Übungsaufgaben zur Katastrophentheorie

1. *(1 Punkte)* Betrachten Sie die $\mathbb{R}$-Algebra $R = \mathbb{R}[x_1, \ldots, x_n]$, das maximale Ideal $m \subset R$, welches von $x_1, \ldots, x_n$ erzeugt wird sowie den $\mathbb{R}$-Untervektorraum $R = \mathbb{R}[x_1, \ldots, x_n]_{\leq d}$ aller Polynome vom Grad höchstens d. Zeigen Sie

   (a) Für alle $k \in \mathbb{N}$ ist die Quotientenalgebra $\mathbb{R}[[\underline{x}]]/\mathbf{m}^k_{\mathbb{R}[[\underline{x}]]}$ ein endlich-dimensionaler $\mathbb{R}$-Vektorraum der Dimension $\binom{n+k-1}{k-1}$.

   (b) Zeigen Sie, dass analog für alle $k \in \mathbb{N}$ gilt, dass

   $$\dim_{\mathbb{R}} \left(\frac{\mathbf{m}^k_{\mathbb{R}[[\underline{x}]]}}{\mathbf{m}^{k+1}_{\mathbb{R}[[\underline{x}]]}} \right) = \binom{n+k-1}{k}$$

   ist.

2. *(3 points)*

   (a) Show that if $(R, \mathbf{m})$ is local, then for any $x \in \mathbf{m}$, the element $1 + x$ is a unit in R.

   (b) Let R be a local ring and $I \subset R$ any ideal. Show that the factor ring R/I is also local.

   (c) Let $\mathbb{R}[[x_1, \ldots, x_n]]$ be the local ring of formal power series over $\mathbb{R}$. Give an explicit expression for the inverse of $1 + x$ for $x \in \mathbf{m}$.

   (d) Show that the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ is not local.

   (e) For any local ring, define $k := R/\mathbf{m}$. Then k is a field, called the residue class field of $(R, \mathbf{m})$. Show that the residue class field of $\mathcal{E}$ is isomorphic to $\mathbb{R}$.

   (f) Let $(R, \mathbf{m})$ be local and define by

   $$H_R(d) := \dim_k(\mathbf{m}^d/\mathbf{m}^{d+1})$$

   the Hilbert function of the local ring R. Calculate the Hilbert function for the following local rings

      i. $R = \mathcal{E}_n$, $R = \mathbb{R}[[x_1, \ldots, x_n]]$,
      ii. $R = \mathbb{R}[[x, y]]/(xy)$,
      iii. $R = \mathbb{R}[[x, y]]/(x^2 - y^3)$.

3. *(2 points)* Consider the local ring $(\mathcal{E}_n, \mathbf{m})$ and let $\Psi := (\Psi_1, \ldots, \Psi_n) \in (\mathbf{m})^n \subset (\mathcal{E}_n)^n = \mathcal{E}_{n,n}$ (caution: $(\mathbf{m})^n$ denotes the direct sum $\mathbf{m} \oplus \ldots \oplus \mathbf{m}$).

   (a) Show that the substitution map (also called pull-back or inverse image)

   $$\begin{array}{rcl} \Psi^* : R & \longrightarrow & R \\ f & \longmapsto & f \circ \Psi \end{array}$$

   is an algebra homomorphism preserving the identity. Show further that $\Psi^*(\mathbf{m}^k) \subset \mathbf{m}^k$.

   (b) Deduce from (a) that Ψ induces linear maps

   $$(\Psi^*)_k : \mathbf{m}^k/\mathbf{m}^{k+1} \longrightarrow \mathbf{m}^k/\mathbf{m}^{k+1}.$$

   Show that Ψ is an automorphism iff $(\Psi^*)_1$ is invertible.

4. *(1 Punkt)* Zeigen Sie, dass eine formale Potenzreihe $\sum_{\underline{\nu} \in \mathbb{N}^n} a_{\underline{\nu}} x^{\underline{\nu}} \in \mathbb{R}[[\underline{x}]] := \mathbb{R}[[x_1, \ldots, x_n]]$ eine Einheit in $\mathbb{R}[[\underline{x}]]$ ist genau dann, wenn $a_{\underline{0}} \neq 0$ gilt.