Math. Z.
DOI 10.1007/s00209-010-0756-8

Mathematische Zeitschrift

An uncertainty principle, Wegner estimates and localization near fluctuation boundaries

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Received: 15 July 2009 / Accepted: 11 June 2010

Abstract We prove a simple uncertainty principle and show that it can be applied to prove Wegner estimates near fluctuation boundaries. This gives new classes of models for which localization at low energies can be proven.

0 Introduction

Starting point of the present paper was the lamentable fact that for certain random models with possibly quite small and irregular support there was a proof of localization via fractional moment techniques (at least for $d \leq 3$) but no proof of Wegner estimates necessary for multiscale analysis. The classes of models include models with surface type random potentials as well as Anderson models with displacement (see [1]) but actually much more classes of examples could be seen in the framework established there which was labelled "fluctuation boundaries". Actually, the big issue in the treatment of random perturbations with small or irregular support is the question, whether the spectrum at low energies really feels the random perturbation. This is exactly what is formalized in the fluctuation boundary framework.

In the present paper we establish the necessary Wegner estimates by using the method from Combes et al. [6] so that we get the correct volume factor and the modulus of continuity of the random variables. One of the main ideas we borrow from the last mentioned work is

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Published online: 10 July 2010

to show that spectral projectors are "spread out", a property we call "uncertainty principle" (see below for further discussion).

The solution to the above mentioned problem is now quite simple in fact. In an abstract framework we show that such an uncertainty principle of the form

$$P_I(H_0)WP_I(H_0) \geq \kappa P_I(H_0), \tag{0.1}$$

where $W \geq 0$ is bounded and $P_I(H_0)$ denotes the spectral projection, is in a sense equivalent to the mobility of

$$\lambda(t) := \inf \sigma(H_0 + tW). \tag{0.2}$$

This is done in Sect. 1.

That fits perfectly with the fluctuation boundary concept and gives the appropriate Wegner estimates. Actually, if the integrated density of states exists, it then must be continuous, provided the distribution of the random variables has a common modulus of continuity. We will prove this in Sect. 2.

Finally, in Sect. 3 we show how to exploit these Wegner estimates for a proof of localization. It lies in the nature of these different methods that we thus get localization under less restrictive conditions than what was needed in [1]. One main point is the dimension restriction of the latter paper, $d \leq 3$, which certainly is not essential but is essential for a proof of digestable length. Clearly, the estimates one gets via the fractional moment method are more powerful.

1 An uncertainty principle and mobility of the ground state energy

In this section we fix a rather abstract setting: $\mathcal{H}$ is a Hilbert space, H_0 is a selfadjoint operator in $\mathcal{H}$ with

$$\lambda(0) := \inf \sigma(H_0) > -\infty. \tag{1.1}$$

Moreover, W is assumed to be bounded and nonnegative.

The uncertainty principle we want to study is the existence of a positive κ such that

$$P_I W P_I \geq \kappa P_I \tag{$*$}$$

where $I \subset \mathbb{R}$ is some compact interval, $I = [\min I, \max I]$ and $P_I = P_I(H_0) = \chi_I(H_0)$ is the corresponding spectral projection.

It is reasonable to call ($*$) an uncertainty principle as a state in the range of P_I cannot be "concentrated where W vanishes". In fact, in the case of $H_0 = -\Delta$ this leads to the uncertainty principles considered in harmonic analysis (see e.g. [8] and discussion in [15]).

In our main application, H_0 will be a Schrödinger operator so that ($*$) is in fact a variant of the usual uncertainty principle.

The use of ($*$) for the proof of Wegner estimates is due to Combes et al., see [5,6]. Its importance lies in the fact that it takes care of random potentials with small support. Our purpose here is to prove a simple criterion that implies ($*$) and can be checked rather easily.

Theorem 1.1 *Let for* $t \geq 0$

$$\lambda(t) := \inf \sigma(H_0 + tW) \tag{1.2}$$

and assume that $\lambda(t_0) > \max I$ *for some* $t_0 > 0$. *Then*

$$P_I W P_I \geq \left[\sup_{t>0} \frac{\lambda(t) - \max I}{t} r \right] P_I. \tag{1.3}$$

Of course, the assumption in the theorem is merely there to guarantee that the square bracket is positive!

Proof Assume that (∗) does not hold for some $\kappa > 0$. Then we find $g \in \operatorname{Ran} P_I$ with $\|g\| = 1$ and

$$\langle Wg, g \rangle = \langle P_I W P_I g, g \rangle < \kappa.$$

Since $\langle H_0 g, g \rangle \leq \max I$, by the functional calculus, we get, for any $t > 0$,

$$\lambda(t) \leq \langle (H_0 + tW)g, g \rangle < \max I + t\kappa,$$

which implies

$$\kappa > \frac{\lambda(t) - \max I}{t}.$$

By contraposition, we find that (∗) must hold with

$$\kappa = \frac{\lambda(t) - \max I}{t}$$

for all $t > 0$. This easily implies validity of (∗) for $\kappa = \sup_{t>0} \frac{\lambda(t) - \max I}{t}$. This is the desired statement. □

Remark (1) One particularly nice aspect of the above result is that the important constant is controlled in a simple way.
(2) Once the ground state energy is pushed up by W we get an uncertainty principle (∗) at least for intervals I near $\lambda(0)$.
(3) The corresponding uncertainty result in [5] for periodic Schrödinger operators does not follow from the preceding theorem.

There is a kind of converse to Theorem 1.1.

Lemma 1.2 *If* (∗) *holds for* I *with* $\min I = \lambda(0) = \inf \sigma(H_0)$ *and* $\max I > \min I$, *then*

$$\lambda(t) > \lambda(0)$$

for all $t > 0$.

Proof We only need to consider small $t > 0$ since $W \geq 0$. For $f \in D(H_0)$, $\|f\| = 1$, let $f_1 := P_I f$ and $f_2 := P_{I^c} f$ so that $\|f_1\|^2 + \|f_2\|^2 = 1$. We consider

$$\begin{aligned} \langle (H_0 + tW)f, f \rangle &= \langle H_0 f_1, f_1 \rangle + \langle H_0 f_2, f_2 \rangle + t\langle Wf, f \rangle \\ &\geq (\max I)\|f_2\|^2 + \lambda(0)\|f_1\|^2 + t\kappa \|f_1\|^2 - 2t \|W\| \|f_1\| \|f_2\| \\ &= \lambda(0)\|f\|^2 + (\max I - \lambda(0))\|f_2\|^2 - 2t \|f_1\| \|f_2\| \|W\| + t\kappa \|f_1\|^2. \end{aligned}$$

Combining this with the Cauchy–Schwarz bound

$$2t\|f_1\|\|f_2\|\|W\| \leq t\|W\|^2 \frac{2}{\kappa}\|f_2\|^2 + t\frac{\kappa}{2}\|f_1\|^2$$

we obtain

$$\langle (H_0 + tW)f, f\rangle \geq \lambda(0)\|f\|^2 + \left(\max I - \lambda(0) - t\|W\|^2 \frac{2}{\kappa}\right)\|f_2\|^2 + t\frac{\kappa}{2}\|f_1\|^2.$$

As $\|f\| = 1$ this can directly be seen to be strictly larger than $\lambda(0)$ for t small enough. □

2 Continuity of the IDS near weak fluctuation boundaries

The main result here is, in fact, rather an "optimal" Wegner estimate meaning that we recover at least the modulus of continuity of the random variables in the Wegner estimate as well as the correct volume factor. The models we consider needn't have a homogeneous background so that the integrated density of states, IDS need not exist. See [16] for a recent survey on how to prove the existence of the IDS in various different settings. We show that a straightforward application of Theorem 1.1 above gives the necessary input to perform the analysis of [6] in a rather general setting which we are going to introduce now.

(A1) The background potential $V_0 \in L^p_{\text{loc,unif}}(\mathbb{R}^d)$ with $p = 2$ if $d \leq 3$, and $p > \frac{d}{2}$ if $d > 3$.

(A2) The set $\mathcal{I} \subset \mathbb{R}^d$, where the random impurities are located, is uniformly discrete, i.e.,

$$\inf_{\substack{\alpha,\beta\in\mathcal{I}\\ \alpha\neq\beta}} |\alpha - \beta| =: r_{\mathcal{I}} > 0.$$

($\widetilde{\text{A3}}$) For the probability measure $\mathbb{P}$ on $\Omega = \prod_{\alpha\in\mathcal{I}}[0, \eta_{\max}]$ we use conditional probabilities to define the following uniform bound

$$s(\varepsilon) = \sup_{\alpha} \operatorname*{ess\,sup}_{E\in\mathbb{R}} \operatorname*{ess\,sup}_{(\omega_\beta)_{\beta\neq\alpha}} \mathbb{P}\left\{\omega_\alpha \in [E, E+\varepsilon] \mid (\omega_\beta)_{\beta\neq\alpha}\right\}.$$

($\widetilde{\text{A4}}$) Let $E_0 := \inf \sigma(H_0)$ and let

$$H_{\mathrm{F}} := H_0 + \eta_{\max} \sum_{\alpha\in\mathcal{I}} U_\alpha$$

the subscript F standing for "full coupling". The single site potentials U_α, $\alpha \in \mathcal{I}$ are measurable functions on $\mathbb{R}^d$ that satisfy

$$c_U \chi_{\Lambda_{r_U}(\alpha)} \leq U_\alpha \leq C_U \chi_{\Lambda_{R_U}(\alpha)}$$

for all $\alpha \in \mathcal{I}$, with $c_U, C_U, r_U, R_U > 0$ independent of α. Here, $\Lambda_s(\beta)$ denotes the box with sidelength $2s$ and center β.

$$V_\omega(x) = \sum_{\alpha\in\mathcal{I}} \omega_\alpha U_\alpha(x)$$

and

$$H := H(\omega) := H_0 + V_\omega \text{ in } L^2(\mathbb{R}^d).$$

Assume that E_0 is a *weak fluctuation boundary* in the sense that $E_{\mathrm{F}} := \inf \sigma(H_{\mathrm{F}}) > E_0$.

Remark (1) In [1] (A3) and (A4) are stronger than their counterparts ($\widetilde{\text{A3}}$) (which actually is not an assumption at all) and ($\widetilde{\text{A4}}$) here.

(2) The modulus of continuity $s(\cdot)$ from (A3) also appears in [6], where, however, the variables appearing in the conditional probabilities are not displayed correctly. For a detailed discussion of regular conditional probabilities see, e.g., [12].

Wegner estimates, named after Wegner's original work [17], are an important tool in random operator theory. They give a bound on the probability that the eigenvalues of a local Hamiltonian come close to a given energy. For a list of some (recent) papers, see [2–6,9–11,13] and the account in the recent Lecture Notes Volume [16]. We consider a box $\Lambda \subset \mathbb{R}^d$ and denote by $H_\Lambda(\omega)$ the restriction of $H(\omega)$ to $L^2(\Lambda)$ with Dirichlet boundary conditions and with H_0^Λ the restriction of H_0 to $L^2(\Lambda)$ with Dirichlet boundary conditions. Here comes our main application of Theorem 1.1.

Theorem 2.1 *Assume* (A1)–(A2) *and* ($\widetilde{\text{A3}}$)–($\widetilde{\text{A4}}$). *Then, for every* $\delta > 0$ *there exists a constant* $C_W = C_W(\delta)$ *such that for any interval* $I = [E_0, E_F - \delta]$ *we have*:

$$\mathbb{P}\{\sigma(H_\Lambda(\omega)) \cap I \neq \varnothing\} \leq \mathbb{E}\{\operatorname{tr} P_I(H_\Lambda(\omega))\} \\ \leq C_W \cdot |\Lambda| \cdot s(\varepsilon).$$

Clearly, any application will need some further assumptions on $s(\cdot)$ for which we a priori just know that $0 \leq s(\varepsilon) \leq 1$ for all $\varepsilon > 0$.

Proof We rely on the analysis from [6]. The main point is to find an estimate

$$P_I(H_0^\Lambda) W_\Lambda P_I(H_0^\Lambda) \geq \kappa P_I(H_0^\Lambda) \tag{$**$}$$

with a constant κ independent of Λ and I (as long as $I \subset [E_0, E_F - \delta]$), and

$$W_\Lambda := \left(\sum_{\alpha \in \mathcal{I}} U_\alpha \right) \cdot \chi_\Lambda.$$

Once ($**$) is established, the proof of [6, Theorem 1.3] takes over, with minor modifications of notation.

But ($**$) follows easily from Theorem 1.1 and ($\widetilde{\text{A4}}$): As Dirichlet boundary conditions shift the spectrum up, for any $t \geq \eta_{\max}$:

$$\lambda(t) = \inf \sigma(H_0^\Lambda + tW_\Lambda) \geq \inf \sigma(H_0 + tW) \geq E_F.$$

For $I \subset [E_0, E_F - \delta]$ we see that

$$\lambda(\eta_{\max}) - \max I > \delta$$

so that we get an uncertainty inequality with

$$\kappa = \frac{\delta}{\eta_{\max}}.$$

□

Remark We should point out that the input from [6] is rather substantial. While the uncertainty principle is important to deal with possibly small support, there is also the issue of the correct volume factor which is settled in [6].

Like in the latter paper, if we assume on top that the IDS $N(\cdot)$ of the random operator H exists, then the preceding theorem implies that $N(\cdot)$ is as continuous as $s(\cdot)$ is.

Corollary 2.2 *Assume* (A1)–(A2) *and* $(\widetilde{\mathrm{A3}})$–$(\widetilde{\mathrm{A4}})$*, and, additionally that the IDS* $N(\cdot)$ *of* H *exists. Then there exists a locally bounded* $c_{\mathrm{W}}(\cdot)$ *on* $[E_0, E_{\mathrm{F}})$ *such that*

$$N(E+\varepsilon) - N(E) \leq c_{\mathrm{W}}(E) \cdot s(\varepsilon)$$

for ε *small enough. In particular,* $N(\cdot)$ *is continuous on* $[E_0, E_{\mathrm{F}})$*, whenever* $s(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

3 Localization near fluctuation boundaries

As mentioned already in the introduction, the validity of a Wegner estimate was missing for a proof of localization via multiscale analysis. Due to Theorem 2.1, this is now resolved. The assumptions we need to make now are weaker than what is found in [1] but stronger than what we needed in the preceding section.

$(\overline{\mathrm{A3}})$ The random variables $\eta_\alpha : \Omega \to \mathbb{R}$, $\omega \mapsto \omega_\alpha$ are independent, supported in $[0, \eta_{\max}]$ and the modulus of continuity

$$s(\varepsilon) := \sup_{\alpha \in \mathcal{I}} \sup_{E \in \mathbb{R}} \mathbb{P}\{\eta_\alpha \in [E, E+\varepsilon]\}$$

satisfies

$$s(\varepsilon) \leq (-\ln \varepsilon)^{-\alpha}$$

for some $\alpha > \frac{4d}{2-m}$, where $m \in (0,2)$ is as in $(\overline{\mathrm{A4}})$.

$(\overline{\mathrm{A4}})$ Additionally to $(\widetilde{\mathrm{A4}})$ assume that there exists $m \in (0,2)$ and L^* such that for some $\xi > 0$, all $L \geq L^*$ and all $x \in \mathbb{Z}^d$:

$$\mathbb{P}\left\{\sigma\left(H_{\Lambda_L(x)}(\omega)\right) \cap [E_0, E_0 + L^{-m}] \neq \varnothing\right\} \leq L^{-\xi}.$$

Remark Clearly, the assumptions (A1)–(A4) from [1] imply (A1)–(A2) and $(\overline{\mathrm{A3}})$–$(\overline{\mathrm{A4}})$ so that the localization result below extends the localization result from the latter paper.

Theorem 3.1 *Assume* (A1)–(A2) *and* $(\overline{\mathrm{A3}})$–$(\overline{\mathrm{A4}})$*. Then there is a* $\delta > 0$ *such that in* $[E_0, E_0+\delta]$ *the spectrum of* $H(\omega)$ *is pure point* $\mathbb{P}$*-a.s. Moreover, for* p *small enough and* $\eta \in L^\infty$ *with* $\operatorname{supp} \eta \subset [E_0, E_0+\delta]$ *it follows that*

$$\mathbb{E}\left\{\left\| |X|^p \eta\left(H(\omega)\right) \cdot \chi_K \right\|\right\} < \infty$$

for every compact $K \subset \mathbb{R}^d$.

Remark (1) Maybe one can strengthen the estimate of Theorem 3.1 in the sense of [7]. Note, however, that in the latter paper a stronger Wegner estimate is supposed to hold.

(2) The theorem provides an extension to $d \geq 4$ of the main result of [1]. Moreover, there is no technique at the moment to include single site distributions as singular as the ones allowed here in the fractional moment methods. In these aspects, our result considerably extends the main result of [1].

(3) At the same time, the estimates that come out of our analysis are weaker than those in the latter paper.

Sketch of the proof We use the multiscale setup from [14]. By now it is quite well understood that homogeneity does not play a major role so that multiscale analysis goes through without much alterations if we can verify the necessary input, i.e., Wegner estimates and initial length scale estimates.

Let us begin with the latter: Combes–Thomas estimates give that $(\overline{\text{A4}})$ implies an initial estimate of the form $G(I, \ell, \gamma, \xi)$ with ξ from $(\overline{\text{A4}})$, $I_\ell = [E_0, E_0 + \frac{1}{2}\ell^{-m}]$, $\gamma_\ell = \ell^{-\frac{m}{2}}$ so that the exponent is of the form $\gamma_\ell = \ell^{\beta-1}$ with $\beta = \beta_m = \frac{2-m}{2}$.

We have to check that an appropriate Wegner estimate is valid as well, i.e., that, for some $q > d, \theta < \frac{\beta}{2}$ we have that

$$\mathbb{P}\left\{\text{dist}\left(\sigma\left(H_\Lambda(\omega)\right), E\right) \leq \exp(-L^\theta)\right\} \leq L^{-q}$$

for L large enough. We check that

$$\begin{aligned}\mathbb{P}\left\{\text{dist}\left(\sigma\left(H_\Lambda(\omega)\right), E\right) \leq \exp(-L^\theta)\right\} &\leq s\left(2\exp(-L^\theta)\right)\\ &\leq \left[-\ln\left(2\exp(-L^\theta)\right)\right]^{-\alpha}\\ &= (-\ln 2 + L^\theta)^{-\alpha}\\ &\sim L^{-\theta\alpha} \leq L^{-\theta\frac{4d}{2-m}}\\ &= L^{-(\frac{2-m}{4}+\kappa)\frac{4d}{2-m}} = L^{-d-\kappa\frac{4d}{2-m}}\end{aligned}$$

where we have chosen $\theta = \frac{\beta}{2} - x = \frac{2-m}{4} - \kappa$ with positive κ. Then, the Wegner estimate is fulfilled for $q = d + \kappa\frac{4d}{2-m}$.

The appropriate p in the strong dynamical localization estimate can be chosen at most $\inf\left\{\kappa\frac{4d}{2-m}, \xi\right\}$ with ξ from $(\overline{\text{A3}})$.

An appeal to [14, Theorems 3.2.2 and 3.4.1] gives the result.

Acknowledgments Fruitful discussions with G. Stolz are gratefully acknowledged. Part of this work was done at a stay of D.L. and P.S. at Paris, financial support by the DFG (German Science Foundation) and the University Paris Diderot Paris 7 are gratefully acknowledged.

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