The Lagrange interpolation for functions of bounded variation

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The solutions of problems in mechanics, physics, chemical kinetics are very often representable by functions of bounded variation or having a derivative of bounded variation.

Indeed is very interesting, from the approximation point of view, to deduce optimal interpolation processes for this class of functions.

We will consider two different situations.

First we will investigate a general interpolation process using the zeros of suitable sequences of polynomials $\{q_m\}_m$. If the function f has the r-th derivative ($r \geq 1$) of bounded variation the following pointwise estimate is stated

$$|f(x) - L_m(f,x)| \leq \frac{C}{m}|q_m(x)| \int_{-1}^{1} \left(\frac{\varphi(t)}{m}\right)^{r-1} |df^{(r)}(t)|, \quad |x| \leq 1.$$

Therefore if the sequence of the polynomials q_m is uniformly bounded, the Lagrange interpolation behaves exactly as the best polynomial approximation for the considered class of functions (and no extra $\log m$ factor appears).

Secondly we will consider a classical interpolatory process based on the zeros of Jacobi polynomials. It is proved that if u, w are two Jacobi weights and $\varphi(x) = \sqrt{1-x^2}$, then the conditions

$$\frac{u}{\sqrt{w\varphi}} \in L^p, \quad \frac{\sqrt{w\varphi}}{u} \in L^q, \quad q = \frac{p}{p-1}$$

are necessary and sufficient in order to obtain

$$\|[f - L_m(w,f)]u\|_p \leq C \int_{-1}^{1} \left(\frac{\varphi(t)}{m}\right)^{r+\frac{1}{p}} u(t)|df^{(r)}(t)|,$$

for any function f such that $f^{(r)}$, $r \geq 0$, is of bounded variation. Therefore once again the best approximation behavior is attained by the proposed interpolation process.