Universität Mannheim
Lehrstuhl für Mathematik VI

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Exercises Algebraic Geometry
Sheet 3

1. Check whether some of the following complex algebraic sets are isomorphic.
   (a) $V(y) \subset \mathbb{C}^2$
   (b) $V(xy) \subset \mathbb{C}^2$
   (c) $V(xy-1) \subset \mathbb{C}^2$
   (d) $V(xy(x-y)) \subset \mathbb{C}^2$
   (e) $V(x^2+y^2) \subset \mathbb{C}^2$
   (f) $V(y^3-x) \subset \mathbb{C}^2$
   (g) $V(xy, yz, xz) \subset \mathbb{C}^3$
   (h) $V(y-x^2, z-x^3) \subset \mathbb{C}^3$

2. Consider a regular map $\varphi : k^n \to k^m$. Are the following statements true or false? Give a short proof (or counterexample).
   (a) For any algebraic set $X \subset k^n$, the image $f(X)$ is algebraic in k^m.
   (b) For any algebraic set $Y \subset k^m$, the inverse image $f^{-1}(Y)$ is algebraic in k^n.
   (c) For any algebraic set $X \subset k^n$, the graph $\Gamma_{X,f} := \{(x, \varphi(x)) \,|\, x \in X\}$ is algebraic in k^{n+m}.

3. (a) Show that the polynomial ring $k[x_1, \ldots, x_n]$ can be equipped with a different grading by fixing $\underline{a} := (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and putting

$$k[x_1, \ldots, x_n]_d := \bigoplus_{\substack{i_1,\ldots i_n \\ \sum_j i_j a_j = d}} k\, x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}$$

   Is the usual grading a particular case of this (i.e., for some specific $\underline{a} \in \mathbb{Z}^n$)? We call a polynomial in $k[\underline{x}]_d$ (for fixed $\underline{a} \in \mathbb{Z}$) quasi-homogenous (or weighted homogenous) of degree d.
   (b) Let $char(k) = 0$. Show that f is quasi-homogenous of degree d if and only if $\sum_{i=1}^n a_i x_i \partial_{x_i} f = d \cdot f$.

4. Let $R = \oplus_{i \geq 0} R_i$ be a graded ring. Show the equivalence of the following statements.
   (a) R is noetherian.
   (b) R_i is noetherian and $R_+ = \oplus_{i>0} R_i$ is a finitely generated ideal in R.
   (c) R_0 is noetherian and R is a finitely generated R_0-algebra.

5. Show that the projective morphism (this means: a regular map between projective varieties) given by

$$\begin{array}{rcl} \varphi : \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ (s:t) & \longmapsto & (s^2 : st : t^2) \end{array}$$

   is an isomorphism (i.e., a biregular map) between $\mathbb{P}^1$ and its image (in particular, show, that the image is a projective subvariety of $\mathbb{P}^2$).