Universität Mannheim, Lehrstuhl für Mathematik VI
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Local Rings and Localization

Local rings

By definition, a local ring is a (commutative) ring (with unit) with exactly one maximal ideal. The following exercises are hints to the proof of the

Lemma 1 *A ring R is local iff (if and only if) the set*

$$NU := \{a \in R \,|\, a \text{ is not a unit}\}$$

is an ideal in R.

1. Let $x \in R$. Show that x is a unit iff $(x) = R$. (Remember that (x) denotes the ideal generated by x in R).
2. Follow from Zorn's Lemma (which implies that any proper ideal in R is contained in a maximal ideal) that
$$NU = \bigcup_{\mathbf{m} \text{ is maximal in R}} \mathbf{m}$$
This is used to show $\Rightarrow$ in the lemma.
3. Conversly, suppose that NU is an ideal in R. Show that it is maximal.
4. Deduce that there cannot be any other maximal ideal, so that R must be local, thus showing $\Leftarrow$ of the above lemma.

To get used to work with local rings, here are some simple statements about them:

1. Show that if $(R, \mathbf{m})$ is local, then for any $x \in \mathbf{m}$, the element $1 + x$ is a unit in R.
2. Let R be a local ring and $I \subset R$ any ideal. Show that the factor ring R/I is also local.
3. Let $\mathbb{R}[[x_1, \ldots, x_n]]$ be the ring of formal power series over $\mathbb{R}$. Show that it is a local ring. Give an explicit expression for the inverse of $1+x$ for $x \in \mathbf{m}$. Show that the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ is not local.

We will now specialize to one particular local ring, which already occurred in the lecture.

1. Let $\mathcal{E}_n$ the ring (more precisely, the $\mathbb{R}$-algebra) of germs of smooth functions on $\mathbb{R}^n$ at the origin. Show that this is a local ring. (Hint: the maximal ideal is given by all functions $f \in \mathcal{E}_n$ with $f(0) = 0$.)
2. For any local ring, define $k := R/\mathbf{m}$. Then k is a field, called the residue class field of $(R, \mathbf{m})$. Show that the residue class field of $\mathcal{E}_n$ is isomorphic to $\mathbb{R}$.
3. Let
$$\mathbf{m}^k := \underbrace{\mathbf{m} \cdot \ldots \mathbf{m}}_{k-times}$$

(recall that the product $I \cdot J$ of ideal is the ideal generated by all elements $f \cdot g$ with $f \in I$ and $g \in J$). Then we have a descending chain of ideals

$$R \supsetneq \mathbf{m}^1 \supsetneq \mathbf{m}^2 \supsetneq \dots$$

(this is called a filtration by ideals). Show that the kernel of the surjective map of rings (even of $\mathbb{R}$-algebras):

$$T^k : \mathcal{E}_n \to \mathbb{R}[x_1, \dots, x_n]_{\leq k}; \quad f \longmapsto \sum_{|\nu| \leq k} \frac{1}{\nu!} (D^\nu f)(0) x^\nu$$

(the Taylor development) is exactly the ideal $\mathbf{m}^{k+1}$. What is the kernel of the full Taylor expansion map $T : \mathcal{E}_n \to \mathbb{R}[[x_1, \dots, x_n]]$?

4. Let $(R, \mathbf{m})$ be local and define by

$$H_R(d) := \dim_k(\mathbf{m}^d / \mathbf{m}^{d+1})$$

the Hilbert function of the local ring R. Calculate the Hilbert function for the following local rings

   (a) $R = \mathcal{E}_n$, $R = \mathbb{R}[[x_1, \dots, x_n]]$,
   (b) $R = \mathbb{R}[[x, y]]/(xy)$,
   (c) $R = \mathbb{R}[[x, y]]/(x^2 - y^3)$.

5. Consider the local ring $(\mathcal{E}_n, \mathbf{m})$ and let $\Psi := (\Psi_1, \dots, \Psi_n) \in (\mathbf{m})^n \subset (\mathcal{E}_n)^n = \mathcal{E}_{n,n}$ (caution: $(\mathbf{m})^n$ denotes the direct sum $\mathbf{m} \oplus \dots \oplus \mathbf{m}$).

   (a) Show that the substitution map (also called pull-back or inverse image)

$$\begin{array}{rcl} \Psi^* : R & \longrightarrow & R \\ f & \longmapsto & f \circ \Psi \end{array}$$

   is an algebra homomorphism preserving the identity. Show further that $\Psi^*(\mathbf{m}^k) \subset \mathbf{m}^k$.

   (b) Deduce from (a) that Ψ induces linear maps

$$(\Psi^*)_k : \mathbf{m}^k / \mathbf{m}^{k+1} \longrightarrow \mathbf{m}^k / \mathbf{m}^{k+1}.$$

   Show that Ψ is an automorphism iff $(\Psi^*)_1$ is invertible.

Localization

Here is a way to construct systematically local rings from arbitrary ones. Let R be a commutative ring with unit and $S \subset R$ be any subset of R. We say that S is mutiplicatively closed iff it contains 1 and iff for any $a, b \in S$ we have $a \cdot b \in S$. Given such a multiplicatively closed subset $S \subset R$, we define the ring of fractions $S^{-1}R$ to be the set of equivalence classes of pairs $(a, b) \in R \times S$ with respect to the relation $(a, b) \sim (a', b')$ iff there exists an $r \in S$ such that $r(a \cdot b' - a' \cdot b) = 0$. This set acquires a ring structure by putting $(a, b) + (a', b') := (ab' + ba', bb')$ and $(a, b) \cdot (a', b') := (aa', bb')$. For notational convenience, we denote by $\frac{a}{b}$ the equivalence class of (a, b).

1. Show that $S^{-1}R$ is again commutative with unit.

2. Let $\mathfrak{p}$ be a prime ideal of R. Show that $S := R \backslash \mathfrak{p}$ is multiplicatively closed, so that we can form the ring $S^{-1}R$, which is denoted by $R_\mathfrak{p}$.

3. Let R be an integral domain, $S \subset R$ multiplicatively closed with $0 \notin S$ and $i : R \to S^{-1}R$ be the map defined by $r \mapsto (r, 1)$. Show that i is an injective ring homomorphism.

4. Let R integral and $S := R \backslash \{0\}$ which is obviously multiplicatively closed. Show that $S^{-1}R$ is a field, which is called the quotient field of R and denoted by $Q(R)$. What are the quotient fields of $\mathbb{Z}$ and of $\overline{\mathbb{Z}}$ (ring of algebraic integers)?

5. Let $\mathfrak{p} \subset R$ prime. Show that $R_\mathfrak{p}$ is a local ring and describe its maximal ideal.

6. Consider again the polynomial ring $R = \mathbb{R}[x_1, \dots, x_n]$ and the maximal ideal $\mathbf{m}$ generated by $x_1, \dots, x_n$. Decide whether the local rings $R_\mathbf{m}$ and $\mathbb{R}[[x_1, \dots, x_n]]$ are isomorphic.