *Harnack principle* for $E \in \mathbb{R}$ if for every relatively compact, connected open subset U of X and every sequence $(\Phi_n)_{n\in\mathbb{N}}$ of nonnegative solutions of $(H_0 + \nu)\Phi = E \cdot \Phi$ in U the following implication holds: If, for some measurable subset $A \subset U$ of positive measure

$$\sup_{n\in\mathbb{N}} \|\Phi_n \mathbb{1}_A\|_2 < \infty$$

then, for all compact $K \subset U$ also

$$\sup_{n\in\mathbb{N}} \|\Phi_n \mathbb{1}_K\|_2 < \infty.$$

(3) We say that $H_0 + \nu$ satisfies the *uniform Harnack principle* if for every bounded intervall $I \subset \mathbb{R}$, every relatively compact, connected open subset U of X and every sequence $(\Phi_n)_{n\in\mathbb{N}}$ of nonnegative solutions of $(H_0 + \nu)\Phi = E_n \cdot \Phi$ in U with $E_n \in I$ the following implication holds: If, for some measurable subset $A \subset U$ of positive measure

$$\sup_{n\in\mathbb{N}} \|\Phi_n \mathbb{1}_A\|_2 < \infty$$

then, for all compact $K \subset U$ also

$$\sup_{n\in\mathbb{N}} \|\Phi_n \mathbb{1}_K\|_2 < \infty.$$

Note that validity of a Harnack principle implies that a nonnegative weak solution Φ must vanish identically if it vanishes on a set of positive measure (as $\Phi_n := n\Phi$ has vanishing L^2 norm on the set of positive measure in question). Note also that validity of an Harnack inequality extends from balls to compact sets by a standard chain of balls argument. This easily shows that $H_0 + \nu$ satisfies the Harnack principle for $E \in \mathbb{R}$ if it obeys a Harnack inequality for $E \in \mathbb{R}$. Therefore, many situations are known in which the Harnack principle is satisfied:

Remark 3.2. (1) *For $\nu \equiv 0$ and $E = 0$ a Harnack inequality holds, whenever $\mathcal{E}$ satisfies a Poincaré and a volume doubling property; cf* [9] *and the discussion there.*

(2) *The most general results for $H_0 = -\Delta$ in terms of the measures ν that are allowed seem to be found in* [17]*, which also contains a thorough discussion of the literature prior to 1999. A crucial condition concerning the measures involved is the Kato condition and the uniformity of the estimates from* [17] *immediately gives that the uniform Harnack principle is satisfied in that context. Of the enormous list of papers on Harnack's inequality, let us mention* [2, 11, 17, 19, 20, 23, 31, 32, 37, 38]

Apart from the Harnack principle there are two more properties that will be important in the proof of existence of positive general eigensolutions at energies below the spectrum.

We say that $\mathcal{E}$ satisfies the *local compactness property* if $D_0(U) := \overline{D \cap C_c(U)}^{\|\cdot\|_{\mathcal{E}}}$ is compactly embedded in $L^2(X)$ for every relatively compact open $U \subset X$. (In case of the classical Dirichlet form this follows from Rellich's Theorem on compactness of the embedding of Sobolev spaces in L^2.)

Theorem 3.3. *Let $(\mathcal{E}, \mathcal{D})$ be a regular, strictly local, irreducible Dirichlet form, H_0 be the associated operator and $\nu \in \mathcal{M}_{R,0} - \mathcal{M}_{R,1}$. Suppose that $\mathcal{E}$ satisfies the local compactness property and X is noncompact. Then, if $E < \inf \sigma(H_0 + \nu)$ and $H_0 + \nu$ satisfies the Harnack principle for E, there is an a.e. positive solution of $(H_0 + \nu)\Phi = E\Phi$.*

Proof. Let $E < \inf \sigma(H_0 + \nu)$. Since X is noncompact, locally compact and σ-compact, it can be written as a countable union

$$X = \bigcup_{R \in \mathbb{N}} U_R,\ U_R \text{ open, relatively compact },\overline{U_R} \subset U_{R+1};$$

where the U_R can be chosen connected, as X is connected, see [21] for details.

For $n \in \mathbb{N}$ let $g_n \in L^2(X)$ with $\mathrm{supp} g_n \subset X \setminus U_{n+2}$, $g_n \geq 0$ and $g_n \neq 0$. It follows that

$$\Phi_n := (H_0 + \nu + E)^{-1} g_n \geq 0$$

is nonzero and is a weak solution of $(H_0 + \nu)\Phi = E\Phi$ on $X \setminus \mathrm{supp} g_n$, in particular on the connected open subset U_{n+2}. Since $(H_0 + \nu + E)^{-1}$ is positivity improving, it follows that $\|\Phi_n 1\!\!1_{U_1}\|_2 > 0$. By multiplying with a positive constant we may and will assume that $\|\Phi_n 1\!\!1_{U_1}\|_2 = 1$ for all $n \in \mathbb{N}$. We want to construct a suitably convergent subsequence of $(\Phi_n)_{n \in \mathbb{N}}$ so that the corresponding limit Φ is a positive weak solution.

First note that by the Harnack principle, for fixed $R \in \mathbb{N}$ and $n \geq R$ we know that

$$\sup_{n \in \mathbb{N}} \|\Phi_n 1\!\!1_{U_R}\|_2 < \infty,$$

since all the corresponding Φ_n are nonnegative solutions on U_{R+2}. In particular, $(\Phi_n 1\!\!1_{U_R})_{n \in \mathbb{N}}$ is bounded in $L^2(X)$ and so has a weakly convergent subsequence. By a standard diagonal argument, we find a subsequence, again denoted by $(\Phi_n)_{n \in \mathbb{N}}$, so that $\Phi_n 1\!\!1_{U_R} \to \Psi_R$ weakly in $L^2(X)$ for all $R \in \mathbb{N}$ and suitable Ψ_R. As multiplication with $1\!\!1_{U_R}$ is continuous and hence also weak-weak continuous, there is $\Phi \in L^2_{loc}(X)$ such that $\Psi_R = \Phi 1\!\!1_{U_R}$. We will now perform some bootstrapping to show that the convergence is, in fact, much better than just local weak convergence in L^2 which will imply that Φ is the desired weak solution.

Since for fixed $R > 0$ and $n \geq R$ the Φ_n are nonnegative solutions on U_{R+2} the Caccioppoli inequality, cf [14] implies that

$$\int_{U_R} d\Gamma(\Phi_n) \leq C \int_{U_{R+1}} \Phi_n^2 dm$$

is uniformly bounded w.r.t. $n \in \mathbb{N}$. Combined with Leibniz rule and Cauchy Schwarz inequality this directly gives that $\int_{U_R} d\Gamma(\psi\Phi_n)$ is uniformly bounded w.r.t. $n \in \mathbb{N}$ for every $\psi \in \mathcal{D}$ with $d\Gamma(\psi) \leq dm$ (see [14] as well). Therefore, for suitable cut-off functions $\eta_R \in \mathcal{D} \cap C_c(X)$ with $\mathbb{1}_{U_R} \leq \eta_R \leq \mathbb{1}_{U_{R+1}}$ the sequence $(\eta_R \Phi_n)$ is bounded in $(D, \|\cdot\|_{\mathcal{E}})$. (In fact, we find $r(R) > 0$ such that $\mathrm{dist}(U_R, (X \setminus U_{R+1})) > 2r(R)$. Then ρ_{U_R} has energy measure bounded by dm. Choosing a smooth $\zeta_R : \mathbb{R} \to \mathbb{R}$ with $\zeta_R(0) = 1$ and $\mathrm{supp}(\zeta_R) \subset (-\infty, r(R))$, the composition $\zeta_R \circ \rho_{U_R}$ does the job by (1).)

The local compactness property implies that $(\eta_R \Phi_n)$ has an L^2-convergent subsequence. Using a diagonal argument again, we see that there is a common subsequence, again denoted by $(\Phi_n)_{n\in\mathbb{N}}$, such that

$$\Phi_n \mathbb{1}_{U_R} \to \Phi \mathbb{1}_{U_R} \text{ in } L^2(X) \text{ as } n \to \infty$$

for all $R \in \mathbb{N}$.

As a first important consequence we note that $\Phi \neq 0$, since $\|\Phi \mathbb{1}_{U_1}\|_2 = \lim_n \|\Phi_n \mathbb{1}_{U_1}\|_2 = 1$.

Another appeal to the Caccioppoli inequality gives that

$$\int_{U_R} d\Gamma(\Phi_n - \Phi_k) \leq C \int_{U_{R+1}} (\Phi_n - \Phi_k)^2 dm \to 0 \text{ as } n, k \to \infty.$$

Therefore, by the same reasoning as above, for every $R \in \mathbb{N}$ the sequence $(\eta_R \Phi_n)$ converges in $(\mathcal{D}, \|\cdot\|_{\mathcal{E}})$. Since this convergence is stronger than weak convergence in $L^2(X)$, its limit must be $\eta_R \Phi$, so that the latter is in $\mathcal{D}$. We have thus proven that $\Phi \in \mathcal{D}_{loc}(X)$. Moreover, we also find that

$$\mathcal{E}[\Phi_n, \varphi] \to \mathcal{E}[\Phi, \varphi] \text{ for all } \varphi \in \mathcal{D} \cap C_c(X),$$

(since, by strong locality, for every cut-off function $\eta \in \mathcal{D} \cap C_c(X)$ that is 1 on $\mathrm{supp}\varphi$, we get

$$\mathcal{E}[\Phi_n, \varphi] = \mathcal{E}[\eta\Phi_n, \varphi] \to \mathcal{E}[\eta\Phi, \varphi] = \mathcal{E}[\Phi, \varphi].)$$

We will now deduce convergence of the potential term. This will be done in two steps. In the first step we infer convergence of the ν_- part from convergence w.r.t. $\|\cdot\|_{\mathcal{E}}$ and the relative boundedness of ν_-. In the second step, we use the fact that Φ is a weak solution to reduce convergence of the ν_+ part to convergence w.r.t. $\|\cdot\|_{\mathcal{E}}$ and convergence of the ν_- part. Here are the details:

Consider cut-off functions η_R for $R \in \mathbb{N}$ as above. Due to convergence in $(\mathcal{D}, \|\cdot\|_{\mathcal{E}})$, we know that there is a subsequence of $(\eta_R \Phi_n)_{n\in\mathbb{N}}$ that converges q.e., see [15] and the discussion in Section 1. One diagonal argument more will give a subsequence, again denoted by $(\Phi_n)_{n\in\mathbb{N}}$, such that the $\tilde{\Phi}_n$ converge to $\tilde{\Phi}$ q.e., where $\tilde{}$ denotes the quasi-continuous representatives. Since ν is absolutely continuous w.r.t capacity we now know that the $\tilde{\Phi}_n$ converge to $\tilde{\Phi}$ ν-a.e. Moreover, again due to convergence in $(\mathcal{D}, \|\cdot\|_{\mathcal{E}})$, we know that $(\widetilde{\eta_R \Phi_n})_{n\in\mathbb{N}}$ is convergent in $L^2(\nu_-)$ as $\nu_- \in \mathcal{M}_{R,1}$. Its limit must coincide with $\eta_R \tilde{\Phi}$, showing that $\tilde{\Phi} d\nu_- \in \mathcal{M}_R$.

We now want to show the analogous convergence for ν_+; we do so by approximation and omit the $\tilde{}$ for notational simplicity. By simple cut-off procedures, every $\varphi \in \mathcal{D}_c(X) \cap L^\infty(X)$ can be approximated w.r.t. $\|\cdot\|_{\mathcal{E}}$

by a uniformly bounded sequence of functions in $\mathcal{D}$ with common compact support. Thus, the equation

$$\mathcal{E}[\Phi, \varphi] + \nu[\Phi, \varphi] = (\Phi|\varphi),$$

initially valid for $\varphi \in \mathcal{D} \cap C_c(X)$ extends to $\varphi \in \mathcal{D}_c(X) \cap L^\infty(X)$ by continuity. Therefore, for arbitrary $k \in \mathbb{N}$,

$$\begin{aligned}
\int_{|\Phi_n - \Phi_m| \leq k} (\Phi_n - \Phi_m)^2 \eta_R d\nu_+ &\leq \int (\Phi_n - \Phi_m)\{(-k) \vee (\Phi_n - \Phi_m) \wedge k\} \eta_R d\nu_+ \\
&= \nu_+[(\Phi_n - \Phi_m), \{(-k) \vee (\Phi_n - \Phi_m) \wedge k\}\eta_R] \\
&= E((\Phi_n - \Phi_m)|\{\ldots\}\eta_R) + \nu_-[(\Phi_n - \Phi_m), \{\ldots\}\eta_R)] \\
&\quad -\mathcal{E}[(\Phi_n - \Phi_m), \{(-k) \vee (\Phi_n - \Phi_m) \wedge k\}\eta_R]
\end{aligned}$$

By what we already know about convergence in $\mathcal{D}$, L^2 and $L^2(\nu_-)$, the RHS goes to zero as $n, m \to \infty$, independently of k. This gives the desired convergence of $\eta_R \tilde{\Phi}_n$, the limit being $\eta_R \tilde{\Phi}$ since this is the limit pointwise.

Finally, an appeal to the Harnack principle gives that Φ is positive a.e. on every U_R and, therefore, a.e. on X. □

Remark 3.4. *That we have to assume thet X is noncompact can easily be seen by looking at the Laplacian on a compact manifold. In that situation any positive weak solution must in fact be in L^2 due to the Harnack principle. Thus the corresponding energy must lie in the spectrum. In fact, the corresponding energy must be the infimum of the spectrum as we will show in the next theorem. The theorem is standard. We include a proof for completeness reasons.*

Theorem 3.5. *Let $(\mathcal{E}, \mathcal{D})$ be a regular, strictly local, irreducible Dirichlet form, H_0 be the associated operator and $\nu \in \mathcal{M}_{R,0} - \mathcal{M}_{R,1}$. Suppose that X is compact and $\mathcal{E}$ satisfies the local compactness property. Then, $H_0 + \nu$ has compact resolvent. In particular, there exists a positive weak solution to $(H_0 + \nu)\Phi = E_0 \Phi$ for $E_0 := \inf \sigma(H_0 + \nu)$. This solution is unique (up to a factor) and belongs to $L^2(X)$. If $H_0 + \nu$ satisfies a Harnack principle, then E_0 is the only value in $\mathbb{R}$ allowing for a positive weak solution.*

Proof. As X is compact, the local compactness property gives that the operator associated to $\mathcal{E}$ has compact resolvent. In particular, the sequence of eigenvalues of H_0 is given by the minmax principle and tends to ∞. As ν_+ is a nonnegative operator and ν_- is form bounded with bound less than one, we can apply the minmax principle to $H_0 + \nu$ as well to obtain empty essential spectrum.

In particular, the infimum of the spectrum is an eigenvalue. By irreducibility and abstract principles, see e.g. [30], XIII.12, the corresponding eigenvector must have constant sign and if a Harnack principle holds then any other energy allowing for a positive weak solution must be an eigenvalue as well (as discussed in the previous remark). As there can not be two different eigenvalues with positive solutions, there can not be another energy with a positive weak solution. □

Combining the results for the compact and noncompact case we get:

Corollary 3.6. *Let $(\mathcal{E}, \mathcal{D})$ be a regular, strictly local, irreducible Dirichlet form, H_0 be the associated operator and $\nu \in \mathcal{M}_{R,0} - \mathcal{M}_{R,1}$. Suppose that*

$\mathcal{E}$ satisfies the local compactness property and $H_0 + \nu$ satisfies the Harnack principle for all $E \in \mathbb{R}$. Then,

$$\inf \sigma(H_0 + \nu) \leq \sup\{E \in \mathbb{R} | \exists \text{ a.e. positive weak solution } (H_0 + \nu)\Phi = E\Phi\}.$$

This doesn't settle the existence of a positive weak solution for the ground-state energy $\inf \sigma(H_0 + \nu)$ in the noncompact case. The uniform Harnack principle settles this question:

Theorem 3.7. *Let $(\mathcal{E}, \mathcal{D})$ be a regular, strictly local, irreducible Dirichlet form, H_0 be the associated operator, $\nu \in \mathcal{M}_{R,0} - \mathcal{M}_{R,1}$. Suppose that $\mathcal{E}$ satisfies the local compactness property and $H_0 + \nu$ satisfies the uniform Harnack principle. Then there is an a.e. positive weak solution of $(H_0 + \nu)\Phi = E\Phi$ for $E = \inf \sigma(H_0 + \nu)$.*

Proof. It suffices to consider the case of noncompact X. Take a sequence (E_n) increasing to $E = \inf \sigma(H_0 + \nu)$. From Theorem 3.3 we know that there is an a.e. positive solution Ψ_n of $(H_0 + \nu)\Phi = E_n\Phi$. We use the exhaustion $(U_R)_{R \in \mathbb{N}}$ from the proof of Theorem 3.3 and assume that

$$\|\Psi_n \mathbb{1}_{U_1}\|_2 = 1 \text{ for all } n \in \mathbb{N}.$$

As in the proof of Theorem 3.3 we can now show that we can pass to a subsequence such that $(\eta_R \Psi_n)$ converges in $\mathcal{D}$, $L^2(m)$ and $L^2(|nu|)$ for every $R \in \mathbb{N}$. The crucial point is that the uniform Harnack principle gives us a control on $\|\eta_R \Psi_n\|_2$, uniformly in n, due to the norming condition above. With aruments analogous to those in the proof of Theorem 3.3, the assertion follows. □

Note that Corollaries 2.4 and 3.6 together almost given

$$\inf \sigma(H_0 + \nu) = \sup\{E \in \mathbb{R} | \exists \text{ a.e. positive weak solution } (H_0 + \nu)\Phi = E\Phi\}.$$

The only problem is that for the "$\geq$" from Corollary 2.4 we would have to replace a.e. positive by a.e. positive and $\Phi^{-1} \in \mathcal{D}_{loc}$. However, if a Harnack inequality is fulfilled, we can use the chain rule and suitable smoothed version of the function $x \mapsto 1/x$ to conclude that Φ^{-1} must belong to $\mathcal{D}_{loc}$. Therefore, we get:

Corollary 3.8. *Let $(\mathcal{E}, \mathcal{D})$ be a regular, strictly local, irreducible Dirichlet form, H_0 be the associated operator and $\nu \in \mathcal{M}_{R,0} - \mathcal{M}_{R,1}$. Suppose that $\mathcal{E}$ satisfies the local compactness property and $H_0 + \nu$ satisfies a Harnack inequality for all $E \in \mathbb{R}$. Then,*

$$\inf \sigma(H_0 + \nu) = \sup\{E \in \mathbb{R} | \exists \text{ a.e. positive weak solution } (H_0 + \nu)\Phi = E\Phi\}.$$

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[1] Mathematisches Institut, Friedrich-Schiller Universität Jena, Ernst-Abbé Platz 2, 03347 Jena, Germany
E-mail address: daniel.lenz@uni-jena.de

[2] Fakultät für Mathematik, Technische Universität, 09107 Chemnitz, Germany
E-mail address: ivan.veselic@mathematik.tu-chemnitz.de

[3] Fakultät für Mathematik, Technische Universität, 09107 Chemnitz, Germany
E-mail address: peter.stollmann@mathematik.tu-chemnitz.de