Universität Mannheim
Lehrstuhl für Mathematik VI

Christian Sevenheck
22.02.2007

Basic notions from topology

The aim of this notes is to introduce as quickly as possible some notions and basic concepts from general topology (also called set-theoretic topology) which are used in almost all domains in mathematics.

Definition 1 *1. A topological space is a pair $(X,\mathcal{U})$ where X is a set and $\mathcal{U}$ is a collection of subsets of X subject to the following properties.*

   (a) $\emptyset \in \mathcal{U}$ and $X \in \mathcal{U}$.

   (b) $U, V \in \mathcal{U}$ implies $U \cap V \in \mathcal{U}$.

   (c) $(U_i)_{i\in I} \in \mathcal{U}$ (I some index set) implies $\bigcup_{i\in I} U_i \in \mathcal{U}$

   We call the elements of $\mathcal{U}$ open sets. Any complement $W := X\backslash U$ of $U \in \mathcal{U}$ is called a closed set (Note that the closed sets satisfy dual axioms, i.e., the any intersection of closed sets is closed and finite union of closed sets are closed. One might as well define a topology by specifying the set of closed subsets and requiring the dual axioms).

2. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be topological spaces and $f : X \to Y$ be a map. Then f is called continuous if for any $V \in \mathcal{V}$ we have that $f^{-1}(V) \in \mathcal{U}$.

Here are some simple exercises which might help to get used to these definitions.

1. Let $(X,\mathcal{U})$ be a topological space and $Y \subset X$ be a subset. Show that

$$\mathcal{V} := \{U \cap Y \,|\, U \in \mathcal{U}\}$$

   defines a collection of open sets in Y making $(Y,\mathcal{V})$ into a topological space. This topology on Y is called *induced by X*.

2. Let (X,d) be a metric space (i.e., let $d : X \times X \to \mathbb{R}_+$ be a function with the usual properties of a metric). Show that $(X,\mathcal{U})$ where $\mathcal{U}$ is the set of open sets of X (i.e., sets such that $\forall x \in U : \exists \epsilon : B^d_\epsilon(x) \subset U$) is a topological space. In particular, any euclidean (resp. hermitian) vector space such as $\mathbb{R}^n$ resp. $\mathbb{C}^n$ (and any subspace of it) is a topological space (this topology is sometimes referred to as the usual or standard topology).

3. For any set X, let $\mathcal{U}_1 := \{X, \emptyset\}$ and $\mathcal{U}_2 := \mathcal{P}^X = \{U \,|\, U \subset X\}$ (the power set of X). Show that both $(X,\mathcal{U}_1)$ and $(X,\mathcal{U}_2)$ are topological spaces. Do there exist metrics d_1 resp. d_2 on X which define the topologies $\mathcal{U}_1$ resp. $\mathcal{U}_2$ as in 2. ?

4. Let X be a topological space and $Y \subset X$ be a subset. The closure of Y in X, denoted by $\overline{Y}$ is the smallest closed subset of X containing Y. Similarly, the interior of Y is by definition $Y^\circ := \cup_{U \subset Y open} U$ (i.e., the largest open subset contained in Y). The border of Y in X is defined as $\partial Y := \overline{Y}\backslash Y^\circ$. Show that $\overline{Y} = (X\backslash(X\backslash Y)^\circ)$. What are the closure and the border in the following situations, where all the time the topology is the one induced from the usual topology on $\mathbb{R}$?

   (a) $X = \mathbb{R}$, $Y = (0,1)$

   (b) $X = (0,2)$, $Y = (0,1)$

   (c) $X = \mathbb{R}$, $Y = \mathbb{Q}$

Some more definitions:

Definition 2 1. *Let X be a topological space and $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X. We say that (x_n) converges to $x \in X$ if for any open neighborhood of x (i.e., any open set U containing x) there is an integer N with $x_k \in U$ for all $k > N$.*

2. *A topological space X is called Hausdorff if for any two distinct points $x, y \in X$ there are open sets $U, V \subset X$ with $x \in U$, $y \in V$ and such that $U \cap V = \emptyset$. In other words, in a Hausdorff space, any two distinct points can be separated.*

3. *A topological space X is called connected if the only sets which are both closed and open are the empty set and X itself.*

4. *A topological space $(X, \mathcal{U})$ is called compact if the following holds: Given any open covering of X, i.e., a subset $\mathcal{V} \subset \mathcal{U}$ such that $X = \bigcup_{U\in\mathcal{V}} U$, then there exists a **finite** subset $\{U_i\}_{i=1,\ldots,n} \subset \mathcal{V}$ with $X = \bigcup_{i=1}^{n} U_i$.*

5. *A continuous map $f : X \to Y$ between topological spaces is called a homeomorphism if it is a bijection and if the inverse map is also continuous.*

Here are some more exercise to think about these notions.

1. Let X be a topological space which is Hausdorff. Show that the limit of a sequence (x_n) in X, if it exists, is uniquely determined. Show that the topology $\mathcal{U}_1 = \{\emptyset, X\}$ is not Hausdorff but that $\mathcal{U}_2 = \mathcal{P}^X$ is (as well as any of the "usual" topologies on (subsets of) $\mathbb{R}^n$ or $\mathbb{C}^n$).

2. Let $f : X \to Y$ be a homeomorphism of topological spaces. Show that Y is compact resp. connected if and only if X is so.

3. Which of the following spaces are compact (in the standard topology)?
   (a) The intervals $[0, 1]$ and $[0, 1)$
   (b) The space $\mathbb{R}$
   (c) The disc $D := \{z \in \mathbb{C} \mid |z| \leq 1\} \subset \mathbb{C}$.
   (d) The area $S := \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1, |y| < 1\} \subset \mathbb{R}^2$.

4. Which of the following spaces are connected (in the standard topology)?
   (a) The intervals $[0, 1)$, $[0, 1]$
   (b) The spaces $[0, 1] \cup [2, 3]$ and $[0, 1] \cup (1, 2]$.
   (c) A convex subset of a metric vector space V, i.e., a subset $U \subset V$ such that for any $x, y \in U$, and any $t \in [0, 1]$, the line $tx + (1 - t)y$ in contained in U

5. Let X be Hausdorff and $A \subset X$ be a compact subset (with the induced topology). Show that A is closed in X.

6. Decide whether the following topological spaces are homeomorphic or not (with respect to the standard topology on $\mathbb{R}^n$).
   (a) The intervals $[0, 1]$ and $[0, 2]$.
   (b) The intervals $[0, 1]$ and $[0, 2)$.
   (c) The intervals $[0, 1]$ and $(0, 1)$.
   (d) The circle $S^1 = \left\{x \in \mathbb{C} \mid |x|^2 = x \cdot \overline{x} = 1\right\}$ and the interval $[0, 1)$.