Universität Mannheim
Lehrstuhl für Mathematik VI

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# Exercises Algebraic Geometry
# Sheet 6 - solutions

1. **Exercise:** (The join of projective varieties) Let $X, Y \subset \mathbb{P}^n$ be *disjoint* projective varieties. Then let $J(X,Y) := \bigcup_l \{l \subset \mathbb{P}^n \text{ line } | l \cap X \neq \emptyset, l \cap Y \neq \emptyset\} \subset \mathbb{P}^n$. Show that $J(X,Y)$ is a closed subset of $\mathbb{P}^n$.

   (a) Let $(\mathbb{P}^n)^* \cong G(2, n+1)$ be the dual projective space. It can be defined either as $\mathbb{P}\left((k^{n+1})^*\right)$ or as the space (the Grassmanian) of two-dimensional subspaces of $k^{n+1}$. Then let

   $$Inc := \{(l,p) \in (\mathbb{P}^n)^* \times \mathbb{P}^n \,|\, p \in l\}$$

   be the *Incidence variety*. We will show that *Inc* is closed in $(\mathbb{P}^n)^* \times \mathbb{P}^n$: We have the canonical projection $\pi : (k^n)^* \times k^n \twoheadrightarrow (\mathbb{P}^n)^* \times \mathbb{P}^n$ and it is sufficient to show that $\pi^{-1}(Inc)$ (the affine cone) is a closed affine algebraic set in $(k^n)^* \times k^n$. But this is obvious, as we have

   $$\pi^{-1}(Inc) = \{(\varphi, v) \in (k^n)^* \times k^n) \,|\, \varphi(v) = 0\}$$

   (b) Consider the following diagram of projective morphisms

   $$\begin{tikzcd} & (\mathbb{P}^n)^* \times \mathbb{P}^n \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ (\mathbb{P}^n)^* & Inc \arrow[u, hook] & \mathbb{P}^n \end{tikzcd}$$

   For any closed subset $Z \subset (\mathbb{P}^n)^*$, denote by $\mathcal{L}_Z$ the union $\cup_l l$ in $\mathbb{P}^n$ of all lines $l$ such that $l \in Z$. Obviously (check this!), we have $\mathcal{L}_Z := \pi_1\left(\pi_2^{-1}(Z) \cap Inc\right)$. This shows (main theorem on projective varieties), that $\mathcal{L}_Z$ is a closed subset of $\mathbb{P}^n$.

   (c) For any subvariety $X$ of $\mathbb{P}^n$, consider $Z_X := \pi_2\left(\pi_1^{-1}(X) \cap Inc\right) \subset (\mathbb{P}^n)^*$. It is clear (check this!) that points of $Z_X$ are the lines in $\mathbb{P}^n$ passing through $X$. Again we see that $Z_X$ is closed in $(\mathbb{P}^n)^*$. On the other hand, we have $J(X,Y) = \mathcal{L}_{Z_X \cap Z_Y}$ so that $J(X,Y)$ is a closed subset of $\mathbb{P}^n$ by part (b).

2. **Exercise:** Prove that for $X, Y \subset \mathbb{P}^n$ closed subvarieties, if $\dim(X) + \dim(X) - n \geq 0$, then $X \cap Y$ is not empty.

   (a) First note that we can chose closed embeddings

   $$\begin{array}{rcl} j_1 : \mathbb{P}^n & \longrightarrow & \mathbb{P}^{2n+1} \\ (x_0 : \ldots : x_n) & \longmapsto & (x_0 : \ldots : x_n : 0 : \ldots : 0) \end{array} \qquad \begin{array}{rcl} j_2 : \mathbb{P}^n & \longrightarrow & \mathbb{P}^{2n+1} \\ (y_0 : \ldots : y_n) & \longmapsto & (0 : \ldots : 0 : y_0 : \ldots : y_n) \end{array}$$

   such that $\widetilde{X} = j_1(X) \cong X$ and $\widetilde{Y} = j_2(Y) \cong Y$ are disjoint. It therefore makes sense to consider the join $J(\widetilde{X}, \widetilde{Y})$. Then it is obvious that $X \cap Y = J(\widetilde{X}, \widetilde{Y}) \cap V\left((x_i - y_i)_{i=0,\ldots,n}\right)$. In order to conclude, it will be sufficient to show that $\dim\left(J(\widetilde{X}, \widetilde{Y})\right) \geq \dim(X) + \dim(Y) + 1$, as then we obtain $\dim(X \cap Y) \geq \dim\left(J(\widetilde{X}, \widetilde{Y})\right) - (n+1) \geq 0$. To show this, we proceed by induction on $k = \dim(\widetilde{X}) + \dim(\widetilde{Y})$. For $k = 0$, the result is clear: A line which is the join of two points has dimension one. Otherwise, let $\dim(\widetilde{X}) = l$ and $\emptyset \neq \widetilde{X}_0 \subsetneq \widetilde{X}_1 \subsetneq \ldots \subsetneq \widetilde{X}_l = \widetilde{X}$ be a chain of closed subvarieties of maximal length. Then $J(\widetilde{X}_{l-1}, \widetilde{Y})$ is a proper closed subset of $J(\widetilde{X}, \widetilde{Y})$ which by induction hypotheses implies that $\dim(J(\widetilde{X}, \widetilde{Y})) > \dim(J(\widetilde{X}_{l-1}, \widetilde{Y}) \geq l - 1 + \dim(\widetilde{Y}) + 1 = \dim(\widetilde{X}) + \dim(\widetilde{Y})$ so that $\dim(J(\widetilde{X}, \widetilde{Y})) \geq \dim(\widetilde{X}) + \dim(\widetilde{Y}) + 1$ as required.

3. **Exercise:** Let $f: X \to Y$ be a dominant morphism between varieties (i.e., the image $f(X)$ is dense in $Y$) which is closed (i.e., which sends closed sets to closed sets). Then there is a non-empty open subset $U \subset Y$ such that for any $y \in U$, we have that $\dim(X) = \dim(Y) + \dim(Z_y)$ for any component $Z_y$ of $f^{-1}(y)$.

   (a) We first discuss a special case, namely, let $X \subset k^{n+1}$ be an affine variety and suppose that $f$ is the restriction to $X$ of the projection $\pi_{n+1} : k^{n+1} \to k^n$ which sends $(x_1, \dots, x_{n+1})$ to $(x_1, \dots, x_n)$. Let $k(X)$ resp. $k(Y)$ the fields of rational functions on $X$ resp. $Y$ (note that $k(Y) = k(\overline{f(X)})$. The induced map $f^* : k(Y) \to k(X)$ turns $k(X)$ into a field extension of $k(Y)$. It is clear that this extension is generated by $x_{n+1}$ as this is already true for the coeffient rings (i.e., $k[X]$ is generated as $k[Y]$-algebra by $x_{n+1}$). There are two possibilities: Either $x_{n+1}$ is algebraic over $k(Y)$ or it is transcendental. In the first case, there are elements $a_0, \dots, a_d \in k(Y)$ such that
   $$a_0 x_{n+1}^d + \ldots + a_d = 0 \quad \text{in } k(X)$$
   (note that by multiplying with the least common multiple of the denominators of the $a_i$, we might assume that $a_i \in k[X]$). For fixed $y \in f(X)$, the fibre $f^{-1}(y)$ is precisely the vanishing locus of $a_0(y)x^d + \ldots + a_d(y) \in k[x_{n+1}]$ which is a finite number of points except if $a_i(y) = 0$ for all $i$. This means that for $k(Y)$ algebraic over $k(X)$, there is a non-empty open set (the $a_i$ are not all constant) $U = f(X) \setminus \bigcup_i V(a_i) \subset Y$ with fibres of constant dimension (namely, of dimension zero).
   Let us now suppose that $x_{n+1}$ is transcendental over $k(Y)$. This means precisely that there is no polynomial $p(t)$ in $k(Y)[t]$ such that $p(x_{n+1}) = 0$ in $k(X)$. In other words, for any $f = a_0 x_{n+1}^d + \ldots + a_d \in I(X)$ (with $a_i \in k[x_1, \dots, x_n]$), we must have $a_i \in I(Y)$ as otherwise we would get a relation (a polynomial $p(t) \in k(Y)[t]$ as above) in $k[X]$ (and thus also in $k(X)$). This means that for any fixed $y \in f(X)$, the whole fibre $f^{-1}(Y)$ is contained in $X$ so that $X = Y \times k$. Again we have an open subset (which is $Y$ itself) with constant fibre dimension.

   (b) The next step is to generalize the situation slightly: Let $X \subset k^{n+m}$ be an affine variety and $f : X \to k^m$ the restriction of the projection sending $(x_1, \dots, n_{n+m})$ to $(x_1, \dots, x_m)$. Denote, as before, by $Y := \overline{f(X)}$ and by $p_i : k^{n+m} \to k^{n+i}$ the intermediate projections. Let finally $X_i := \overline{p_i(X)} \subset k^{n+i}$. The we have a tower of maps
   $$f : X = X_m \xrightarrow{\pi_{m-1}} X_{m-1} \xrightarrow{\pi_{m-2}} \ldots \xrightarrow{\pi_0} X_0 = Y \subset k^n$$
   where at each step, $\pi_i$ is the restriction of the projection $k^{n+i+1} \to k^{n+i}$ to $X_{i+1}$. For any $\pi_i : X_{i+1} \to X_i$, either $X_{i+1} \cong X_i \times k$ or there is an open set $U_i \subset X_i$ such that $\pi_i^{-1}(y)$ is of constant dimension zero for all $y \in U_i$, denote the set of indices $i$ where the former hypothesis occurs by $I$. Then $U := f(\cap_{i \notin I} p_i^{-1}(U_i)) \subset Y$ is open (here the closedness of $f$ is needed) and $f$ has constant fibre dimension over $U$, namely, $\#I = \dim(X) - \dim(Y)$.

   (c) The next more general case is that of an arbitrary dominant and closed morphism of affine varieties $f : X \to Y$ with $X \subset k^n$ and $Y \subset k^m$. But this can easily be reduced to the case just treated by considering the graph $\Gamma_f \subset k^{n+m} \to Y \subset k^m$.
   Now the general case is obtained by choosing an open affine cover $Y = \cup_i V_i$ and $X = \cup_i f^{-1}(V_i)$. We obtain open subsets $U_i$ of $V_i$ with constant fibre dimension equal to $\dim(X) - \dim(Y)$. These $U_i$ can be patched together to give an open subset $U$ of $Y$ with the desired properties.

4. **Exercise:** Suppose that we are given a closed morphism of varieties $f : X \to Y$ and a closed subset $Z \subset X$ with $f_{|Z}$ dominant such that for all $y \in Y$, the sets $f^{-1}(y) \cap Z$ are irreducible and of constant dimension $n$. Then $Z$ itself is irreducible.

   (a) An argument similar to the last exercise (see, e.g., Shafarevich, volume 1, 6.3) shows that the sets
   $$Y_r := \{y \in Y \mid \dim(f^{-1}(y) \cap Z) \geq r\}$$
   are closed in $Y$. Suppose now that $Z = \bigcup_i Z_i$ where $Z_i$ are the irreducible components of $Z$. For any $y \in Y$, let $d_i(y)$ be the dimension of the fibre over $y$ of $f_{|Z_i}$. Then by hypothesis we have that for all $y \in Y$, $\max_i d_i(Y) = n$. This implies that $Y = \bigcup_i \{y \in Y \mid d_i(y) \geq n\}$. As these sets are closed and $Y$ is irreducible, there must be an index $i$ such that $Y = \{y \in Y \mid d_i(y) \geq n\}$. Therefore, for all $y \in Y$, $f_{|Z_i}^{-1}(y) \subset f^{-1}(y) \cap Z$ is of dimension $n$, which implies that $f_{|Z_i}^{-1}(y) = f^{-1}(y) \cap Z$ so that $Z = Z_i$.

   (b) The following examples shows that the assumption of constant fibre dimension of $f_{|Z} : Z \to Y$ is essential: Let $Z = V(z) \cup V(x, y) \subset k^3 =: X$ be the union of a plane with a line, and let $f : X \to Y := k^2$ be the projection $f(x, y, z) = (x, y)$. Then for each $(x, y) \neq (0, 0)$, the fibre $f_{|Z}^{-1}(x, y)$ is $(x, y)$ itself, thus irreducible (and of dimension zero), and for $(x, y) = (0, 0)$, the fibre $f_{|Z}^{-1}(0, 0)$ is the whole line $V(x, y)$, which is also irreducible, but of dimension one. We see that in this case, all fibres are irreducible but $Z$ itself is not.