Oberseminar Kaiserslautern-Mannheim WS 2012/2013

Hodge theory, $\mathcal{D}$-Modules and non-isolated singularities

Introduction

The overall aim of the seminar is a deeper understanding of topological, differential and Hodge theoretic invariants associated to certain non-isolated hypersurfaces. The main case of interest will be the so-called **free divisors**, defined back in 1980 by K. Saito ([Sai80]), which provide a rich class of examples of such hypersurfaces. As free divisors generalize normal crossing divisors they are supposed to have similar properties. The idea of the seminar is to study some recent papers on these divisors, and to work actively on open problems in the field. We hope to make some substantial progress for at least some of the questions mentioned below. The seminar can also serve as a starting point for a master or Ph.D. thesis in a related area.

Free divisors appear in many areas of algebraic geometry, mainly as various types of discriminants. Generally speaking these are loci of points in algebraic or analytic varieties with exceptional behavior. Among varieties containing discriminants are moduli or deformation spaces of given geometric objects, and representation spaces of algebraic groups. In particular, the so-called **linear free divisors** ([BM06]), or more general discriminants in prehomogenous vector spaces ([GS10] and [GMS11]) provide a combinatorial device (namely, quiver representations) to construct many examples of free divisors.

One central topic of the seminar is the *logarithmic comparison theorem* (LCT for short), which states, roughly speaking, that the cohomology of the complement of a sufficiently good divisor (which should be at least free) can be computed by the complex of differential forms with logarithmic poles along the divisor. This theorem is well-known in the case of a normal crossing divisor. However, although there is a notion of logarithmic forms for any divisors (with reasonable behavior at least for free ones), it does not hold in general. There are a number of cases where it is true, most prominently those which are *locally (weakly) quasi-homogenous* (see [CJNMM96]). As an interesting application of the LCT, one can show for instance that *Aluffi's conjecture* holds true: the Chern–Schwarz–MacPherson class of the complement of a locally quasihomogeneous free divisor in complex projective space is given by the total Chern class of the module of logarithmic derivations (see [Lia12b]).

For LCT free divisors, one can ask for a far-reaching generalization of a classical fact for the normal crossing case, namely, to describe the Hodge filtration on the cohomology of the complement of a free divisor by the logarithmic de Rham complex. More precisely, let $D \subset X$ be a free divisor in a complex manifold such that the LCT hold true, that is, such that the complex $\Omega^\bullet_X(\log D)$ is quasi-isomorphic to $Rj_*\mathbb{C}_{X\backslash D}$, where $j : X\backslash D \hookrightarrow X$ is the natural open embedding. Then we seek to study the following

Problem 1. *Is the natural morphism*

$$\Omega^\bullet_X(\log D) \longrightarrow \Omega^\bullet_X(*D)$$

a (strict) filtered quasi-isomorphism, where both complexes are equipped with the so-called "filtration bête", that is, the filtration defined by putting

$$F^k\mathcal{K}^\bullet := \left[0 \to \mathcal{K}^k \to \mathcal{K}^{k+1} \to \ldots \to \mathcal{K}^n \to 0\right]$$

for $\mathcal{K}^\bullet = \Omega_X^\bullet(\log\ D)$ *resp.* $\mathcal{K}^\bullet = \Omega_X^\bullet(*D)$.

Although it is unlikely that the seminar will answer this question in full generality, we hope to grasp some ideas on how to attack it.
Another problem in this context concerns a symmetry property of the roots of the b-function. In the classical case of an quasihomogeneous isolated hypersurface there is such a symmetry coming from the mixed Hodge structure of the Milnor fibre. Similar symmetries have been observed for reductive linear free divisors [GS10] and for free divisors of linear Jacobian type [NM12]. However, a unifying underlying principle is still missing.

Problem 2. *For which (free) divisors is there a symmetry of the roots of the b-function?*

Another (related) problem concerns a generalization of the Lê–Saito theorem proved in its basic form in [Sai80], [TS84], and [GS11]. We would like to understand whether the duality approach of [GS11] can be applied to

Problem 3. *Let D be a divisor in a complex in a complex manifold X with normalization $\pi : \tilde{D} \to D$ and let $\rho : \Omega_X^\bullet(\log D) \to \Omega_D^{\bullet-1} \otimes \mathcal{M}_D$ be K. Saito's logarithmic residue map. Let $\nabla : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{O}_X} \Omega_X^1(\log D)$ be an integrable logarithmic connection over an $\mathcal{O}_X$-free module $\mathcal{H}$. Does $\rho(\nabla(\mathcal{H})) \subset \mathcal{H} \otimes_{\mathcal{O}_X} \pi_*\mathcal{O}_{\tilde{D}}$ imply that the local monodromies of the connection abelian?*

In the course of the seminar, several other open problems will be discussed.
Below is a tentative list of talks; the precise topics can and shall be adapted to the specific interests of the audience.

1 Overview (1 talk)

This talk should give an overview on free divisors. It shall briefly mention the definition and main properties without going into details, and then concentrate on key results and open problems in the field. Here is a possible list of topics:

1. logarithmic vector fields and logarithmic differential forms
2. classes of examples: discriminants, linear free divisors, hyperplane arrangements, etc.
3. related properties: logarithmic stratification, local quasihomogeneity, Euler homogeneity, etc.
4. related topics: logarithmic residues, b-functions, characteristic classes, etc.

Possible references: [Sai80], [OT92, §4], [BM06], [GMS11], [GS10], [NM12], etc.

2 Aspects of $\mathcal{D}$-modules and Hodge theory (1-2 talks)

In this talk basics from the theory of $\mathcal{D}$-modules and from Hodge theory shall be presented to prepare the audience for the later talks, in which these techniques will be used. Probably the definition of vector bundles with connections (holomorphic and meromorphic one's) should be mentioned, the definition and some basic material on $\mathcal{D}$-modules (such as characteristic varieties, holonomicity, regularity, direct and inverse images etc.). The speaker can introduce pure and mixed Hodge structures, variations of them and how this gives rise to filtered $\mathcal{D}$-modules. If time permits, the idea of Deligne's proof of the existence of a mixed Hodge structure on a smooth quasi-projective variety should be presented, in order to highlight the use of the logarithmic de Rham complex in the classical case of (the complement of) a normal crossing divisor.

3 Free and linear free divisors, discriminants and quiver representations (1-2 talks)

This talk aims at a more detailed presentation of the case of linear free divisors and more generally of quiver discriminants. The basic constructions from [BM06] should be explained. More precisely, the talk will cover the following topics (see also [GMNS09] and [GMS11]).

1. Reminder on definitions of free and linear free divisors
2. Saito's criterion of freeness
3. Prehomogenous vector spaces
4. quiver representations, discriminants and how this leads to linear free divisors
5. examples, Dynking quivers, examples of calculation of equations

4 The logarithmic comparison theorem (1 talk)

This talk should give an overview on the current state-of-art on the LCT. In particular, it should report on the main result from [CJNMM96] (namely, theorem 1.1. of loc.cit.), if possible with some details on the proof. If time permits, the generalization to the case of locally *weakly* quasi-homogenous divisors should be mentioned, without proof.

5 Duality theory for logarithmic connections (2-3 talks)

This aim of these talks is to cover the main results of [CMNM05]. The basic idea is to look at *logarithmic connections* as mentioned in the introduction above. They can be viewed as modules $\mathcal{E}$ over $V_0\mathcal{D}_X := \{P \in \mathcal{D}_X \,|\, \mathcal{P}(I) \subset \mathcal{I}\}$ (where $\mathcal{I}$ is the defining ideal of the divisor) which are free as $\mathcal{O}_X$-modules. Then any logarithmic connection $\mathcal{E}$ gives rise to a logarithmic de Rham resp. logarithmic Spencer complex, and one can define a logarithmic dual object. On the other hand, the tensor product $\mathcal{D}_X \otimes_{V_0(\mathcal{D}_X)} \mathcal{E}$ is a $\mathcal{D}_X$-module and there is the usual holonomic dual module. The main result of loc. cit. gives a compatibility between these two dual constructions. There is the notion of an admissible logarithmic connection, which turns out to be equivalent to the logarithmic de Rham complex being perverse. As an application, one can prove the above mentioned extension of the LCT to the weakly quasi-homogenous case.
Other relevant references are: [CM99] for some generalities on $V_0(\mathcal{D}_X)$ (see also [Sch07]) and on the logarithmic de Rham complex, [CMNM09] for "LCT-type"-results for logarithmic connections and a calculation of the intersection cohomology $\mathcal{D}$-modules of (the local system corresponding to an) logarithmic connection and [CJU02] which contains a "predecessor" of the main result of [CMNM05].

6 b-Functions (1-2) talks

In a first talk the symmetry of the b-function for linear free divisors will be covered explaining results in [GS10] and [Sev11]. A second talk is dedicated to explain the symmetry in the case of free divisors of linear Jacobian type based on [NM12].
Gaining some insight in Problem 2 on the way would be great.

7 Lê–Saito theorem (1-2 talks)

The classical Lê–Saito theorem (equivalence of toplological and geometrical property) is formulated and the proof of the easy implication is given [Sai80]. Ideally an idea of the difficult implication is given [TS84].
Then logarithmic residues are introduced and the recent algebraic extension of the Lê–Saito theorem from [GS11] is discussed.
Ideally some connection with Problem 3 is developped.

8 Characteristic classes (1-2 talks)

Recent results by Xia Liao [Lia12b, Lia12a] on Aluffi's conjecture on Chern–Schwarz–MacPherson classes of complements of free divisors are discussed.

References

[BM06] Ragnar-Olaf Buchweitz and David Mond, *Linear free divisors and quiver representations*, Singularities and computer algebra (Cambridge) (Christoph Lossen and Gerhard Pfister, eds.), London Math. Soc. Lecture Note Ser., vol. 324, Cambridge Univ. Press, 2006, Papers from the conference held at the University of Kaiserslautern, Kaiserslautern, October 18–20, 2004, pp. 41–77.

[CJNMM96] Francisco J. Castro-Jiménez, Luis Narváez-Macarro, and David Mond, *Cohomology of the complement of a free divisor*, Trans. Amer. Math. Soc. **348** (1996), no. 8, 3037–3049.

[CJU02] F. J. Castro-Jiménez and J. M. Ucha, *Free divisors and duality for $\mathcal{D}$-modules*, Tr. Mat. Inst. Steklova **238** (2002), no. Monodromiya v Zadachakh Algebr. Geom. i Differ. Uravn., 97–105.

[CJUE05] F. J. Castro-Jiménez and J. M. Ucha-Enríquez, *Logarithmic comparison theorem and some Euler homogeneous free divisors*, Proc. Amer. Math. Soc. **133** (2005), no. 5, 1417–1422 (electronic).

[CM99] Francisco J. Calderón-Moreno, *Logarithmic differential operators and logarithmic de Rham complexes relative to a free divisor*, Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 5, 701–714.

[CMNM05] Francisco Javier Calderón Moreno and Luis Narváez Macarro, *Dualité et comparaison sur les complexes de de Rham logarithmiques par rapport aux diviseurs libres*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 47–75.

[CMNM09] F. J. Calderón Moreno and L. Narváez Macarro, *On the logarithmic comparison theorem for integrable logarithmic connections*, Proc. Lond. Math. Soc. (3) **98** (2009), no. 3, 585–606.

[GMNS09] Michel Granger, David Mond, Alicia Nieto, and Mathias Schulze, *Linear free divisors and the global logarithmic comparison theorem.*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 1, 811–850.

[GMS11] Michel Granger, David Mond, and Mathias Schulze, *Free divisors in prehomogeneous vector spaces*, Proc. Lond. Math. Soc. (3) **102** (2011), no. 5, 923–950. MR 2795728 (2012h:14052)

[GS10] Michel Granger and Mathias Schulze, *On the symmetry of b-functions of linear free divisors*, Publ. Res. Inst. Math. Sci. **46** (2010), no. 3, 479–506.

[GS11] ———, *Dual logarithmic residues and free complete intersections*, arXiv.org **1109.2612** (2011), Submitted.

[Lia12a] Xia Liao, *Chern classes of logarithmic derivations for free divisors with jacobian ideal of linear type*, Preprint arXiv:1210.6079, 2012.

[Lia12b] ———, *Chern classes of logarithmic vector fields for locally-homogenous free divisors*, Preprint arXiv:1205.3843, 2012.

[NM08] L. Narváez Macarro, *Linearity conditions on the Jacobian ideal and logarithmic-meromorphic comparison for free divisors*, Singularities I (Jean-Paul Brasselet, José Luis Cisneros-Molina, David Massey, José Seade, and Bernard Teissier, eds.), Contemp. Math., vol. 474, Amer. Math. Soc., Providence, RI, 2008, Algebraic and analytic aspects, pp. 245–269.

[NM12] ———, *A duality approach to the symmetry of bernstein-sato polynomials of free divisors*, Preprint arXiv:1201.3594, 2012.

[OT92] Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 300, Springer-Verlag, Berlin, 1992.

[Sai80] Kyoji Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291.

[Sch07] Mathias Schulze, *A criterion for the logarithmic differential operators to be generated by vector fields*, Proc. Amer. Math. Soc. **135** (2007), no. 11, 3631–3640 (electronic).

[Sev11] Christian Sevenheck, *Bernstein polynomials and spectral numbers for linear free divisors*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 1, 379–400.

[Tor04] Tristan Torrelli, *On meromorphic functions defined by a differential system of order 1*, Bull. Soc. Math. France **132** (2004), no. 4, 591–612.

[TS84] Lê Dũng Tráng and Kyoji Saito, *The local* π_1 *of the complement of a hypersurface with normal crossings in codimension* 1 *is abelian*, Ark. Mat. **22** (1984), no. 1, 1–24.