Universität Mannheim
Lehrstuhl für Mathematik VI

Christian Sevenheck
14.03.2007

Exercises Algebraic Geometry
Sheet 4

1. **The tensor product and product of varieties:** We first introduce a rather basic algebraic construction, discuss some properties and then use it to describe the coordinate rings of products of affine varieties. Let A be a ring and M and N two A-modules (one particular case that you should keep in mind is when A is a field k and M and N are (finitely generated) k-algebras, e.g., coordinate rings of algebraic sets). The we define $M \otimes_A N$ (called the tensor product of M and N over A) to be the quotient of the free A-module generated by all symbols $m \otimes n$ with $m \in M$ and $n \in N$ by the submodule generated by all elements of the form

$$\begin{array}{cc} (m_1 + m_2) \otimes n - m_1 \otimes n - m_2 \otimes n; & m \otimes (n_1 + n_2) - m \otimes n_1 - m \otimes n_2 \\ r(m \otimes n) - m \otimes rn; & rm \otimes n - m \otimes rn \end{array}$$

for any $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in A$. Show that

(a) The quotient map $\varphi : M \times N \to M \otimes_A N$ given by $(m, n) \mapsto m \otimes n$ is bilinear. (Recall that given A-modules P, Q, R, then a map of sets $\varphi : P \times Q \to R$ is called bilinear iff $\varphi(r_1 x_1 + s_1 y_1, r_2 x_2 + s_2 y_2) = r_1 r_2 \varphi(x_1, x_2) + r_1 s_2 \varphi(x_1, y_2) + s_1 r_2 \varphi(y_1, x_2) + s_1 s_2 \varphi(y_2, y_2)$ for all $r_1, r_2, s_1, s_2 \in A$ and all $x_1, x_2 \in P$, $y_1, y_2 \in Q$).

(b) The tensor product has the following universal property: Given any bilinear map $\psi : M \times N \to R$, where R is an A-module, then it factors uniquely over φ, i.e., there is a unique homomorphism of A-modules $\phi : M \otimes_A N \to R$ such that $\psi = \phi \circ \varphi$.

(c) We have isomorphisms $M \otimes_A A \cong M \cong A \otimes_A M$.

(d) Functoriality: Let $M' \xrightarrow{\gamma} M$ be a homomorphism of A-modules and P be any A-module, then there is an induced homomorphism

$$M' \otimes_A P \xrightarrow{\gamma \otimes id} M \otimes_A P$$

(e) Right-exactness: Let $M' \to M$ be a surjective homomorphism of A-modules, then the induced homomorphism $M' \otimes_A P \to M \otimes_A P$ is also surjective. More generally, given a short exact sequence of A-modules

$$0 \xrightarrow{\alpha} M'' \xrightarrow{\beta} M' \xrightarrow{\gamma} M \xrightarrow{\delta} 0$$

(this means that for any two successive arrows, such as β and α or γ and β, we have that the image of the first one equal the kernel of the second one, e.g. $Im(\beta) = Ker(\gamma)$ etc. In particular, any composition $\beta \circ \alpha$, $\gamma \circ \beta$, $\delta \circ \gamma$ is zero. It also implies (check this!) that β is injective and that γ is surjective) one obtains an exact sequence

$$M'' \otimes_A P \xrightarrow{\beta \otimes id} M' \otimes_A P \xrightarrow{\gamma \otimes id} M \otimes_A P \xrightarrow{\delta \otimes id} 0$$

(f) Let $X \subset k^n$ and $Y \subset k^m$ be algebraic sets. Show that the product

$$X \times Y := \{(x, y) \in k^n \times k^m \mid x \in X, y \in Y\} \subset k^{n+m}$$

is algebraic (Hint: the defining ideal of $X \times Y$ must be given in terms of the vanishing ideals $I(X)$ and $I(Y)$). Show further that $k[X \times Y] \cong k[X] \otimes_k k[Y]$. (Hint: show first the isomorphism $k[x_1, \ldots, x_n, y_1, \ldots, y_m] \cong k[x_1, \ldots, x_n] \otimes_k k[y_1, \ldots, y_m]$).

2. **Another description of a sheaf:** Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space X. Let $\mathcal{F}_x$ be the stalk of $\mathcal{F}$ at $x \in X$. Define

$$\widetilde{\mathcal{F}} := \bigcup_{x \in X} \mathcal{F}_x$$

to be the union of all stalks. We have the obvious projection map $\pi : \widetilde{\mathcal{F}} \to X$ which associates to any germ $[f]_x$ in $\mathcal{F}_x$ its base point x. We put a topology on $\widetilde{\mathcal{F}}$ generated by the following open neighborhoods of a given germ $[f]_x$: Let $f \in \mathcal{F}(U)$ be a representative of $[f]_x$ then the set

$$\widetilde{U}_{[f]_x} := \{[f]_y \,|\, y \in U\}$$

is by definition open in $\widetilde{\mathcal{F}}$.

   (a) Verify that π is continuous with respect to this topology.

   (b) We call a continuous map $s : U \to \widetilde{\mathcal{F}}$ (where U is open in X) a section (of π or of $\widetilde{\mathcal{F}}$) over U if $\pi \circ s = id_U$. Show that any map $s : U \to \widetilde{\mathcal{F}}$ satisfying $\pi \circ s = id_U$ is a section (i.e., continuous) iff the following holds: Denote for any $x \in U$ by $f_x \in \mathcal{F}(U_x)$ a representative for the image (germ) $s(x) \in \mathcal{F}_x$. Then for any $y \in U_x$, we must have $s(y) = [f_y]_y = [f_x]_y$.

   We denote by $\Gamma(U, \widetilde{\mathcal{F}})$ the set of all continuous sections of $\mathcal{F}$ over U. Show that any additional structure of the presheaf descends to $\Gamma(U, \widetilde{\mathcal{F}})$, that is, $\Gamma(U, \widetilde{\mathcal{F}})$ is an ring resp. k-algebra resp. module etc. if $\mathcal{F}$ is a presheaf of rings resp. k-algebras resp. modules and so on.

   (c) Define for any open set $U \subset X$ the following map

$$\begin{array}{rcl} \phi_U : \mathcal{F}(U) & \longrightarrow & \Gamma(U, \widetilde{\mathcal{F}}) \\ f & \longmapsto & (x \mapsto [f]_x) \end{array}$$

   Show that ϕ_U is an isomorphism of abelian groups (rings, ...) iff $\mathcal{F}$ is a sheaf (and not only a presheaf). This shows that for a sheaf, the data $\mathcal{F}(U)$ are equivalent to $\widetilde{\mathcal{F}}$ (which is called "espace étalé" of the (pre)sheaf $\mathcal{F}$).

   (d) Show that for any presheaf $\mathcal{F}$, the rule $U \mapsto \Gamma(U, \widetilde{\mathcal{F}})$ defines a sheaf (which is by (c) the same as $\mathcal{F}$ in case $\mathcal{F}$ is itself a sheaf). $\widetilde{\mathcal{F}}$ is called the sheafification of $\mathcal{F}$. What is the sheafification of the constant presheaf $\mathcal{F}_X(U) := \mathbb{R}$ on a topological space X?

3. **The Grassmann varieties:** This exercise gives a first glimpse of a huge part of algebraic geometry, called the theory of moduli spaces. The basic idea is that whenever you want to classify or describe any kind of (geometric) objects, you just gather them together in a set and try to put an extra structure on it, e.g., that of an algebraic variety. This is what is called a moduli space. Here is the most basic example, a variety parametrizing sub-vector spaces of fixed dimension of a given space.

   (a) Let V be an n-dimensional vector space over any field k. Define (for any $l \leq n$) $\mathrm{Gr}(l, n)$ to be the set $\{L \subset V \,|\, L \text{ sub-vector space of } V;\ \dim_k(L) = l\}$. Chose any basis $e_1, \ldots, e_n$ of V and consider the map

$$\begin{array}{rcl} \mathrm{Pl} : \mathrm{Gr}(l, n) & \longrightarrow & \mathbb{P}^{\binom{n}{l}-1} \\ L = span_k \langle (e_1, \ldots, e_n) \cdot A \rangle & \longmapsto & (\ldots, A_{i_1,\ldots,i_k}, \ldots)_{1 \leq i_1 < \ldots < i_k \leq n} \end{array}$$

   where $A_{i_1,\ldots,i_k}$ is the $l \times l$-minor of $A \in M(n \times l, k)$ obtained by taking the determinant of the matrix consisting of the rows $i_1, \ldots, i_k$ of A. Show that this map is well-defined and injective. (It is called the Plücker embedding.)

   (b) Show that the image $\mathrm{Pl}(\mathrm{Gr}(2, n))$ is a projective subvariety in $\mathbb{P}^{\binom{n}{2}-1}$. (This statement is true for any l but the proof is slightly more involved, although elementary). Therefore the Plücker embedding endows $\mathrm{Gr}(2, n)$ with the structure of a projective variety.

   (c) Show that the projective variety $\mathrm{Pl}(\mathrm{Gr}(2, n))$ has an open affine cover by affine spaces of dimension $2(n-2)$ (which implies that the dimension of $\mathrm{Pl}(\mathrm{Gr}(2, n))$ is $2(n-2)$, but we have not yet discussed this).

   (d) Show that $\mathrm{Gr}(2, 2) = \{pt\}$, $\mathrm{Gr}(1, 3) \cong \mathrm{Gr}(2, 3) \cong \mathbb{P}^2$ and that $\mathrm{Gr}(2, 4)$ is a quadric in $\mathbb{P}^5$ (i.e., a hypersurface cut out by a quadratic polynomial in $k[x_0, \ldots, x_5]$).