Spectra and Finite Sections of Band Operators

Marko Lindner

TU Chemnitz & TU Bergakademie Freiberg

26.-30. September 2011

TECHNISCHE UNIVERSITÄT CHEMNITZ

TECHNISCHE UNIVERSITÄT BERGAKADEMIE FREIBERG
Die Ressourcenuniversität. Seit 1765.

What is this course about?

- **spectral theory**
- **Fredholm theory**
- **stable approximation**

of **infinite matrices** (a_{ij}), understood as bounded linear operators on a sequence space E.

What is this course about?

- **spectral theory**
- **Fredholm theory**
- **stable approximation**

of **infinite matrices** (a_{ij}), understood as bounded linear operators on a sequence space E.

Simplest example: $E = \ell^2(\mathbb{Z}, \mathbb{C})$

$$\begin{pmatrix} \ddots & \vdots & \iddots \\ \cdots & a_{ij} & \cdots \\ \iddots & \vdots & \ddots \end{pmatrix} : \begin{pmatrix} \vdots \\ x_j \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ b_i \\ \vdots \end{pmatrix}$$

with indices $i, j \in \mathbb{Z}$ and entries $a_{ij}, x_j, b_i \in \mathbb{C}$

Table of Contents

Table of Contents

Table of Contents

Table of Contents

Table of Contents

Table of Contents

ℓ^p sequence spaces

In what follows, we consider sequence spaces such as

$$E = \ell^2(\mathbb{Z}, \mathbb{C})$$

ℓ^p sequence spaces

In what follows, we consider sequence spaces such as

$$E = \ell^p(\mathbb{Z}, \mathbb{C})$$

with

- $p \in [1, \infty]$

In what follows, we consider sequence spaces such as

$$E = \ell^p(\mathbb{Z}^N, \mathbb{C})$$

with

- $p \in [1, \infty]$
- $N \in \mathbb{N}$

In what follows, we consider sequence spaces such as

$$E = \ell^p(\mathbb{Z}^N, X)$$

with

- $p \in [1, \infty]$
- $N \in \mathbb{N}$
- X ... complex **Banach space**

Clearly, the entries of a matrix (a_{ij}) acting on E also need to be indexed by $i, j \in \mathbb{Z}^N$, and the entries a_{ij} are themselves bounded linear **operators** $X \to X$.

Vector-valued ℓ^p-spaces

$x \in E = \ell^p(\mathbb{Z}^N, X)$ iff $x = (x_k)_{k\in\mathbb{Z}^N}$ with $x_k \in X$ for $k \in \mathbb{Z}^N$ and

$$\begin{aligned} \|x\|_E &:= \sqrt[p]{\sum_{k\in\mathbb{Z}^N} \|x_k\|_X^p} < \infty, \qquad p < \infty, \\ \|x\|_E &:= \sup_{k\in\mathbb{Z}^N} \|x_k\|_X < \infty, \qquad p = \infty. \end{aligned}$$

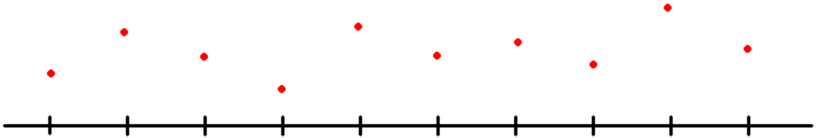
Vector-valued ℓ^p-spaces

$x \in E = \ell^p(\mathbb{Z}^N, X)$ iff $x = (x_k)_{k\in\mathbb{Z}^N}$ with $x_k \in X$ for $k \in \mathbb{Z}^N$ and

$$\begin{aligned} \|x\|_E &:= \sqrt[p]{\sum_{k\in\mathbb{Z}^N} \|x_k\|_X^p} < \infty, \qquad p < \infty, \\ \|x\|_E &:= \sup_{k\in\mathbb{Z}^N} \|x_k\|_X < \infty, \qquad p = \infty. \end{aligned}$$

Simplest case

$$E = \ell^p := \ell^p(\mathbb{Z}, \mathbb{C}), \qquad N = 1, \; X = \mathbb{C}$$



Vector-valued ℓ^p-spaces

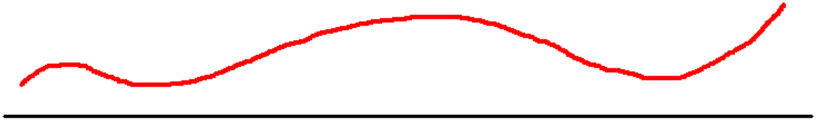
$x \in E = \ell^p(\mathbb{Z}^N, X)$ iff $x = (x_k)_{k\in\mathbb{Z}^N}$ with $x_k \in X$ for $k \in \mathbb{Z}^N$ and

$$\begin{aligned} \|x\|_E &:= \sqrt[p]{\sum_{k\in\mathbb{Z}^N} \|x_k\|_X^p} < \infty, & p < \infty, \\ \|x\|_E &:= \sup_{k\in\mathbb{Z}^N} \|x_k\|_X < \infty, & p = \infty. \end{aligned}$$

Somewhat more complicated case

$$E = L^p(\mathbb{R}^N) \cong \ell^p(\mathbb{Z}^N, X), \qquad X = L^p([0,1]^N)$$

via identification of $f \in L^p(\mathbb{R}^N)$ with $(f|_{\alpha+[0,1]^N})_{\alpha\in\mathbb{Z}^N}$



Vector-valued ℓ^p-spaces

$x \in E = \ell^p(\mathbb{Z}^N, X)$ iff $x = (x_k)_{k\in\mathbb{Z}^N}$ with $x_k \in X$ for $k \in \mathbb{Z}^N$ and

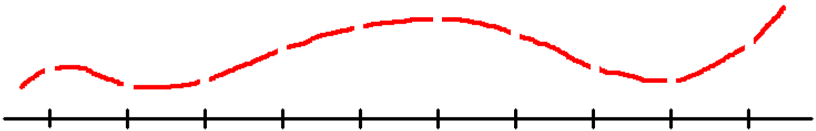
$$\|x\|_E := \sqrt[p]{\sum_{k\in\mathbb{Z}^N} \|x_k\|_X^p} < \infty, \qquad p < \infty,$$

$$\|x\|_E := \sup_{k\in\mathbb{Z}^N} \|x_k\|_X < \infty, \qquad p = \infty.$$

Somewhat more complicated case

$$E = L^p(\mathbb{R}^N) \cong \ell^p(\mathbb{Z}^N, X), \qquad X = L^p([0,1]^N)$$

via identification of $f \in L^p(\mathbb{R}^N)$ with $(f|_{\alpha+[0,1]^N})_{\alpha\in\mathbb{Z}^N}$



Operators: Some first notations

Let $E = \ell^p(\mathbb{Z}^N, X)$ be one of our sequence spaces.
Then we denote by

$$L(E) \quad \ldots \quad \text{space of all } \textbf{bounded linear} \text{ operators } E \to E,$$
$$K(E) \quad \ldots \quad \text{space of all } \textbf{compact} \text{ operators } E \to E.$$

Important:

$L(E)$ is a Banach algebra and
$K(E)$ is a closed two-sided ideal in $L(E)$.

Two basic types of operators

- **shift operators**, V_k

$$V := V_1 = \begin{pmatrix} \ddots & & & & & \\ \ddots & 0 & & & & \\ & 1 & 0 & & & \\ & & 1 & 0 & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix} : \begin{pmatrix} \vdots \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ x_{-3} \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{pmatrix}$$

in general: $(V_k x)_i = x_{i-k}$ with $i, k \in \mathbb{Z}^N$
shift operators are isometric and invertible

Two basic types of operators

- **shift operators**, V_k
- **multiplication operators**, M_b

$$\begin{pmatrix} \ddots & & & & & & \\ & b_{-2} & & & & & \\ & & b_{-1} & & & & \\ & & & b_0 & & & \\ & & & & b_1 & & \\ & & & & & b_2 & \\ & & & & & & \ddots \end{pmatrix} : \begin{pmatrix} \vdots \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ b_{-2}x_{-2} \\ b_{-1}x_{-1} \\ b_0x_0 \\ b_1x_1 \\ b_2x_2 \\ \vdots \end{pmatrix}$$

M_b is bounded if $b = (b_k)$ is so; $\|M_b\| = \|b\|_\infty$

Two basic types of operators

- **shift operators**, V_k
- **multiplication operators**, M_b

Now let them mingle: Take

- scalar multiples
- sums
- products

or combinations of those.

$\Longrightarrow$ an **operator algebra**

Typical elements of the algebra look like this:

$$A \;=\; V^2 M_b \;+\; 3M_b \;+\; M_a V^{-1}$$

i.e.

$$A \;=\; \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & 3b_{-2} & a_{-2} & & & & \\ \ddots & 0 & 3b_{-1} & a_{-1} & & & \\ & b_{-2} & 0 & 3b_0 & a_0 & & \\ & & b_{-1} & 0 & 3b_1 & a_1 & \\ & & & b_0 & 0 & 3b_2 & \ddots \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

Note: $M_a M_b = M_{a \cdot b}, \;\; V M_b = M_{Vb} V$

Typical elements of the algebra look like this:

$$A \;=\; V^2 M_b \;+\; 3M_b \;+\; M_a V^{-1}$$

i.e.

$$A \;=\; \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & 3b_{-2} & a_{-2} & & & & \\ \ddots & 0 & 3b_{-1} & a_{-1} & & & \\ & b_{-2} & 0 & 3b_0 & a_0 & & \\ & & b_{-1} & 0 & 3b_1 & a_1 & \\ & & & b_0 & 0 & 3b_2 & \ddots \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

Note: $M_a M_b = M_{a \cdot b}$, $VM_b = M_{Vb} V$

An algebra of shifts and multiplications

Typical elements of the algebra look like this:

$$A \;=\; V^2 M_b \;+\; 3M_b \;+\; M_a V^{-1}$$

i.e.

$$A \;=\; \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & 3b_{-2} & a_{-2} & & & & \\ \ddots & 0 & 3b_{-1} & a_{-1} & & & \\ & b_{-2} & 0 & 3b_0 & a_0 & & \\ & & b_{-1} & 0 & 3b_1 & a_1 & \\ & & & b_0 & 0 & 3b_2 & \ddots \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

Note: $M_a M_b = M_{a \cdot b}$, $VM_b = M_{Vb} V$

Typical elements of the algebra look like this:

$$A \;=\; V^2 M_b \;+\; 3M_b \;+\; M_a V^{-1}$$

i.e.

$$A \;=\; \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & 3b_{-2} & a_{-2} & & & & \\ \ddots & 0 & 3b_{-1} & a_{-1} & & & \\ & b_{-2} & 0 & 3b_0 & a_0 & & \\ & & b_{-1} & 0 & 3b_1 & a_1 & \\ & & & b_0 & 0 & 3b_2 & \ddots \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

Note: $M_a M_b = M_{a \cdot b}$, $VM_b = M_{Vb} V$

It is easy to see that

$$A \text{ is a finite sum-product of shifts and multiplications}$$
$$\Updownarrow$$
$$A \text{ acts as a band matrix}$$

The set of these operators is denoted by $BO(E)$, where BO is short for **band operator**.

It is easy to see that

A is a finite sum-product of shifts and multiplications

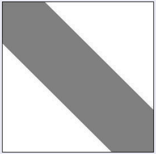
$$\Updownarrow$$

A acts as a band matrix

The set of these operators is denoted by $BO(E)$, where BO is short for **band operator**.

Band operators

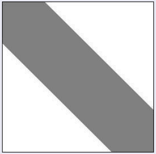
$$\underline{A \in BO(E):} \qquad A = \sum_{k=-w}^{w} M_{b^{(k)}} V^k$$



The number w is called the **band-width** of A.

Band operators

$$\underline{A \in BO(E) :} \qquad A = \sum_{k=-w}^{w} M_{b^{(k)}} V^k$$



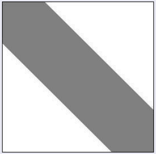
The number w is called the **band-width** of A.

It is nice to have an algebra of operators (i.e. a set that is closed under addition, multiplication and taking scalar multiples) but it is even nicer to have a **Banach algebra** (also closed w.r.t. $\|\cdot\|$).

Algebras of shifts and multiplications

Band operators

$$\underline{A \in BO(E):} \qquad A = \sum_{k=-w}^{w} M_{b^{(k)}} V^k$$

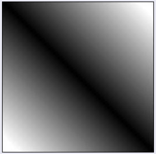


The number w is called the **band-width** of A.

It is nice to have an algebra of operators (i.e. a set that is closed under addition, multiplication and taking scalar multiples) but it is even nicer to have a **Banach algebra** (also closed w.r.t. $\|\cdot\|$).

Band-dominated operators

$$BDO(E) := \text{clos}_{L(E)} BO(E)$$



The matrices have a certain off-diagonal decay.

Algebras of shifts and multiplications

The norm under which $BDO(E)$ is closed is the usual **operator norm**

$$\|A\| \;:=\; \sup_{\|x\|=1} \|Ax\|.$$

Here is **another norm**: For

$$A \;=\; \sum_{k=-w}^{w} M_{b^{(k)}} V^k \quad \in \quad BO(E),$$

we have

$$\|A\| \;\leq\; \sum_{k=-w}^{w} \|M_{b^{(k)}}\|\,\|V^k\| \;=\; \sum_{k=-w}^{w} \|b^{(k)}\|_\infty \;=:\; [\![A]\!]$$

Wiener norm

$$[\![A]\!] \;:=\; \sum_{k=-w}^{w} \|b^{(k)}\|_\infty$$

Now let $\mathcal{W}(E)$ denote the completion of $BO(E)$ w.r.t. $[\![\cdot]\!]$.

Algebras of shifts and multiplications

Wiener norm

$$[\![A]\!] := \sum_{k=-w}^{w} \|b^{(k)}\|_\infty$$

Now let $\mathcal{W}(E)$ denote the completion of $BO(E)$ w.r.t. $[\![\cdot]\!]$.
This also gives a Banach algebra (w.r.t. $[\![\cdot]\!]$):

Wiener algebra

$$\mathcal{W}(E) = \Big\{ \underbrace{\sum_{k=-\infty}^{+\infty} M_{b^{(k)}} V^k}_{A} : \underbrace{\sum_{k=-\infty}^{+\infty} \|b^{(k)}\|_\infty}_{[\![A]\!]} < \infty \Big\}$$

Clearly: $BO(E) \subset \mathcal{W}(E) \subset BDO(E) \subset L(E)$

Example: Laurent operator

Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$ and fix a function $a \in L^\infty(\mathbb{T})$.

Fourier series of a: $$\sum_{k=-\infty}^{+\infty} a_k t^k, \qquad t \in \mathbb{T}.$$

This function is closely related to the associated operator

$$L(a) \;:=\; \sum_{k=-\infty}^{+\infty} a_k V^k,$$

which is a so-called **Laurent operator** (constant matrix diagonals):

$$L(a) \;=\; \begin{pmatrix} \ddots & \ddots & \ddots & & ⋰ \\ \ddots & a_0 & a_{-1} & a_{-2} & \\ \ddots & a_1 & a_0 & a_{-1} & \ddots \\ & a_2 & a_1 & a_0 & \ddots \\ ⋰ & & \ddots & \ddots & \ddots \end{pmatrix}$$

Example: Laurent operator

The function a is called the **symbol** of the operator $L(a)$.

$$L(a) \;=\; \text{discrete convolution by } (a_k)_{k\in\mathbb{Z}} \;\overset{\text{Fourier}}{\cong}\; \text{multiplication by } a$$

For simplicity, suppose $E = \ell^2$. Then $E \overset{\text{Fourier}}{\cong} L^2(\mathbb{T})$.

Example: Laurent operator

The function a is called the **symbol** of the operator $L(a)$.

$$L(a) \;=\; \text{discrete convolution by } (a_k)_{k\in\mathbb{Z}} \;\overset{\textit{Fourier}}{\cong}\; \text{multiplication by } a$$

For simplicity, suppose $E = \ell^2$. Then $E \overset{\textit{Fourier}}{\cong} L^2(\mathbb{T})$.
Correspondence between $L(a)$ and its symbol a on $\mathbb{T}$:

Laurent operator $L(a)$	symbol a
bounded operator	bounded function
$\|L(a)\| \;=\; \|a\|_\infty$	

Example: Laurent operator

The function a is called the **symbol** of the operator $L(a)$.

$$L(a) \;=\; \text{discrete convolution by } (a_k)_{k\in\mathbb{Z}} \;\overset{\textit{Fourier}}{\cong}\; \text{multiplication by } a$$

For simplicity, suppose $E = \ell^2$. Then $E \overset{\textit{Fourier}}{\cong} L^2(\mathbb{T})$.
Correspondence between $L(a)$ and its symbol a on $\mathbb{T}$:

Laurent operator $L(a)$	symbol a
bounded operator	bounded function
$\|L(a)\| = \|a\|_\infty$	
invertible operator	no zeros on $\mathbb{T}$
$L(a)^{-1} = L(a^{-1})$	
$\operatorname{spec} L(a) = a(\mathbb{T})$	

Example: Laurent operator

The function a is called the **symbol** of the operator $L(a)$.

$$L(a) \;=\; \text{discrete convolution by } (a_k)_{k\in\mathbb{Z}} \;\overset{\text{Fourier}}{\cong}\; \text{multiplication by } a$$

For simplicity, suppose $E = \ell^2$. Then $E \overset{\text{Fourier}}{\cong} L^2(\mathbb{T})$.
Correspondence between $L(a)$ and its symbol a on $\mathbb{T}$:

Laurent operator $L(a)$	symbol a
bounded operator	bounded function
$\|L(a)\| = \|a\|_\infty$	
invertible operator	no zeros on $\mathbb{T}$
$L(a)^{-1} = L(a^{-1})$	
$\operatorname{spec} L(a) = a(\mathbb{T})$	
in $BO(E)$	trig. polynomial
in $\mathcal{W}(E)$	Wiener function
in $BDO(E)$	continuous function

Example: Laurent operator

Here we call $a : \mathbb{T} \to \mathbb{C}$ a Wiener function and write $a \in W(\mathbb{T})$ if its Fourier coefficients are summable, i.e. $(a_k) \in \ell^1$. Note that

$$\|a\|_W \;:=\; \sum_{k=-\infty}^{+\infty} |a_k| \;=\; [\![L(a)]\!].$$

Wiener's theorem

If $a \in W(\mathbb{T})$ has no zeros then also $a^{-1} \in W(\mathbb{T})$.

Wiener's theorem in Laurent operator language

If $L(a) \in \mathcal{W}(E)$ is invertible then $L(a)^{-1} = L(a^{-1}) \in \mathcal{W}(E)$.

Operator algebras: inverse closedness

This theorem

Wiener's theorem in Laurent operator language

If $L(a) \in \mathcal{W}(E)$ is invertible then $L(a)^{-1} = L(a^{-1}) \in \mathcal{W}(E)$.

has an amazing generalisation:

The Wiener algebra is inverse closed

If $A \in \mathcal{W}(E)$ is an invertible operator then $A^{-1} \in \mathcal{W}(E)$.

And now that we're at it:

Also $BDO(E)$ is inverse closed

If $A \in BDO(E)$ is an invertible operator then $A^{-1} \in BDO(E)$.

The last two theorems hold in the general case $E = \ell^p(\mathbb{Z}^N, X)$.

Infinite systems

We look at linear equations $Ax = b$ in infinitely many variables:

$$\underbrace{\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & a_{-1,-1} & a_{-1,0} & a_{-1,1} & \cdots \\ \cdots & a_{0,-1} & a_{0,0} & a_{0,1} & \cdots \\ \cdots & a_{1,-1} & a_{1,0} & a_{1,1} & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \vdots \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} \vdots \\ b_{-1} \\ b_0 \\ b_1 \\ \vdots \end{pmatrix}}_{b}$$

Assumption: $A \in BDO(E)$ with $E = \ell^p$, i.e. $N = 1$ and $X = \mathbb{C}$.

Task: Given such an A and a RHS $b \in E$, find $x \in E$.

The Finite Section Method

Let A be **invertible** (bijective) as a map $E \to E$, so that $Ax = b$ is **uniquely solvable** for **every** RHS b.

How do we compute this unique solution x of $Ax = b$, i.e.

$$\sum_{j=-\infty}^{+\infty} a_{ij} x_j \;=\; b_i, \qquad i \in \mathbb{Z} \qquad ? \tag{1}$$

Replace the infinite system (1) by the sequence of finite systems

$$\sum_{j=-n}^{n} a_{ij} x_j \;=\; b_i, \qquad i = -n, \ldots, n$$

for $n = 1, 2, \ldots$

The Finite Section Method

Let A be **invertible** (bijective) as a map $E \to E$, so that $Ax = b$ is **uniquely solvable** for **every** RHS b.

How do we compute this unique solution x of $Ax = b$, i.e.

$$\sum_{j=-\infty}^{+\infty} a_{ij} x_j \;=\; b_i, \qquad i \in \mathbb{Z} \qquad ? \tag{1}$$

Or, more flexible: Take two monotonous sequences of integers

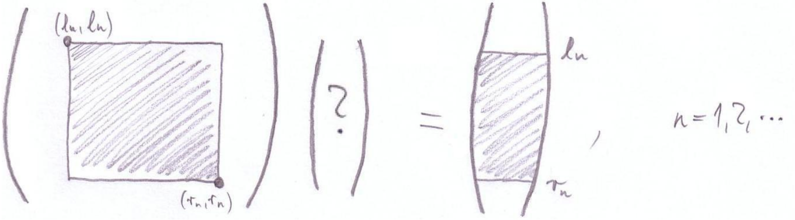
$$-\infty \leftarrow \cdots < l_2 < l_1 \qquad < \qquad r_1 < r_2 < \cdots \to +\infty$$

and replace the infinite system (1) by the sequence of finite systems

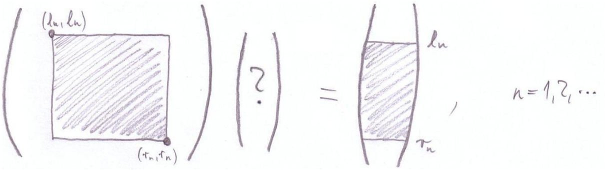
$$\sum_{j=l_n}^{r_n} a_{ij} x_j \;=\; b_i, \qquad i = l_n, \ldots, r_n \tag{2}$$

for $n = 1, 2, \ldots$

Graphically, (2) means

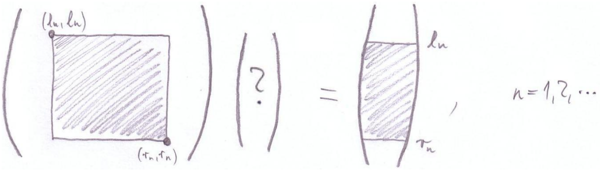


We say the **finite section method** is **applicable** to A if the truncated equations (2) are uniquely solvable for all $n > n_0$ and their solutions converge componentwise to the unique solution x of (1).



We say the **finite section method** is **applicable** to A if the truncated equations (2) are uniquely solvable for all $n > n_0$ and their solutions converge componentwise to the unique solution x of (1).

All of this should happen **independently** of the right-hand side b. So applicability of the method only depends on A.



Precisely: The finite section method (FSM) is applicable to A

iff

A is invertible and its so-called **finite sections**

$$A_n \; := \; (a_{ij})_{i,j=l_n}^{r_n}, \qquad n = 1, 2, \dots$$

form a stable sequence.

Here we call a sequence (A_n) **stable** if there exists n_0 such that

$$\sup_{n > n_0} \|A_n^{-1}\| < \infty.$$

When the Finite Section Method goes wrong

There are very simple examples where the FSM fails to apply.

Example 1: a block-flip

$$A = \begin{pmatrix} \ddots & & & & & & \\ & 0 & 1 & & & & \\ & 1 & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & 1 & \\ & & & & 1 & 0 & \\ & & & & & & \ddots \end{pmatrix} : \begin{pmatrix} \vdots \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ x_{-1} \\ x_{-2} \\ x_0 \\ x_2 \\ x_1 \\ \vdots \end{pmatrix}$$

When the Finite Section Method goes wrong

There are very simple examples where the FSM fails to apply.

Example 1: a block-flip

$$A = \begin{pmatrix} \ddots & & & & & & \\ & \boxed{\begin{matrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{matrix}} & \\ & & \ddots \end{pmatrix} : \begin{pmatrix} \vdots \\ \hline x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ \hline \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ \hline x_{-1} \\ x_{-2} \\ x_0 \\ x_2 \\ x_1 \\ \hline \vdots \end{pmatrix}$$

When the Finite Section Method goes wrong

There are very simple examples where the FSM fails to apply.

Example 1: a block-flip

$$A = \begin{pmatrix} \ddots &&&&&& \\ & 0 & 1 &&&& \\ & 1 & 0 &&&& \\ &&& 1 &&& \\ &&&& 0 & 1 & \\ &&&& 1 & 0 & \\ &&&&&& \ddots \end{pmatrix} : \begin{pmatrix} \vdots \\ x_{-2} \\ \hline x_{-1} \\ x_0 \\ x_1 \\ \hline x_2 \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ x_{-1} \\ \hline 0 \\ x_0 \\ 0 \\ \hline x_1 \\ \vdots \end{pmatrix}$$

When the Finite Section Method goes wrong

There are very simple examples where the FSM fails to apply.

Example 1: a block-flip

$$A = \begin{pmatrix} \ddots & & & & & & \\ & 0 & 1 & & & & \\ & 1 & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & 1 & \\ & & & & 1 & 0 & \\ & & & & & & \ddots \end{pmatrix} : \begin{pmatrix} \vdots \\ x_{-2} \\ \hline x_{-1} \\ x_0 \\ x_1 \\ \hline x_2 \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ x_{-1} \\ \hline 0 \\ x_0 \\ 0 \\ \hline x_1 \\ \vdots \end{pmatrix}$$

Only 50% of the finite systems are uniquely solvable.

When the Finite Section Method goes wrong

There are very simple examples where the FSM fails to apply.

Example 1: a block-flip

$$A = \left(\begin{array}{cc|ccc|cc} \ddots & & & & & & \\ & 0 & 1 & & & & \\ \cline{3-5} & 1 & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & 1 & \\ \cline{3-5} & & & & 1 & 0 & \\ & & & & & & \ddots \end{array}\right) : \left(\begin{array}{c} \vdots \\ x_{-2} \\ \hline x_{-1} \\ x_0 \\ x_1 \\ \hline x_2 \\ \vdots \end{array}\right) \mapsto \left(\begin{array}{c} \vdots \\ x_{-1} \\ \hline 0 \\ x_0 \\ 0 \\ \hline x_1 \\ \vdots \end{array}\right)$$

Only 50% of the finite systems are uniquely solvable.

Choosing good cut-off intervals $[l_n, r_n]$ will solve the problem!

When the Finite Section Method goes wrong

Example 2: the shift

$$A = \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & 1 & 0 & & & & \\ & & 1 & 0 & & & \\ & & & 1 & 0 & & \\ & & & & 1 & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix} : \begin{pmatrix} \vdots \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ x_{-3} \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{pmatrix}$$

When the Finite Section Method goes wrong

Example 2: the shift

$$A = \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & 1 & 0 & & & & \\ & & 1 & 0 & & & \\ & & & 1 & 0 & & \\ & & & & 1 & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix} : \begin{pmatrix} \vdots \\ x_{-2} \\ \hline x_{-1} \\ x_0 \\ x_1 \\ \hline x_2 \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ x_{-3} \\ \hline 0 \\ x_{-1} \\ x_0 \\ \hline x_1 \\ \vdots \end{pmatrix}$$

Example 2: the shift

$$A = \begin{pmatrix} \ddots & & & & & & \\ \ddots & \boxed{\begin{matrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{matrix}} & & \\ & & \ddots & \ddots \end{pmatrix} : \begin{pmatrix} \vdots \\ \hline x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ \color{red}{x_2} \\ \hline \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ \hline \color{red}{0} \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ \hline \vdots \end{pmatrix}$$

When the Finite Section Method goes wrong

Example 2: the shift

$$A = \begin{pmatrix} \ddots & & & & & & \\ \ddots & \boxed{\begin{matrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{matrix}} & & \\ & & \ddots & \ddots \end{pmatrix} : \begin{pmatrix} \vdots \\ \hline x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ \color{red}{x_2} \\ \hline \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ \hline \color{red}{0} \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ \hline \vdots \end{pmatrix}$$

No one of the finite systems is uniquely solvable.

Example 2: the shift

$$A = \begin{pmatrix} \ddots & & & & & & \\ \ddots & \boxed{\begin{matrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{matrix}} & & \\ & & \ddots & \ddots \end{pmatrix} : \begin{pmatrix} \vdots \\ \hline x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ \color{red}{x_2} \\ \hline \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ \hline \color{red}{0} \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ \hline \vdots \end{pmatrix}$$

No one of the finite systems is uniquely solvable.

Adapting the cut-off points $[l_n, r_n]$ will **not** help here!
Instead, place the corners of A_n **along another diagonal**!

The Finite Section Method

Clearly, there is a lot of room in a bi-infinite matrix and therefore a lot of **freedom** to place the finite sections.

The previous examples have shown that sometimes one **needs** to make use of that freedom by

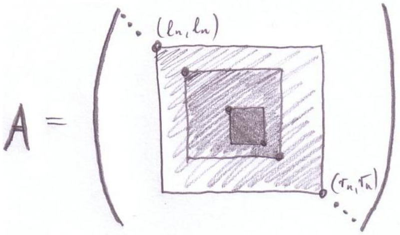
- picking the appropriate "main" diagonal (which one is it?),
- choosing good cut-off sequences (l_n) and (r_n)

We will learn how to do both of that.

The Finite Section Method

We need theorems that tell us when the FSM works and when not.

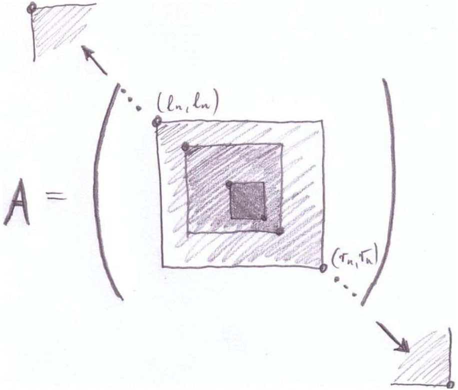
One can show that applicability of the finite section method



The Finite Section Method

We need theorems that tell us when the FSM works and when not.

One can show that applicability of the finite section method

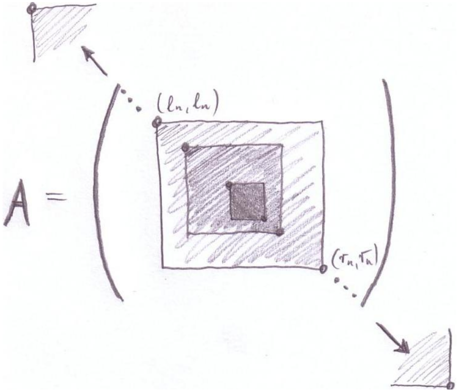


is controlled by certain **limits** of the upper left and lower right **corners** of the finite sections A_n as $n \to \infty$.

So we have to follow the two "corners" (semi-infinite matrices)

$$\begin{pmatrix} a_{l_n,l_n} & \cdots \\ \vdots & \ddots \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \ddots & \vdots \\ \cdots & a_{r_n,r_n} \end{pmatrix}$$

of A_n as $n \to \infty$ and find (partial) limits of these matrix sequences:

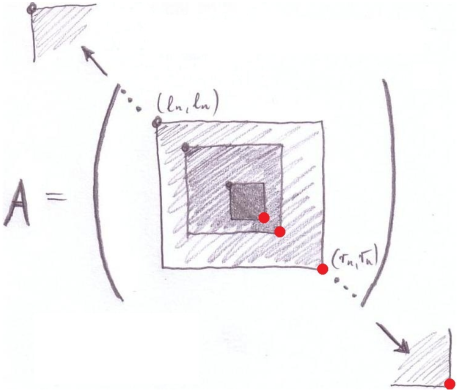


Following the corners as they move out to infinity

So we have to follow the two "corners" (semi-infinite matrices)

$$\begin{pmatrix} a_{l_n,l_n} & \cdots \\ \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \ddots & \vdots \\ \cdots & a_{r_n,r_n} \end{pmatrix}$$

of A_n as $n \to \infty$ and find (partial) limits of these matrix sequences:

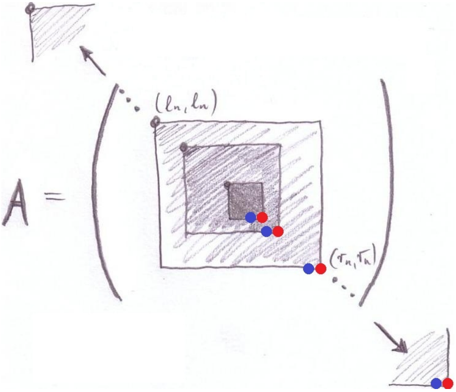


Following the corners as they move out to infinity

So we have to follow the two "corners" (semi-infinite matrices)

$$\begin{pmatrix} a_{l_n,l_n} & \cdots \\ \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \ddots & \vdots \\ \cdots & a_{r_n,r_n} \end{pmatrix}$$

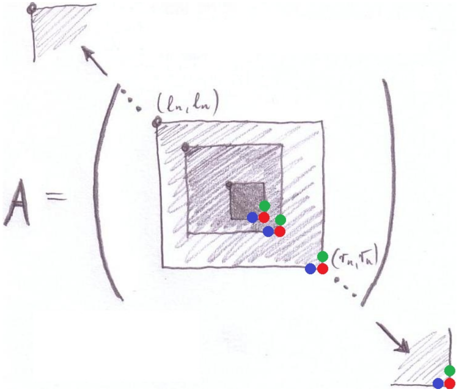
of A_n as $n \to \infty$ and find (partial) limits of these matrix sequences:



So we have to follow the two "corners" (semi-infinite matrices)

$$\begin{pmatrix} a_{l_n,l_n} & \cdots \\ \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \ddots & \vdots \\ \cdots & a_{r_n,r_n} \end{pmatrix}$$

of A_n as $n \to \infty$ and find (partial) limits of these matrix sequences:



Limit Operators: Definition

This leads to the study of so-called limit operators.

Definition: Limit Operator

For a given sequence $h_1, h_2, ... \in \mathbb{Z}$ with $|h_n| \to \infty$ and a matrix $A = (a_{ij})_{i,j\in\mathbb{Z}}$, we call $B = (b_{ij})_{i,j\in\mathbb{Z}}$ a **limit operator** of A with respect to that sequence $h = (h_1, h_2, ...)$ if for all $i, j \in \mathbb{Z}$,

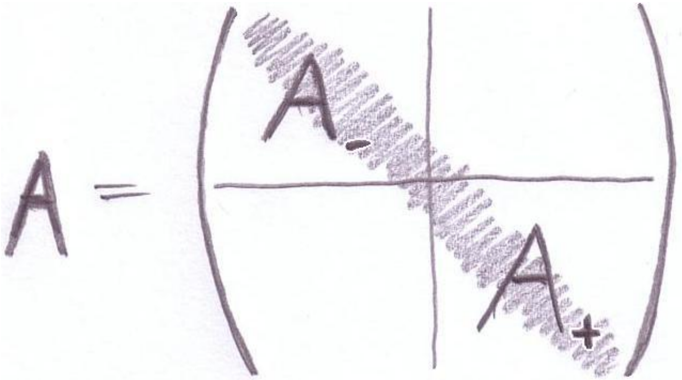
$$a_{i+h_n, j+h_n} \;\to\; b_{ij} \qquad \text{as} \qquad n \to \infty.$$

We write A_h instead of B, where $h = (h_1, h_2, ...)$.

For our FSM, we will use $l = (l_1, l_2, ...)$ and $r = (r_1, r_2, ...)$ (or subsequences of those) in place of $h = (h_1, h_2, ...)$.

One more notation: A_+ and A_-

Think of a bi-infinite band matrix A as 2×2 block matrix:



So A_+ and A_- are one-sided infinite submatrices of A.

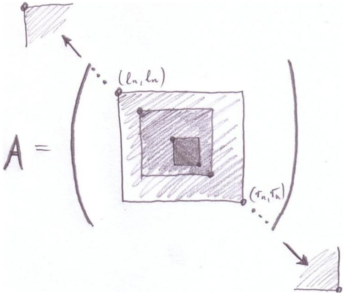
Theorem

ML, Roch 2010; Seidel, Silbermann 2011

The finite section method (2) is applicable to A **iff** the following operators are invertible:

$$A, \quad B_-, \quad C_+$$

for **all** limit operators B of A w.r.t. a subsequence of r and **all** limit operators C of A w.r.t. a subsequence of l.



Strategy:
Choose the sequences $r = (r_1, r_2, \cdots)$ and $l = (l_1, l_2, \cdots)$ so that B_- and C_+ are **invertible**!

The FSM for **one-sided infinite** matrices

For a banded and one-sided infinite matrix $A = A_+ = (a_{ij})_{i,j\in\mathbb{N}}$, acting boundedly on $\ell^p(\mathbb{N})$, the situation is similar:

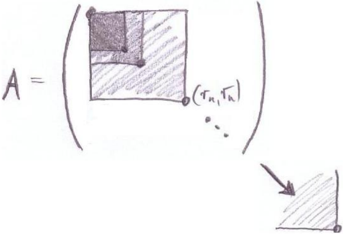
Now $l_n \equiv 1$ is fixed and $r_1 < r_2 < \cdots \to +\infty$.

Theorem

The FSM (2) is applicable **iff**

$$A \quad \text{and} \quad B_-$$

are invertible for **all** limops B of A w.r.t. a subsequence of r.



Strategy: Again, make sure B_- is/are invertible – by choosing the sequence $r = (r_1, r_2, ...)$.

Ok, limit operators seem to be useful.

Time to learn more about them!

(We come back to the FSM at some later point.)

Limit Operators: Definition

Let $A \in BDO(E)$. Recall:

Definition: Limit Operator

For a given sequence $h_1, h_2, ... \in \mathbb{Z}$ with $|h_n| \to \infty$ and a matrix $A = (a_{ij})_{i,j\in\mathbb{Z}}$, we call $B = (b_{ij})_{i,j\in\mathbb{Z}}$ a **limit operator** of A with respect to that sequence $h = (h_1, h_2, ...)$ if for all $i, j \in \mathbb{Z}$,

$$a_{i+h_n, j+h_n} \to b_{ij} \qquad \text{as} \qquad n \to \infty.$$

We write A_h instead of B, where $h = (h_1, h_2, ...)$.

In short: A_h is the entrywise limit of $V_{-h_n} A V_{h_n}$ as $n \to \infty$.

By $\sigma^{\mathrm{op}}(A)$ we denote the set of all limit operators of A.

Example: Discrete Schrödinger operator

The Schrödinger operator $-\Delta + M_b$ with a bounded potential $b \in L^\infty(\mathbb{R})$ is usually discretized as

$$A \; = \; V_{-1} + M_c + V_1 \; = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & c_{-2} & 1 & & & & \\ & 1 & c_{-1} & 1 & & & \\ & & 1 & c_0 & 1 & & \\ & & & 1 & c_1 & 1 & \\ & & & & 1 & c_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}$$

where $c = (\ldots, c_{-1}, c_0, c_1, \ldots) \in \ell^\infty(\mathbb{Z})$.

Example: Discrete Schrödinger operator

The Schrödinger operator $-\Delta + M_b$ with a bounded potential $b \in L^\infty(\mathbb{R})$ is usually discretized as

$$A \;=\; V_{-1} + M_c + V_1 \;=\; \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & c_{-2} & 1 & & & & \\ & 1 & c_{-1} & 1 & & & \\ & & 1 & c_0 & 1 & & \\ & & & 1 & c_1 & 1 & \\ & & & & 1 & c_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}$$

where $c = (\ldots, c_{-1}, c_0, c_1, \ldots) \in \ell^\infty(\mathbb{Z})$. Clearly,

$$A_h \;=\; (V_{-1})_h \,+\, (M_c)_h \,+\, (V_1)_h \;=\; V_{-1} \,+\, (M_c)_h \,+\, V_1,$$

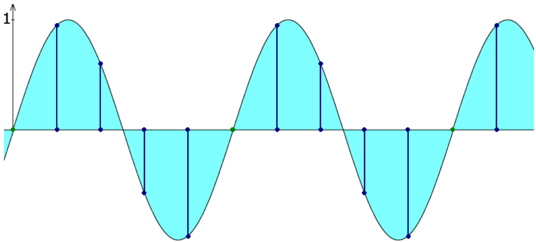
so that everything depends on the limit operators of M_c only.

Example 1: Periodic potential

If there is a $P \in \mathbb{N}$ such that

$$c_{k+P} = c_k \qquad \text{for every} \qquad k \in \mathbb{Z},$$

then $\sigma^{\mathrm{op}}(M_c) = \Big\{ M_{V_k c} \; : \; k \in \{0, 1, ..., P-1\} \Big\}.$

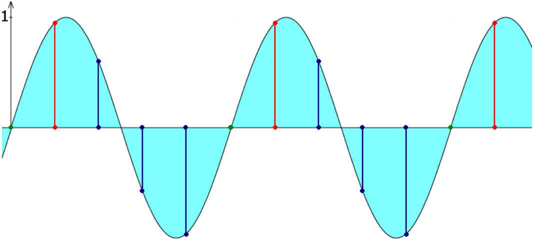


Example 1: Periodic potential

If there is a $P \in \mathbb{N}$ such that

$$c_{k+P} = c_k \qquad \text{for every} \qquad k \in \mathbb{Z},$$

then $\sigma^{\mathrm{op}}(M_c) = \Big\{ M_{V_k c} \; : \; k \in \{0, 1, ..., P-1\} \Big\}$.

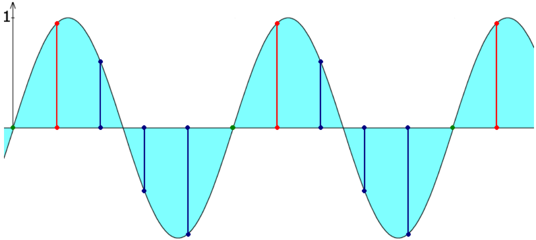


Example 1: Periodic potential

If there is a $P \in \mathbb{N}$ such that

$$c_{k+P} = c_k \qquad \text{for every} \qquad k \in \mathbb{Z},$$

then $\sigma^{\mathrm{op}}(M_c) = \Big\{ M_{V_k c} \;:\; k \in \{0, 1, ..., P-1\} \Big\}$.



But then $\sigma^{\mathrm{op}}(A) = \Big\{ V_{-k} A V_k \;:\; k \in \{0, 1, ..., P-1\} \Big\}$.

Example 2: Almost-periodic potential

Note: $c \in \ell^\infty$ is **periodic** iff the set $\{V_k c : k \in \mathbb{Z}\}$ of all its translates is **finite**.

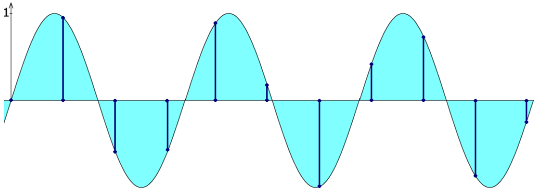
E.g. if $c = (\cdots, c_1, c_2, c_3, c_1, c_2, c_3, \cdots)$ then

$$\{V_k c : k \in \mathbb{Z}\} = \{ \begin{array}{l} V_0 c = (\cdots, c_1, c_2, c_3, c_1, c_2, c_3, \cdots), \\ V_1 c = (\cdots, c_3, c_1, c_2, c_3, c_1, c_2, \cdots), \\ V_2 c = (\cdots, c_2, c_3, c_1, c_2, c_3, c_1, \cdots) \} \end{array}$$

Example 2: Almost-periodic potential

Note: $c \in \ell^\infty$ is **periodic** iff the set $\{V_k c : k \in \mathbb{Z}\}$ of all its translates is **finite**.

Definition: $c \in \ell^\infty$ is **almost-periodic** iff the set $\{V_k c : k \in \mathbb{Z}\}$ of all its translates is **relatively compact in** ℓ^∞.



The set $h(c) := \mathrm{clos}\{V_k c \ : \ k \in \mathbb{Z}\} \subset \ell^\infty$ is called the **hull** of c.

In that case $\sigma^{\mathrm{op}}(M_c) = \{M_d \ : \ d \in h(c)\} = \mathrm{clos}\{M_{V_k c} \ : \ k \in \mathbb{Z}\}$,
$\sigma^{\mathrm{op}}(A) = \{V_{-1} + M_d + V_1 \ : \ d \in h(c)\} = \mathrm{clos}\{V_{-k} A V_k \ : \ k \in \mathbb{Z}\}$.

Example 2: Almost-periodic potential (continued)
Fix $\lambda \in \mathbb{R}$ and look at $c = (c_k)_{k\in\mathbb{Z}} \in \ell^\infty$ with entries

$$c_k \;:=\; e^{i\lambda k}, \qquad k \in \mathbb{Z}.$$

If λ/π is rational then c is periodic;
otherwise it is almost-periodic and its hull is

$$h(c) \;=\; \left\{\; d = \left(e^{i(\lambda k+\gamma)}\right)_{k\in\mathbb{Z}} \;:\; \gamma \in [0, 2\pi) \;\right\}.$$

Example 2: Almost-periodic potential (continued)

Fix $\lambda \in \mathbb{R}$ and look at $c = (c_k)_{k\in\mathbb{Z}} \in \ell^\infty$ with entries

$$c_k \;:=\; e^{i\lambda k}, \qquad k \in \mathbb{Z}.$$

If λ/π is rational then c is periodic;
otherwise it is almost-periodic and its hull is

$$h(c) \;=\; \Big\{\; d = \Big(e^{i(\lambda k+\gamma)}\Big)_{k\in\mathbb{Z}} \;:\; \gamma \in [0, 2\pi) \;\Big\}.$$

Slightly more advanced: If

$$c_k \;:=\; \alpha e^{i\lambda k} \;+\; \beta e^{i\mu k}, \qquad k \in \mathbb{Z}$$

with λ/π, μ/π and λ/μ all irrational then the hull is

$$h(c) \;=\; \Big\{\; d = \Big(\alpha e^{i(\lambda k+\gamma)} + \beta e^{i(\mu k+\delta)}\Big)_{k\in\mathbb{Z}} \;:\; \gamma, \delta \in [0, 2\pi) \;\Big\}.$$

Example 2: Almost-periodic potential (continued 2)

For the so-called Almost-Mathieu operator, the potential c is the real part of what we studied earlier:

Fix $\alpha, \lambda \in \mathbb{R}$ and take $c = (c_k)_{k\in\mathbb{Z}} \in \ell^\infty$ with entries

$$c_k \;:=\; \alpha \cos(\lambda k), \qquad k \in \mathbb{Z}.$$

If λ/π is irrational then c is almost-periodic (but non-periodic) and its hull is

$$h(c) \;=\; \Big\{ \, d = \big(\alpha \cos(\lambda k + \gamma)\big)_{k\in\mathbb{Z}} \;:\; \gamma \in [0, 2\pi) \, \Big\}.$$

Ten-Martini problem: Show that spec$\,(V_{-1} + M_c + V_1)$ is a Cantor set. (Puig 2003, Avila&Jitomiskaya 2005)

Example 3: Slowly oscillating potential

If

$$c_{k+1} - c_k \to 0 \qquad \text{as} \qquad k \to \pm\infty,$$

then $\sigma^{\mathrm{op}}(M_c) = \{aI \ : \ a \in c(\infty)\}$.

Now all limit operators A_h are Laurent operators (i.e., they have constant diagonals).

Example 3: Slowly oscillating potential

If

$$c_{k+1} - c_k \to 0 \qquad \text{as} \qquad k \to \pm\infty,$$

then $\sigma^{\mathrm{op}}(M_c) = \{aI \ : \ a \in c(\infty)\}$.

Now all limit operators A_h are Laurent operators (i.e., they have constant diagonals).

For example, let $c = (c_k)_{k\in\mathbb{Z}}$ with

$$c_k \ = \ \sin\sqrt{|k|}, \qquad k \in \mathbb{Z}.$$

Then c is slowly oscillating and $c(\infty) = [-1, 1]$.

Example 4: Pseudo-ergodic potential

Let Σ be a compact subset of $\mathbb{C}$.

Definition (Davies 2001)

A sequence $c = (c_k)_{k\in\mathbb{Z}}$ over Σ is called **pseudoergodic** over Σ if every finite vector $f = (f_i)_{i\in F}$ with values $f_i \in \Sigma$ can be found, up to arbitrary precision $\varepsilon > 0$, somewhere inside the infinite sequence c, i.e.

$$\forall\varepsilon > 0 \quad \exists m \in \mathbb{Z} : \quad \max_{i\in F} |c_{i+m} - f_i| \; < \; \varepsilon.$$

Idea: Model random behaviour by a purely deterministic concept.

Example 4: Pseudo-ergodic potential

Let Σ be a compact subset of $\mathbb{C}$.

Definition — Davies 2001

A sequence $c = (c_k)_{k\in\mathbb{Z}}$ over Σ is called **pseudoergodic** over Σ if every finite vector $f = (f_i)_{i\in F}$ with values $f_i \in \Sigma$ can be found, up to arbitrary precision $\varepsilon > 0$, somewhere inside the infinite sequence c, i.e.

$$\forall\varepsilon > 0 \quad \exists m \in \mathbb{Z} : \quad \max_{i\in F} |c_{i+m} - f_i| \ < \ \varepsilon.$$

Idea: Model random behaviour by a purely deterministic concept.

Example: $dec(\pi) = 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, \ldots$ is pseudo-ergodic over $\Sigma = \{0, 1, 2, \ldots, 9\}$ (conjecture)

Example: $bin(1), bin(2), bin(3), \ldots$ is pseudo-erg. over $\Sigma = \{0, 1\}$.

Example 4: Pseudo-ergodic potential (continued)

If c is pseudo-ergodic over Σ then **every** multiplication operator M_d with a sequence $d = (..., d_{-1}, d_0, d_1, ...)$ over Σ is a limit operator of M_c:

$$\sigma^{\mathrm{op}}(M_c) \;=\; \{\, M_d \;:\; d : \mathbb{Z} \to \Sigma \,\} \tag{3}$$

The statement also holds the other way round!

So c is pseudo-ergodic **iff** (3) holds.

Example 5: A locally constant potential

$$c = (\ldots, \underbrace{\beta, \beta, \beta, \beta}_{4}, \underbrace{\alpha, \alpha, \alpha}_{3}, \underbrace{\beta, \beta}_{2}, \underbrace{\alpha}_{1}, \underbrace{\beta, \beta}_{2}, \underbrace{\alpha, \alpha, \alpha}_{3}, \underbrace{\beta, \beta, \beta, \beta}_{4}, \ldots).$$

Example 5: A locally constant potential

$$c = (\ldots, \underbrace{\beta,\beta,\beta,\beta}_{4}, \underbrace{\alpha,\alpha,\alpha}_{3}, \underbrace{\beta,\beta}_{2}, \underbrace{\alpha}_{1}, \underbrace{\beta,\beta}_{2}, \underbrace{\alpha,\alpha,\alpha}_{3}, \underbrace{\beta,\beta,\beta,\beta}_{4}, \ldots).$$

Then all limit operators of A are of the form

$$\begin{pmatrix} \ddots & \ddots & & \\ \ddots & \beta & 1 & \\ & 1 & \beta & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad \begin{pmatrix} \ddots & \ddots & & \\ \ddots & \alpha & 1 & \\ & 1 & \alpha & \ddots \\ & & \ddots & \ddots \end{pmatrix},$$

Example 5: A locally constant potential

$$c = (\dots, \underbrace{\beta,\beta,\beta,\beta}_{4}, \underbrace{\alpha,\alpha,\alpha}_{3}, \underbrace{\beta,\beta}_{2}, \underbrace{\alpha}_{1}, \underbrace{\beta,\beta}_{2}, \underbrace{\alpha,\alpha,\alpha}_{3}, \underbrace{\beta,\beta,\beta,\beta}_{4}, \dots).$$

Then all limit operators of A are of the form

$$\begin{pmatrix} \ddots & \ddots & & \\ \ddots & \beta & 1 & \\ & 1 & \beta & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad \begin{pmatrix} \ddots & \ddots & & \\ \ddots & \alpha & 1 & \\ & 1 & \alpha & \ddots \\ & & \ddots & \ddots \end{pmatrix},$$

$$\begin{pmatrix} \ddots & \ddots & & \\ \ddots & \beta & 1 & \\ & 1 & \alpha & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad \begin{pmatrix} \ddots & \ddots & & \\ \ddots & \alpha & 1 & \\ & 1 & \beta & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

Example 5: A locally constant potential

$$c = (\ldots, \underbrace{\beta,\beta,\beta,\beta}_{4}, \underbrace{\alpha,\alpha,\alpha}_{3}, \underbrace{\beta,\beta}_{2}, \underbrace{\alpha}_{1}, \underbrace{\beta,\beta}_{2}, \underbrace{\alpha,\alpha,\alpha}_{3}, \underbrace{\beta,\beta,\beta,\beta}_{4}, \ldots).$$

Then all limit operators of A are of the form

$$\begin{pmatrix} \ddots & \ddots & & \\ \ddots & \beta & 1 & \\ & 1 & \beta & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad \begin{pmatrix} \ddots & \ddots & & \\ \ddots & \alpha & 1 & \\ & 1 & \alpha & \ddots \\ & & \ddots & \ddots \end{pmatrix},$$

$$\begin{pmatrix} \ddots & \ddots & & \\ \ddots & \beta & 1 & \\ & 1 & \alpha & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad \begin{pmatrix} \ddots & \ddots & & \\ \ddots & \alpha & 1 & \\ & 1 & \beta & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

or they are translates of the latter two matrices.

Limit Operators: Some rules

Let $A, B \in BDO(E)$ and $h = (h_1, h_2, \cdots)$ be a sequence of integers going to $\pm\infty$.

Recall: limit op A_h is the entrywise limit of $V_{-h_n} A V_{h_n}$ as $n \to \infty$

Basic rules

If the right-hand side exists then also the left-hand side exists and equality holds:

$$
\begin{aligned}
(A+B)_h &= A_h + B_h \\
(AB)_h &= A_h\, B_h \\
(\alpha A)_h &= \alpha A_h \\
(\lim A_n)_h &= \lim\, (A_n)_h \quad (\text{w.r.t. op-}\|\cdot\|)
\end{aligned}
$$

$\Rightarrow$ Compute limit operators of elements of an operator algebra in terms of limit operators of generators of the algebra.

Moreover, $\|A_h\| \le \|A\|$.

Besides studying the **FSM**, limit operators are also important for determining the **essential spectrum** of A.

Here we define

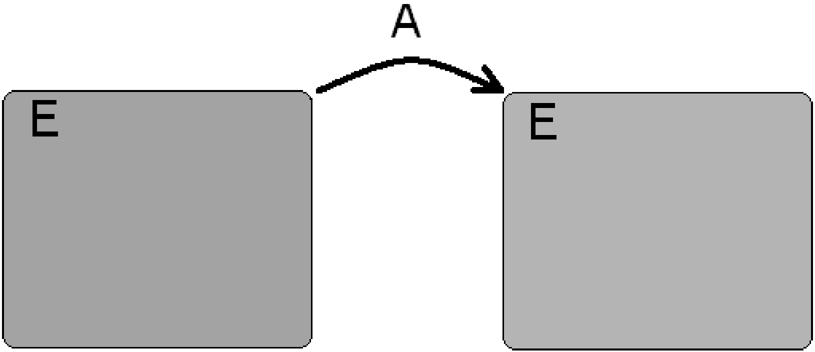
Definition: Essential Spectrum

For a (not necessarily self-adjoint) operator A, we denote by

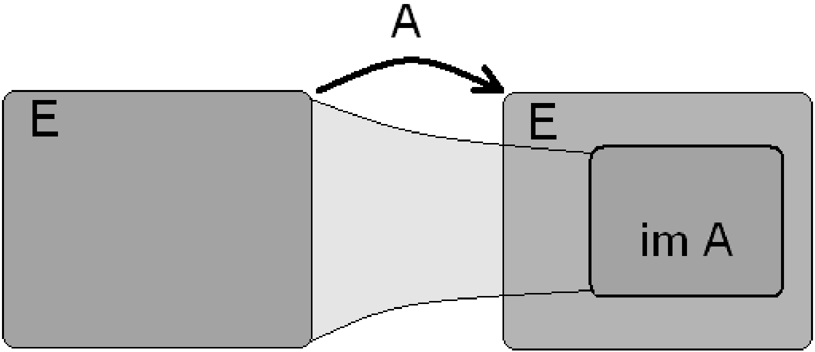
$$\mathrm{spec}_{\mathrm{ess}} A \;:=\; \{\lambda \in \mathbb{C} \;:\; A - \lambda I \text{ is not Fredholm } \}$$

the **essential spectrum** of A.

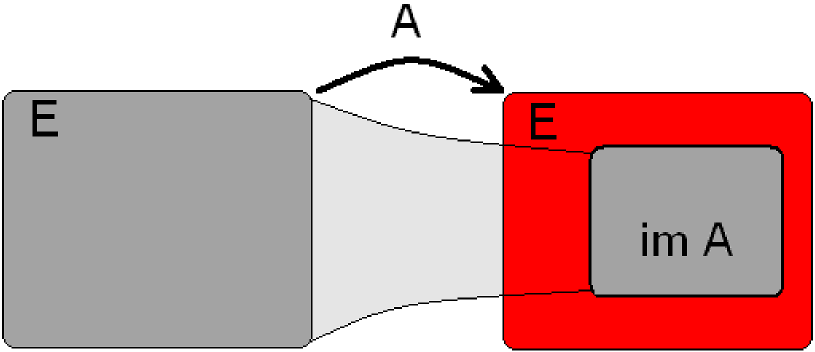
Fredholm operators

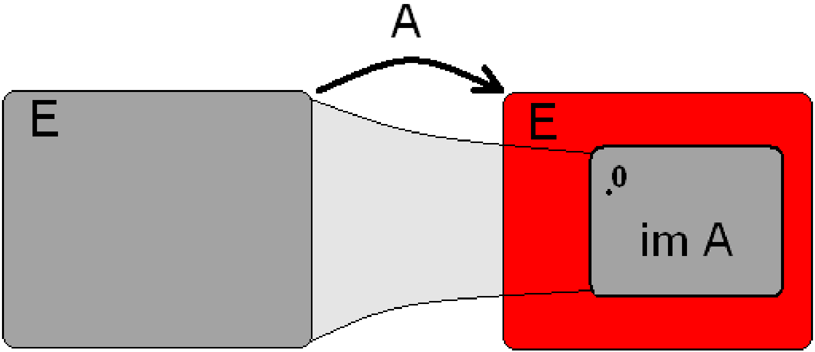


Fredholm operators

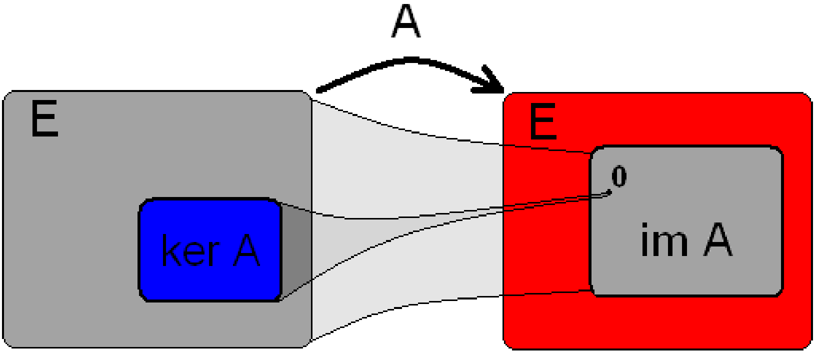


Fredholm operators

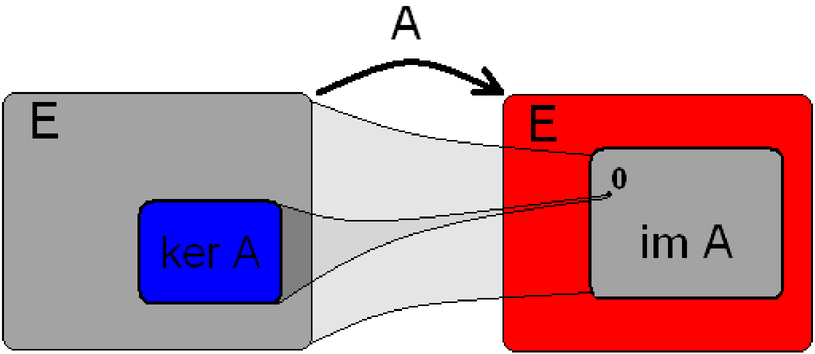



A
E
E
0
im A

Fredholm operators



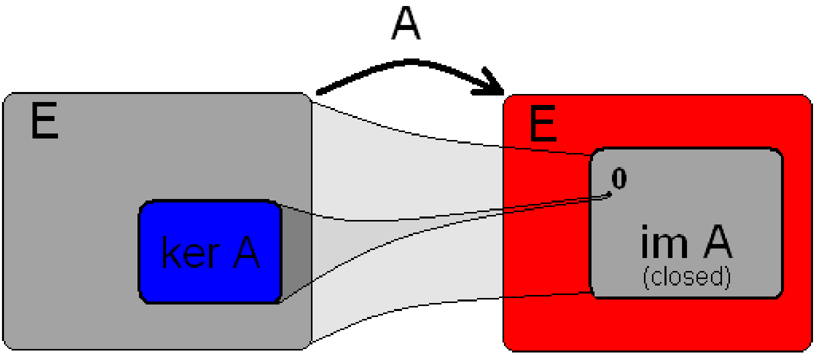
Fredholm operators



Definition

$A : E \to E$ is a **Fredholm operator**
if $\alpha := \dim(\ker A)$ and $\beta := \mathrm{codim}(\mathrm{im}\ A)$ are both finite.
The difference $\alpha - \beta$ is then called the **index of** A.

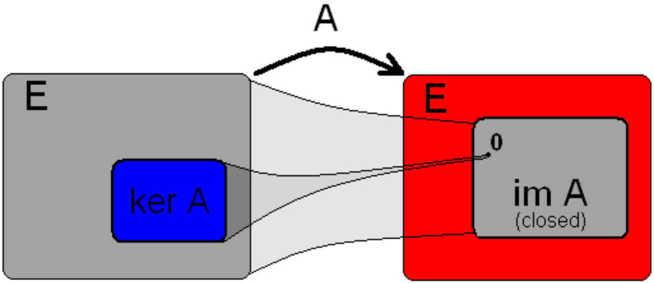
Fredholm operators



Definition

$A : E \to E$ is a **Fredholm operator**
if $\alpha := \dim(\ker A)$ and $\beta := \operatorname{codim}(\text{im } A)$ are both finite.
The difference $\alpha - \beta$ is then called the **index of** A.

Fredholm operators



Definition

$A : E \to E$ is a **Fredholm operator**
if $\alpha := \dim(\ker A)$ and $\beta :=$ codim(im A) are both finite.
The difference $\alpha - \beta$ is then called the **index of** A.

$A \in L(E)$ is Fredholm iff $A + K(E)$ is invertible in $L(E)/K(E)$.

Take $A \in \mathcal{W}(E)$.

Then it is not hard to see that

A **Fredholm** $\Longrightarrow$ all limit operators of A are **invertible**.

Limit Operators vs. Fredholmness

Take $A \in \mathcal{W}(E)$.

Then it is not hard to see that

A **Fredholm** $\Longrightarrow$ all limit operators of A are **invertible**.

In fact, also the reverse implication holds:

Theorem — Rabinovich, Roch, Silbermann 1998; ML 2003

The following are equivalent for all $p \in [1, \infty]$:

- A is **Fredholm** on ℓ^p,
- all limit operators of A are **invertible** on ℓ^p,

Limit Operators vs. Fredholmness

Take $A \in \mathcal{W}(E)$.

Then it is not hard to see that

A **Fredholm** $\Longrightarrow$ all limit operators of A are **invertible**.

In fact, also the reverse implication holds:

Theorem Rabinovich, Roch, Silbermann 1998; ML 2003; Chandler-Wilde, ML 2007

The following are equivalent for all $p \in [1, \infty]$:

- A is **Fredholm** on ℓ^p,
- all limit operators of A are **invertible** on ℓ^p,
- all limit operators of A are **injective** on ℓ^∞.

...moreover, the **Fredholm index** of A does **not** depend on p.

Theorem Rabinovich, Roch, Silbermann 1998; ML 2003; Chandler-Wilde, ML 2007

The following are equivalent for all $p \in [1,\infty]$:

- A is **Fredholm** on ℓ^p,
- all limit operators of A are **invertible** on ℓ^p,
- all limit operators of A are **injective** on ℓ^∞.

...moreover, the **Fredholm index** of A does **not** depend on p.

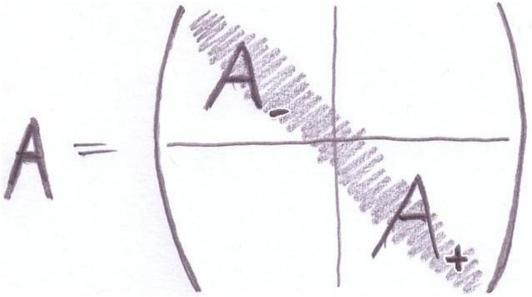
If we repeat the same argument with $A - \lambda I$ in place of A, we get:

Essential Spectrum Rabinovich, Roch, Silbermann 1998; ML 2003; Chandler-Wilde, ML 2007

$$\mathrm{spec}^p_{\mathrm{ess}}(A) \;=\; \bigcup_h \mathrm{spec}^p(A_h) \;=\; \bigcup_h \mathrm{spec}^\infty_{\mathrm{point}}(A_h), \qquad p \in [1,\infty]$$

Limit operators vs. Fredholm index

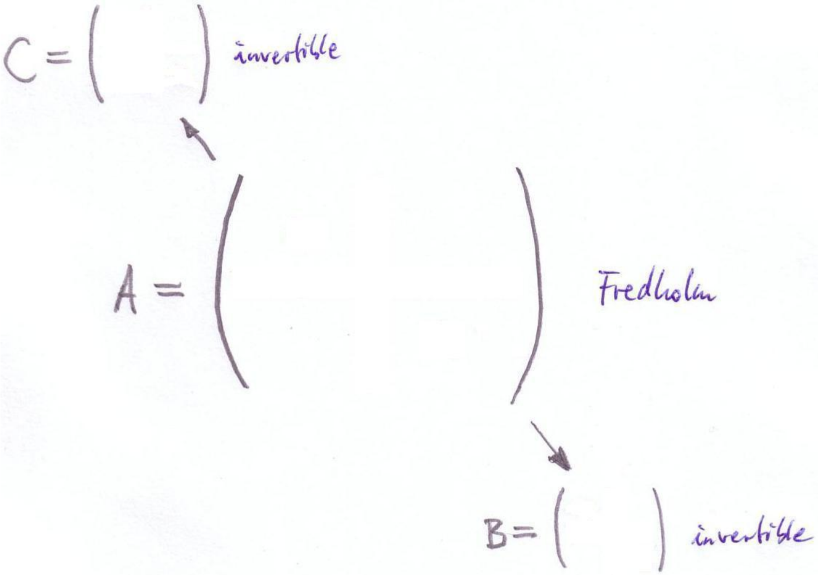
Think of a bi-infinite band matrix A as 2×2 block matrix:



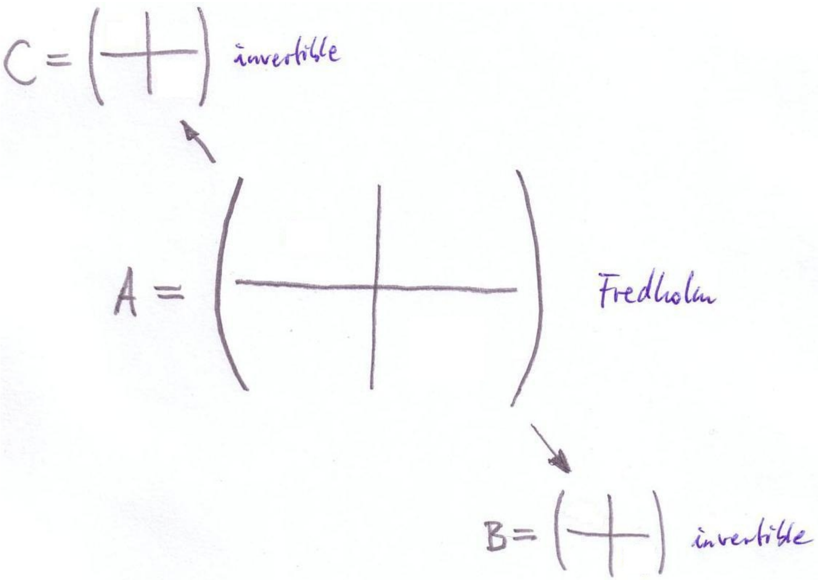
Then

$$\begin{aligned} A \text{ is Fredholm} \quad &\text{iff} \quad A_+ \text{ and } A_- \text{ are Fredholm} \\ \operatorname{spec}_{\text{ess}} A \quad &= \quad \operatorname{spec}_{\text{ess}} A_+ \cup \operatorname{spec}_{\text{ess}} A_- \\ \operatorname{ind} A \quad &= \quad \operatorname{ind} A_+ + \operatorname{ind} A_- \end{aligned}$$

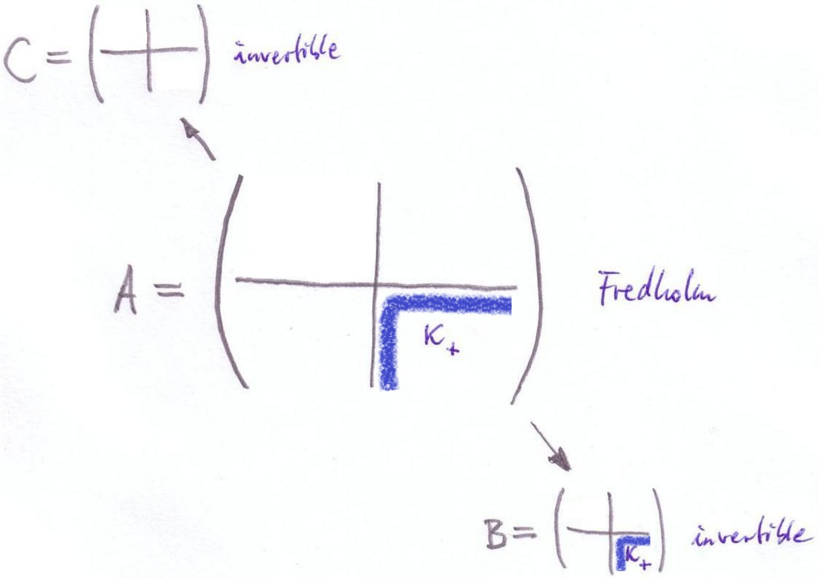
Limit operators vs. Fredholm index



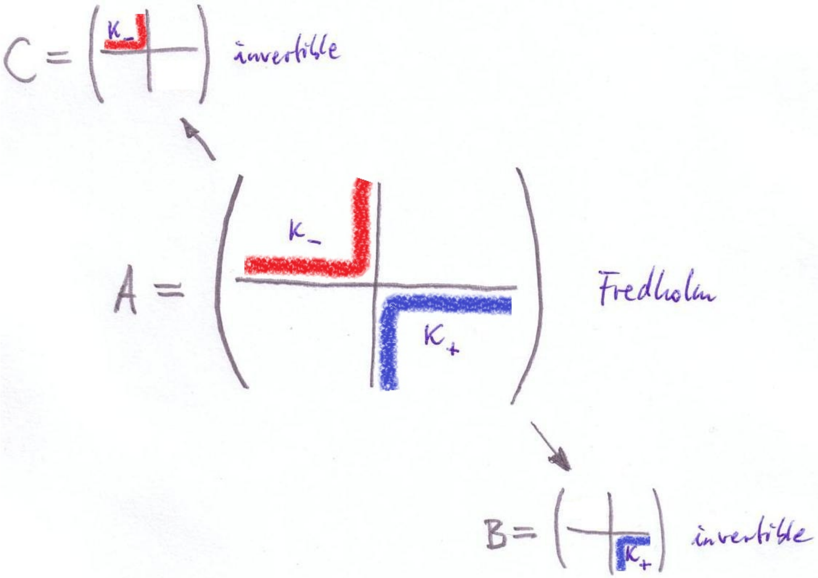
Limit operators vs. Fredholm index

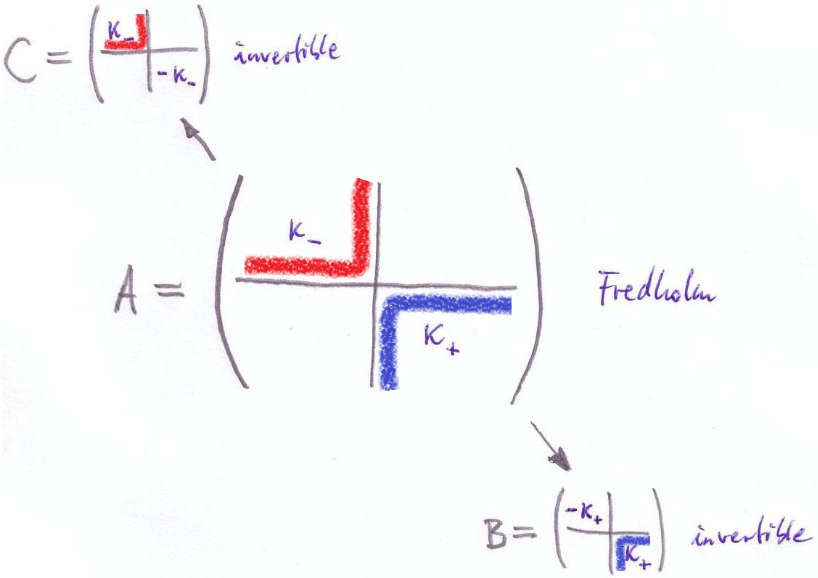


Limit operators vs. Fredholm index



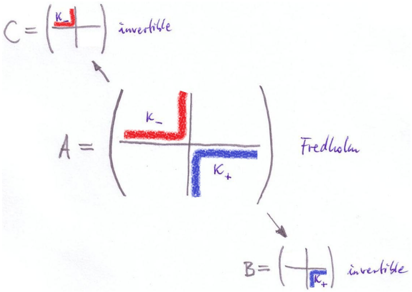
Limit operators vs. Fredholm index




C =
K_
-K_
invertible
A =
K_
K+
Fredholm
B =
-K+
K+
invertible

Limit operators vs. Fredholm index

$$\operatorname{ind} A \;=\; \operatorname{ind} A_+ \;+\; \operatorname{ind} A_-$$



Theorem Rabinovich, Roch, Roe 2004

$$\operatorname{ind} A_+ \;=\; \operatorname{ind} B_+ \quad \text{for all limops } B \text{ of } A \text{ at } +\infty$$
$$\operatorname{ind} A_- \;=\; \operatorname{ind} C_- \quad \text{for all limops } C \text{ of } A \text{ at } -\infty$$

Example: Random Jacobi Operator

Let U, V and W be compact sets in $\mathbb{C}$ and

$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & v_{-2} & w_{-1} & & & & \\ & u_{-2} & v_{-1} & w_0 & & & \\ & & u_{-1} & v_0 & w_1 & & \\ & & & u_0 & v_1 & w_2 & \\ & & & & u_1 & v_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix} \tag{4}$$

with iid entries

$$u_i \in U, \ v_i \in V \text{ and } w_i \in W. \tag{5}$$

Example: Random Jacobi Operator

Let U, V and W be compact sets in $\mathbb{C}$ and

$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & v_{-2} & w_{-1} & & & & \\ & u_{-2} & v_{-1} & w_0 & & & \\ & & u_{-1} & v_0 & w_1 & & \\ & & & u_0 & v_1 & w_2 & \\ & & & & u_1 & v_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix} \tag{4}$$

with iid entries

$$u_i \in U, \;\; v_i \in V \text{ and } w_i \in W. \tag{5}$$

Then, almost surely, A is **pseudoergodic** (i.e. it contains **all** finite tridiagonal matrices with diagonals in U, V and W).

Example: Random Jacobi Operator

Let U, V and W be compact sets in $\mathbb{C}$ and

$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & v_{-2} & w_{-1} & & & & \\ & u_{-2} & v_{-1} & w_0 & & & \\ & & u_{-1} & v_0 & w_1 & & \\ & & & u_0 & v_1 & w_2 & \\ & & & & u_1 & v_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix} \tag{4}$$

with iid entries

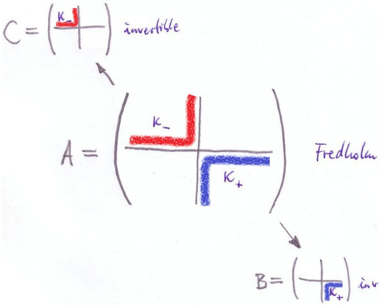
$$u_i \in U, \; v_i \in V \text{ and } w_i \in W. \tag{5}$$

$\Rightarrow$ The set of all limops of A equals the set of **all** operators of the form (4) with entries (5). That includes all Laurent operators.

So, if the random (meaning pseudoergodic) operator A is Fredholm then, for **all** choices $u \in U$, $v \in V$ and $w \in W$:

$$\mathrm{ind}\, A_- = \mathrm{ind} \begin{pmatrix} \ddots & \ddots & \\ \ddots & \ddots & w \\ & u & v \end{pmatrix}$$

$$\mathrm{ind}\, A_+ = \mathrm{ind} \begin{pmatrix} v & w & \\ u & \ddots & \ddots \\ & \ddots & \ddots \end{pmatrix}$$



Example: Random Jacobi Operator

So, if the random (meaning pseudoergodic) operator A is Fredholm then, for **all** choices $u \in U$, $v \in V$ and $w \in W$:

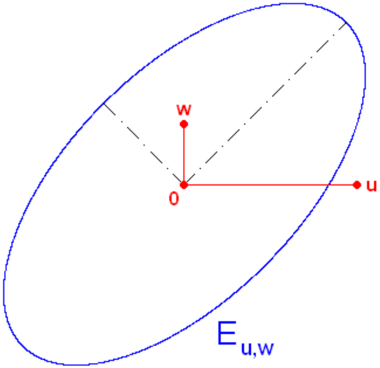
$$\operatorname{ind} A_- = \operatorname{ind} \begin{pmatrix} \ddots & \ddots & \\ \ddots & \ddots & w \\ & u & v \end{pmatrix} = \operatorname{wind}(E_{u,w}, v)$$

$$\operatorname{ind} A_+ = \operatorname{ind} \begin{pmatrix} v & w & \\ u & \ddots & \ddots \\ & \ddots & \ddots \end{pmatrix} = -\operatorname{wind}(E_{u,w}, v)$$

Example: Random Jacobi Operator

So, if the random (meaning pseudoergodic) operator A is Fredholm then, for **all** choices $u \in U$, $v \in V$ and $w \in W$:

$$\operatorname{ind} A_- \;=\; \operatorname{wind}(E_{u,w}, v), \qquad \operatorname{ind} A_+ \;=\; -\operatorname{wind}(E_{u,w}, v)$$



Example: Random Jacobi Operator

The previous slide has shown how 'hard' it is for a pseudoergodic Jacobi operator to be Fredholm. Let us underline this. Put

$$
\begin{aligned}
J(U,V,W) &:= \{\text{ Jacobi ops (4) } : u_i \in U,\ v_i \in V,\ w_i \in W\},\\
\Psi E(U,V,W) &:= \{\ A \in J(U,V,W)\ :\ A \text{ pseudoergodic }\}.
\end{aligned}
$$

$$
A = \begin{pmatrix}
\ddots & \ddots & & & & & \\
\ddots & v_{-2} & w_{-1} & & & & \\
 & u_{-2} & v_{-1} & w_0 & & & \\
 & & u_{-1} & v_0 & w_1 & & \\
 & & & u_0 & v_1 & w_2 & \\
 & & & & u_1 & v_2 & \ddots \\
 & & & & & \ddots & \ddots
\end{pmatrix} \tag{4}
$$

Example: Random Jacobi Operator

The previous slide has shown how ‘hard’ it is for a pseudoergodic Jacobi operator to be Fredholm. Let us underline this. Put

$$\begin{aligned} J(U,V,W) &:= \{ \text{ Jacobi ops (4) } : \ u_i \in U,\ v_i \in V,\ w_i \in W \}, \\ \Psi E(U,V,W) &:= \{ \ A \in J(U,V,W) \ : \ A \text{ pseudoergodic } \}. \end{aligned}$$

Example: Random Jacobi Operator

The previous slide has shown how ‘hard’ it is for a pseudoergodic Jacobi operator to be Fredholm. Let us underline this. Put

$$\begin{aligned} J(U,V,W) &:= \{ \text{ Jacobi ops (4) } : u_i \in U,\ v_i \in V,\ w_i \in W \}, \\ \Psi E(U,V,W) &:= \{ A \in J(U,V,W) \ : \ A \text{ pseudoergodic } \}. \end{aligned}$$

For $A \in \Psi E(U,V,W)$, the set of limops is **all of** $J(U,V,W)$. Hence, the following are equivalent:

- A is Fredholm,
- A is invertible,
- all $B \in J(U,V,W)$ are Fredholm,
- all $B \in J(U,V,W)$ are invertible.

Example: Random Jacobi Operator

$$\begin{aligned} J(U,V,W) &:= \{\text{ Jacobi ops (4) } : u_i \in U,\ v_i \in V,\ w_i \in W\}, \\ \Psi E(U,V,W) &:= \{\ A \in J(U,V,W)\ :\ A \text{ pseudoergodic }\}. \end{aligned}$$

For $A \in \Psi E(U,V,W)$, the set of limops is **all of** $J(U,V,W)$.
Hence, the following are equivalent:

- A is Fredholm,
- A is invertible,
- all $B \in J(U,V,W)$ are Fredholm,
- all $B \in J(U,V,W)$ are invertible.

In other words:

$$\mathrm{spec}_{\mathrm{ess}} A \;=\; \mathrm{spec}\, A \;=\; \bigcup \mathrm{spec}_{\mathrm{ess}} B \;=\; \bigcup \mathrm{spec}\, B,$$

with the unions taken over all $B \in J(U,V,W)$.

Example: Random Jacobi Operator

$$
\begin{aligned}
J(U,V,W) &:= \{\text{ Jacobi ops (4) } : u_i \in U,\ v_i \in V,\ w_i \in W\},\\
\Psi E(U,V,W) &:= \{ A \in J(U,V,W) : A \text{ pseudoergodic } \}.
\end{aligned}
$$

In other words:

$$
\operatorname{spec}_{\mathrm{ess}} A = \operatorname{spec} A = \bigcup \operatorname{spec}_{\mathrm{ess}} B = \bigcup \operatorname{spec} B,
$$

with the unions taken over all $B \in J(U,V,W)$.

In particular,

$$
\operatorname{spec} A \supseteq \bigcup_{\text{Laurent}} \operatorname{spec} L = \bigcup_{u,v,w} (v + E_{u,w}).
$$

Example: Discrete Schrödinger operator

We come back to discrete Schrödinger operators (i.e. discretisations of $-\Delta + M_b$):

$$A \;=\; V_{-1} + M_c + V_1 \;=\; \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & c_{-2} & 1 & & & & \\ & 1 & c_{-1} & 1 & & & \\ & & 1 & c_0 & 1 & & \\ & & & 1 & c_1 & 1 & \\ & & & & 1 & c_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}$$

where $c = (\dots, c_{-1}, c_0, c_1, \dots) \in \ell^\infty(\mathbb{Z})$. Clearly,

$$A_h \;=\; (V_{-1})_h + (M_c)_h + (V_1)_h \;=\; V_{-1} + (M_c)_h + V_1,$$

so that everything depends on the limit operators of M_c only.

Example 1: Periodic potential

If

$$c_{k+P} = c_k \qquad \text{for every} \qquad k \in \mathbb{Z},$$

then $\sigma^{\mathrm{op}}(M_c) = \left\{ M_{V_k c} \;:\; k \in \{0, 1, ..., P-1\} \right\}.$

Example 1: Periodic potential

If

$$c_{k+P} = c_k \qquad \text{for every} \qquad k \in \mathbb{Z},$$

then $\sigma^{\mathrm{op}}(M_c) = \Big\{ M_{V_k c} \ : \ k \in \{0, 1, ..., P-1\} \Big\}$.

But then $\sigma^{\mathrm{op}}(A) = \Big\{ V_{-k} A V_k \ : \ k \in \{0, 1, ..., P-1\} \Big\}$.

Consequently, A is invertible iff any/all of its limit operators are invertible. So in this case, A is **Fredholm** iff it is **invertible**, and

$$\mathrm{spec}_{\mathrm{ess}} A \;=\; \mathrm{spec}\, A \;=\; \mathrm{spec}^{\infty}_{\mathrm{point}} A$$

Example 2: Almost-periodic potential

Let c be almost-periodic with hull $h(c) = \mathrm{clos}\{V_k c : k \in \mathbb{Z}\}$.

$$\sigma^{\mathrm{op}}(M_c) = \{M_d \ : \ d \in h(c)\} = \mathrm{clos}\{M_{V_k c} \ : \ k \in \mathbb{Z}\},$$
$$\sigma^{\mathrm{op}}(A) = \{V_{-1} + M_d + V_1 \ : \ d \in h(c)\} = \mathrm{clos}\{V_{-k} A V_k \ : \ k \in \mathbb{Z}\}.$$

Example 2: Almost-periodic potential

Let c be almost-periodic with hull $h(c) = \mathrm{clos}\{V_k c : k \in \mathbb{Z}\}$.

$$\sigma^{\mathrm{op}}(M_c) = \{M_d \,:\, d \in h(c)\} = \mathrm{clos}\{M_{V_k c} \,:\, k \in \mathbb{Z}\},$$
$$\sigma^{\mathrm{op}}(A) = \{V_{-1} + M_d + V_1 \,:\, d \in h(c)\} = \mathrm{clos}\{V_{-k} A V_k \,:\, k \in \mathbb{Z}\}.$$

If any $A_h = \lim V_{-h_n} A V_{h_n}$ is invertible then $V_{-h_n} A V_{h_n}$ is invertible for large n, so that A itself is invertible!

Hence

$$A \text{ Fredholm} \iff A \text{ invertible} \iff \text{any } A_h \text{ invertible}$$

$$\mathrm{spec}_{\mathrm{ess}} A \;=\; \mathrm{spec}\, A \;=\; \mathrm{spec}\, A_h \;=\; \bigcup_h \mathrm{spec}^{\infty}_{\mathrm{point}} A_h$$

Example 3: Slowly oscillating potential

If
$$c_{i+1} - c_i \to 0 \qquad \text{as} \qquad i \to \pm\infty,$$
then $\sigma^{\mathrm{op}}(M_c) = \{\alpha I \,:\, \alpha \in c(\infty)\}$.

Example 3: Slowly oscillating potential

If

$$c_{i+1} - c_i \to 0 \qquad \text{as} \qquad i \to \pm\infty,$$

then $\sigma^{\mathrm{op}}(M_c) = \{\alpha I \,:\, \alpha \in c(\infty)\}$.

But then all limit operators A_h are Laurent operators (i.e., they have constant diagonals), for which invertibility & spectrum are well-understood.

$$\mathrm{spec}_{\mathrm{ess}} A \;=\; \bigcup \mathrm{spec}\, A_h \;=\; c(\infty) + [-2, 2]$$

Summary: Limit operators help us to determine the essential spectrum. So they give us **lower bounds** on the spectrum.

It would be good to also have **upper bounds**!

Classical upper bounds

- Gershgorin circles
- numerical range

We will discuss another approach.

Upper spectral bounds

We study bi-infinite matrices of the form

$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & \beta_{-2} & \gamma_{-1} & & & & \\ & \alpha_{-2} & \beta_{-1} & \gamma_0 & & & \\ & & \alpha_{-1} & \beta_0 & \gamma_1 & & \\ & & & \alpha_0 & \beta_1 & \gamma_2 & \\ & & & & \alpha_1 & \beta_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix},$$

where $\alpha = (\alpha_i)$, $\beta = (\beta_i)$ and $\gamma = (\gamma_i)$ are bounded sequences of complex numbers (more general: operators on a Banach space).

Upper spectral bounds

$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & \beta_{-2} & \gamma_{-1} & & & & \\ & \alpha_{-2} & \beta_{-1} & \gamma_0 & & & \\ & & \alpha_{-1} & \beta_0 & \gamma_1 & & \\ & & & \alpha_0 & \beta_1 & \gamma_2 & \\ & & & & \alpha_1 & \beta_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}$$

Task

Compute **upper bounds on spectrum and pseudospectra** of A, understood as a bounded linear operator $\ell^2 \to \ell^2$.

One more notion: Pseudospectrum

For $A \in L(E)$ and $\varepsilon > 0$, we put

$$\operatorname{spec} A \quad = \quad \{\lambda \in \mathbb{C} \ : \ A - \lambda I \text{ not invertible on } E\},$$

One more notion: Pseudospectrum

For $A \in L(E)$ and $\varepsilon > 0$, we put

$$\begin{aligned} \operatorname{spec} A &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ not invertible on } E\}, \\ \operatorname{spec}_{\text{ess}} A &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ not Fredholm on } E\}, \end{aligned}$$

One more notion: Pseudospectrum

For $A \in L(E)$ and $\varepsilon > 0$, we put

$$\begin{aligned}
\operatorname{spec} A &= \{\lambda \in \mathbb{C} \;:\; A - \lambda I \text{ not invertible on } E\}, \\
\operatorname{spec}_{\text{ess}} A &= \{\lambda \in \mathbb{C} \;:\; A - \lambda I \text{ not Fredholm on } E\}, \\
\operatorname{spec}_\varepsilon A &= \operatorname{spec} A \;\cup\; \{\lambda \in \mathbb{C} \;:\; \|(A - \lambda I)^{-1}\| > 1/\varepsilon\}
\end{aligned}$$

One more notion: Pseudospectrum

For $A \in L(E)$ and $\varepsilon > 0$, we put

$$\begin{aligned}
\operatorname{spec} A &= \{\lambda \in \mathbb{C} \;:\; A - \lambda I \text{ not invertible on } E\}, \\
\operatorname{spec}_{\mathrm{ess}} A &= \{\lambda \in \mathbb{C} \;:\; A - \lambda I \text{ not Fredholm on } E\}, \\
\operatorname{spec}_\varepsilon A &= \operatorname{spec} A \;\cup\; \{\lambda \in \mathbb{C} \;:\; \|(A - \lambda I)^{-1}\| > 1/\varepsilon\} \\
&= \bigcup_{\|T\| < \varepsilon} \operatorname{spec}(A + T)
\end{aligned}$$

One more notion: Pseudospectrum

For $A \in L(E)$ and $\varepsilon > 0$, we put

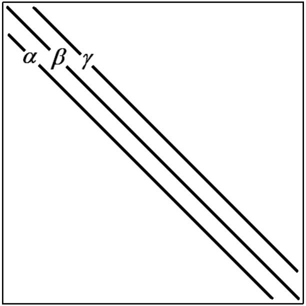
$$\begin{aligned}
\operatorname{spec} A &= \{\lambda \in \mathbb{C} \;:\; A - \lambda I \text{ not invertible on } E\},\\
\operatorname{spec}_{\mathrm{ess}} A &= \{\lambda \in \mathbb{C} \;:\; A - \lambda I \text{ not Fredholm on } E\},\\
\operatorname{spec}_{\varepsilon} A &= \operatorname{spec} A \;\cup\; \{\lambda \in \mathbb{C} \;:\; \|(A - \lambda I)^{-1}\| > 1/\varepsilon\}\\
&= \bigcup_{\|T\| < \varepsilon} \operatorname{spec}(A + T)\\
&\supseteq \operatorname{spec} A \;+\; \varepsilon\mathbb{D} \qquad (\text{“}{=}\text{” if } A \text{ is normal}).
\end{aligned}$$

The sets $\operatorname{spec}_{\varepsilon} A$, $\varepsilon > 0$, are the so-called $\varepsilon-$**pseudospectra** of A. It holds that

$$\operatorname{spec} A \;=:\; \operatorname{spec}_0 A \;\subset\; \operatorname{spec}_{\varepsilon_1} A \;\subset\; \operatorname{spec}_{\varepsilon_2} A, \qquad 0 < \varepsilon_1 < \varepsilon_2.$$

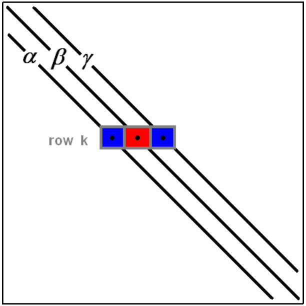
Reminder: Gershgorin's circles

Here is our tridiagonal bi-infinite matrix:



Reminder: Gershgorin's circles

Here is our tridiagonal bi-infinite matrix:

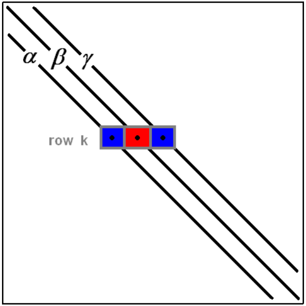


For every row k, consider the **disk** with

center at $a_{k,k}$ and radius $|a_{k,k-1}| + |a_{k,k+1}|$

Reminder: Gershgorin's circles

Here is our tridiagonal bi-infinite matrix:

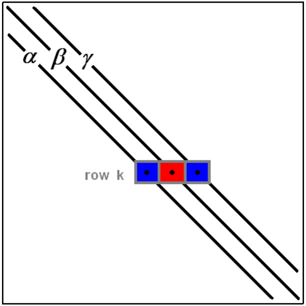


For every row k, consider the **disk** with

center at $a_{k,k}$ and radius $|a_{k,k-1}| + |a_{k,k+1}|$

Now take the union over all $k \in \mathbb{Z}$. $\Rightarrow$ **upper bound** on spec A.

Here is our tridiagonal bi-infinite matrix:



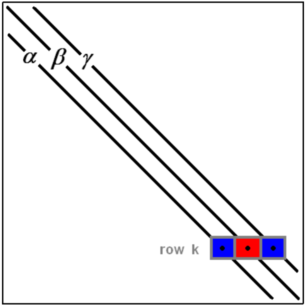
For every row k, consider the **disk** with

center at $a_{k,k}$ and radius $|a_{k,k-1}| + |a_{k,k+1}|$

Now take the union over all $k \in \mathbb{Z}$. $\Rightarrow$ **upper bound** on spec A.

Reminder: Gershgorin's circles

Here is our tridiagonal bi-infinite matrix:



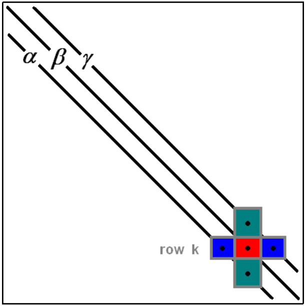
For every row k, consider the **disk** with

center at $a_{k,k}$ and radius $|a_{k,k-1}| + |a_{k,k+1}|$

Now take the union over all $k \in \mathbb{Z}$. $\Rightarrow$ **upper bound** on spec A.

Reminder: Gershgorin's circles

Here is our tridiagonal bi-infinite matrix:

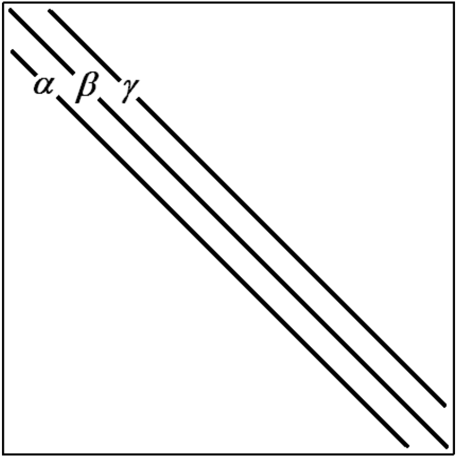


For every row k, consider the **disk** with

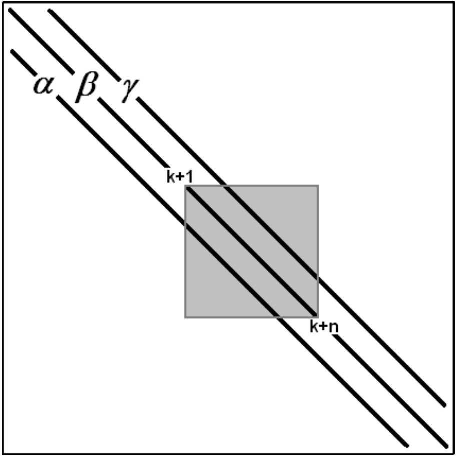
center at $a_{k,k}$ and radius $|a_{k,k-1}| + |a_{k,k+1}|$

Now take the union over all $k \in \mathbb{Z}$. $\Rightarrow$ **upper bound** on spec A.

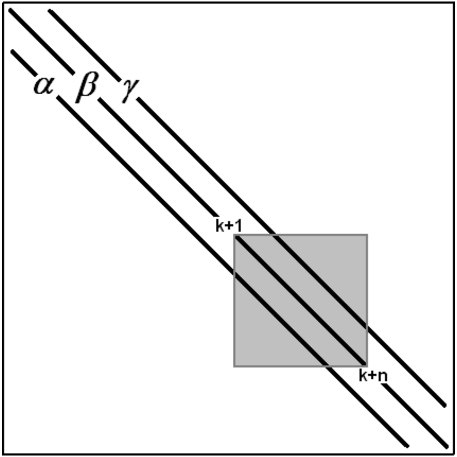
Look at (pseudo)spectra of the **finite principal submatrices** of A:



Look at (pseudo)spectra of the **finite principal submatrices** of A:

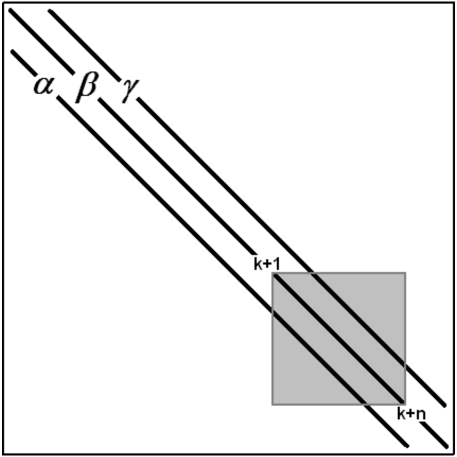


Look at (pseudo)spectra of the **finite principal submatrices** of A:



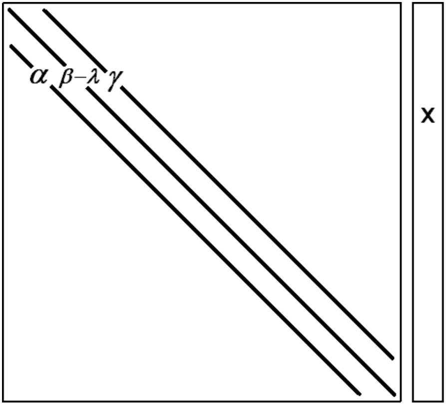
Our new strategy

Look at (pseudo)spectra of the **finite principal submatrices** of A:



Method 1: Finite principal submatrices

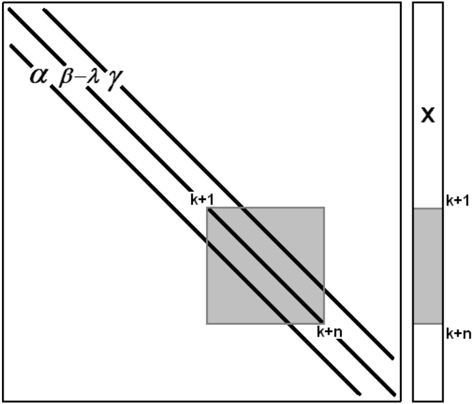
Let $\lambda \in \mathrm{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\, x\| \;<\; \varepsilon\, \|x\|$$

Method 1: Finite principal submatrices

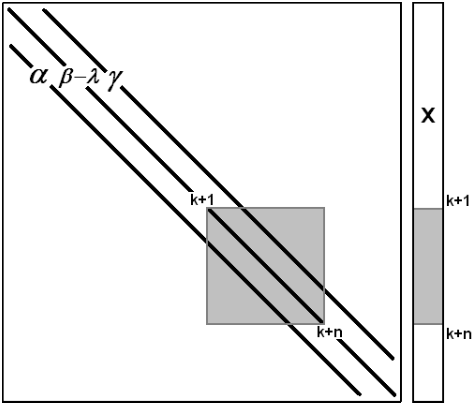
Let $\lambda \in \text{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\, x\| \;<\; \varepsilon\, \|x\|$$

Method 1: Finite principal submatrices

Let $\lambda \in \mathrm{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.



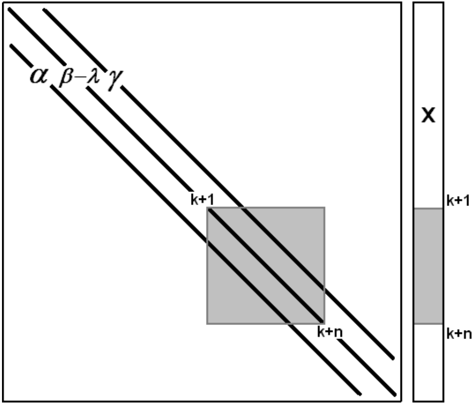
$$\|(A - \lambda I)\,x\| \;<\; \varepsilon\,\|x\|$$

Claim: $\exists k \in \mathbb{Z}:$

$$\|(A_{n,k} - \lambda I_n)\,x_{n,k}\| < (\varepsilon + \varepsilon_n)\,\|x_{n,k}\|$$

Method 1: Finite principal submatrices

Let $\lambda \in \text{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\, x\| \ < \ \varepsilon\, \|x\|$$

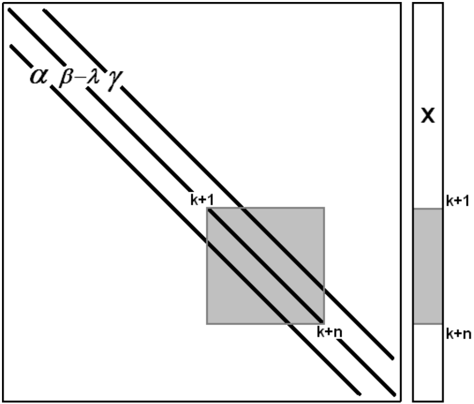
Claim: $\exists k \in \mathbb{Z}$:

$$\|(A_{n,k} - \lambda I_n)\, x_{n,k}\| < (\varepsilon + \varepsilon_n)\, \|x_{n,k}\|$$

$$\sum_k \|(A_{n,k} - \lambda I_n)\, x_{n,k}\|^2 < (\varepsilon + \varepsilon_n)^2 \sum_k \|x_{n,k}\|^2$$

Method 1: Finite principal submatrices

Let $\lambda \in \operatorname{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\,x\| \;<\; \varepsilon\,\|x\|$$

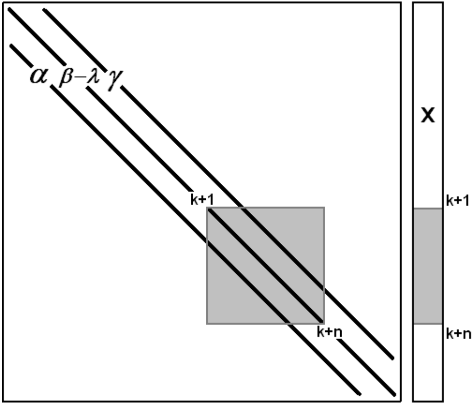
Claim: $\exists k \in \mathbb{Z}$:

$$\|(A_{n,k} - \lambda I_n)\,x_{n,k}\| < (\varepsilon + \varepsilon_n)\,\|x_{n,k}\|$$

$$\|(A_{n,k} - \lambda I_n)\,x_{n,k}\|^2 < (\varepsilon + \varepsilon_n)^2\,\|x_{n,k}\|^2$$

Method 1: Finite principal submatrices

Let $\lambda \in \mathrm{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.

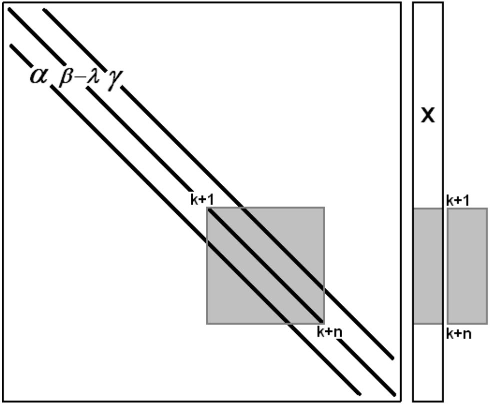


$$\|(A - \lambda I)\, x\| \;<\; \varepsilon\, \|x\|$$

$$\|(A_{n,k} - \lambda I_n)\, x_{n,k}\| < (\varepsilon + \varepsilon_n)\, \|x_{n,k}\|$$

Method 1: Finite principal submatrices

Let $\lambda \in \mathrm{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.

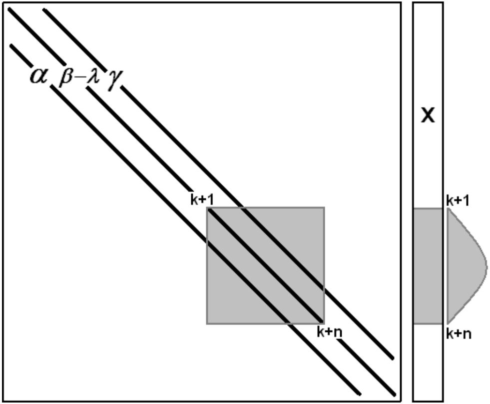


$$\|(A - \lambda I)\, x\| \;<\; \varepsilon\, \|x\|$$

$$\|(A_{n,k} - \lambda I_n)\, x_{n,k}\| < (\varepsilon + \varepsilon_n)\, \|x_{n,k}\|$$

$$\varepsilon_n \;<\; \frac{1}{\sqrt{n}}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

Let $\lambda \in \mathrm{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.



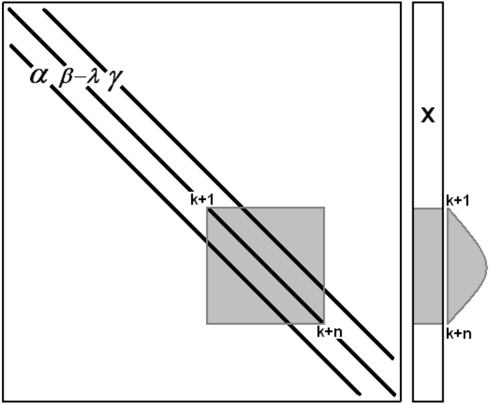
$$\|(A - \lambda I)\, x\| \;<\; \varepsilon\, \|x\|$$

$$\|(A_{n,k} - \lambda I_n)\, x_{n,k}\| < (\varepsilon + \varepsilon_n)\, \|x_{n,k}\|$$

$$\varepsilon_n \;<\; \frac{\pi}{n}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

Method 1: Finite principal submatrices

Let $\lambda \in \mathrm{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.



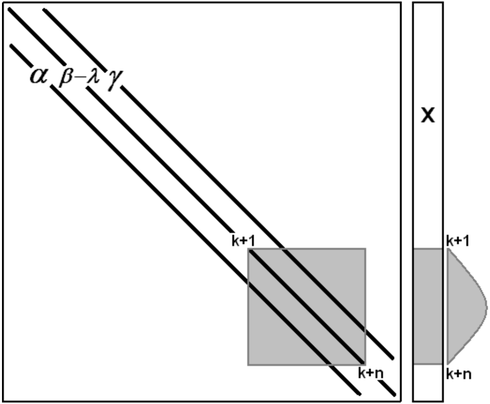
$$\|(A - \lambda I)\, x\| \;<\; \varepsilon\, \|x\|$$

$$\|(A_{n,k} - \lambda I_n)\, x_{n,k}\| < (\varepsilon + \varepsilon_n)\, \|x_{n,k}\|$$

$$\varepsilon_n \;<\; \frac{\pi}{n}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

$$\Rightarrow\; \lambda \in \mathrm{spec}_{\varepsilon+\varepsilon_n}(A_{n,k})$$

Let $\lambda \in \mathrm{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.



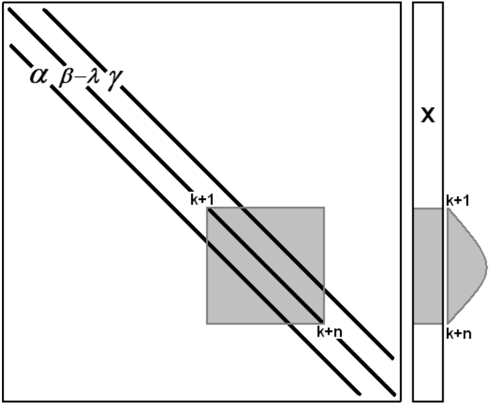
$$\|(A - \lambda I)\, x\| \;<\; \varepsilon\, \|x\|$$

$$\|(A_{n,k} - \lambda I_n)\, x_{n,k}\| \;<\; (\varepsilon + \varepsilon_n)\, \|x_{n,k}\|$$

$$\varepsilon_n \;<\; \frac{\pi}{n}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

$$\Rightarrow\; \lambda \in \mathrm{spec}_{\varepsilon+\varepsilon_n}(A_{n,k})$$

Let $\lambda \in \mathrm{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\,x\| \;<\; \varepsilon\,\|x\|$$

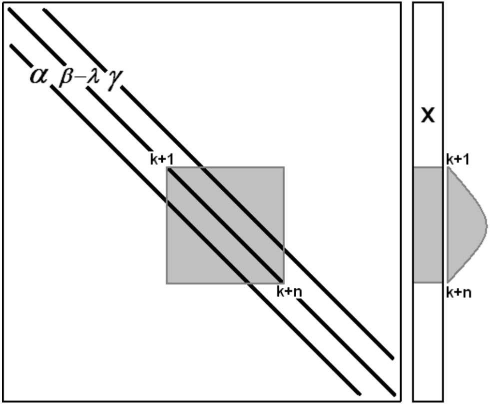
$$\|(A_{n,k} - \lambda I_n)\,x_{n,k}\| < (\varepsilon + \varepsilon_n)\,\|x_{n,k}\|$$

$$\varepsilon_n \;<\; \frac{\pi}{n}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

$$\Rightarrow \;\lambda \in \mathrm{spec}_{\varepsilon+\varepsilon_n}(A_{n,k})$$

Method 1: Finite principal submatrices

Let $\lambda \in \mathrm{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\,x\| \;<\; \varepsilon\,\|x\|$$

$$\|(A_{n,k} - \lambda I_n)\,x_{n,k}\| < (\varepsilon + \varepsilon_n)\,\|x_{n,k}\|$$

$$\varepsilon_n \;<\; \frac{\pi}{n}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

$$\Rightarrow \;\lambda \in \mathrm{spec}_{\varepsilon+\varepsilon_n}(A_{n,k})$$

Method 1: Finite principal submatrices

So one gets

Upper Bound

$$\mathrm{spec}_\varepsilon(A) \quad \subset \quad \bigcup_{k\in\mathbb{Z}} \mathrm{spec}_{\varepsilon+\varepsilon_n}(A_{n,k}),$$

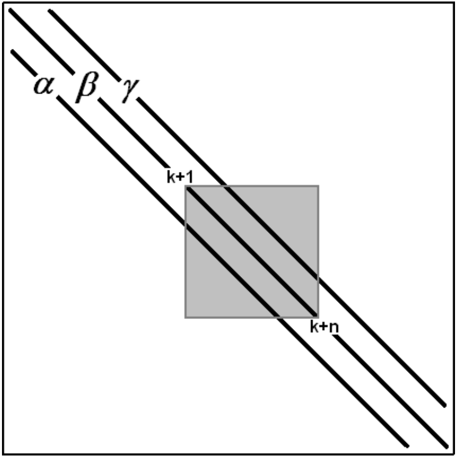
where

$$\varepsilon_n \;<\; \frac{\pi}{n}(\|\alpha\|_\infty + \|\gamma\|_\infty).$$

In particular, $\varepsilon_n \to 0$ as $n \to \infty$.

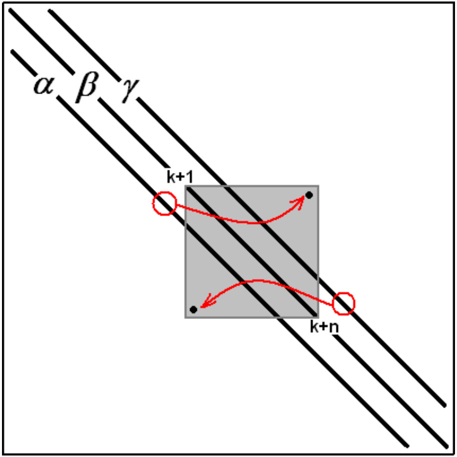
Method 2: **Periodised** finite principal submatrices

If the finite submatrices $A_{n,k}$ are "periodised",



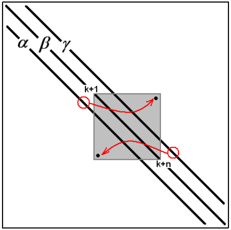
Method 2: **Periodised** finite principal submatrices

If the finite submatrices $A_{n,k}$ are "periodised",



Method 2: **Periodised** finite principal submatrices

If the finite submatrices $A_{n,k}$ are "periodised",



very similar computations show that, again,

$$\mathrm{spec}_\varepsilon(A) \subset \bigcup_{k\in\mathbb{Z}} \mathrm{spec}_{\varepsilon+\varepsilon_n}(A_{n,k}^{\mathrm{per}})$$

$$\text{with} \qquad \varepsilon_n < \frac{\pi}{n}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

but this upper bound on $\mathrm{spec}_\varepsilon(A)$ generally seems sharper than in method 1.

Method 1 vs. Method 2: An Example

Look at the shift operator

$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

Method 1 vs. Method 2: An Example

Look at the shift operator

$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

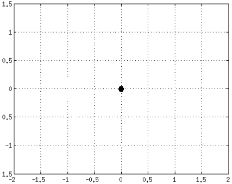
$$A_{n,k} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

Method 1 vs. Method 2: An Example

Look at the shift operator

$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

$$A_{n,k} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \qquad \operatorname{spec} A_{n,k} =$$

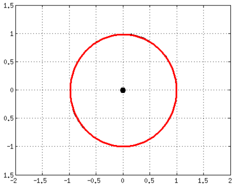


Method 1 vs. Method 2: An Example

Look at the shift operator

$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

$$A_{n,k} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \qquad \operatorname{spec} A_{n,k} =$$



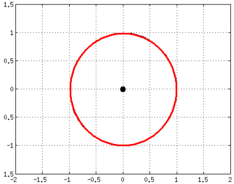
Method 1 vs. Method 2: An Example

Look at the shift operator

$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

$$A_{n,k} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

$$\operatorname{spec} A_{n,k} =$$



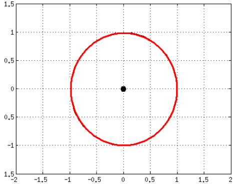
$$A_{n,k}^{\text{per}} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}$$

Method 1 vs. Method 2: An Example

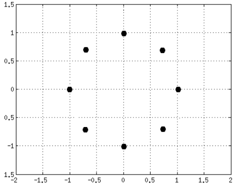
Look at the shift operator

$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

$$A_{n,k} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \qquad \operatorname{spec} A_{n,k} =$$



$$A^{\text{per}}_{n,k} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix} \qquad \operatorname{spec} A^{\text{per}}_{n,k} =$$

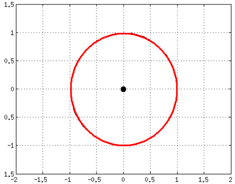


Method 1 vs. Method 2: An Example

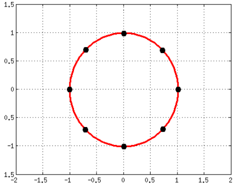
Look at the shift operator

$$(Ax)(i) = x(i+1), \quad \text{i.e.} \quad A = \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

$$A_{n,k} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \qquad \operatorname{spec} A_{n,k} =$$



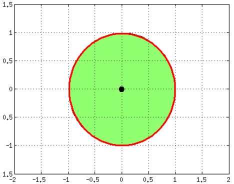
$$A^{\text{per}}_{n,k} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix} \qquad \operatorname{spec} A^{\text{per}}_{n,k} =$$



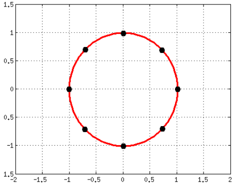
Look at the shift operator

$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

$$A_{n,k} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \qquad \mathrm{spec}_{\varepsilon_n} A_{n,k} =$$



$$A^{\mathrm{per}}_{n,k} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix} \qquad \mathrm{spec}\, A^{\mathrm{per}}_{n,k} =$$

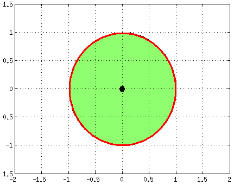


Method 1 vs. Method 2: An Example

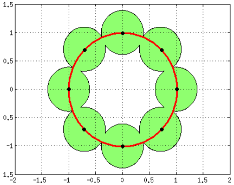
Look at the shift operator

$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

$$A_{n,k} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \qquad \mathrm{spec}_{\varepsilon_n} A_{n,k} =$$



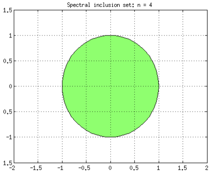
$$A^{\mathrm{per}}_{n,k} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix} \qquad \mathrm{spec}_{\varepsilon_n} A^{\mathrm{per}}_{n,k} =$$



Method 1 vs. Method 2: An Example

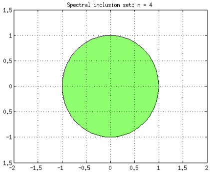
Look at the shift operator

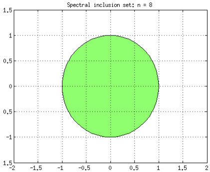
$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$



Look at the shift operator

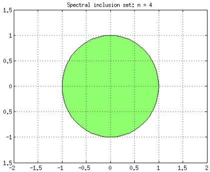
$$(Ax)(i) = x(i+1), \quad \text{i.e.} \quad A = \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

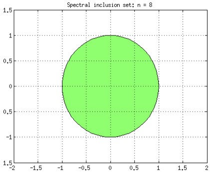


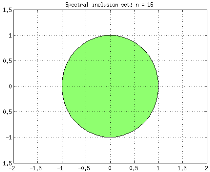


Look at the shift operator

$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$



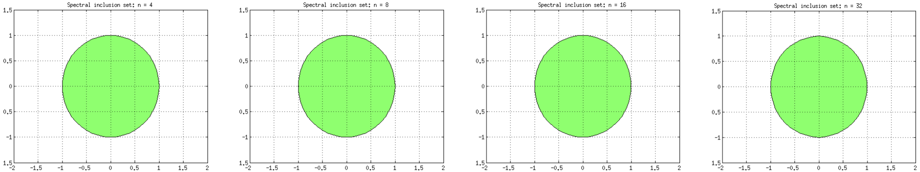




Method 1 vs. Method 2: An Example

Look at the shift operator

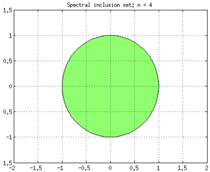
$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

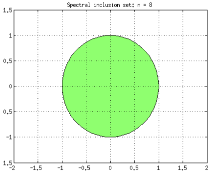


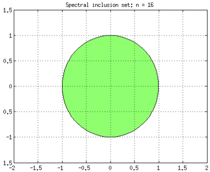
Method 1 vs. Method 2: An Example

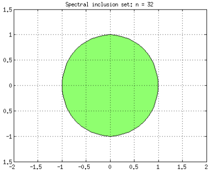
Look at the shift operator

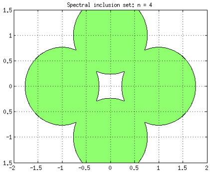
$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$







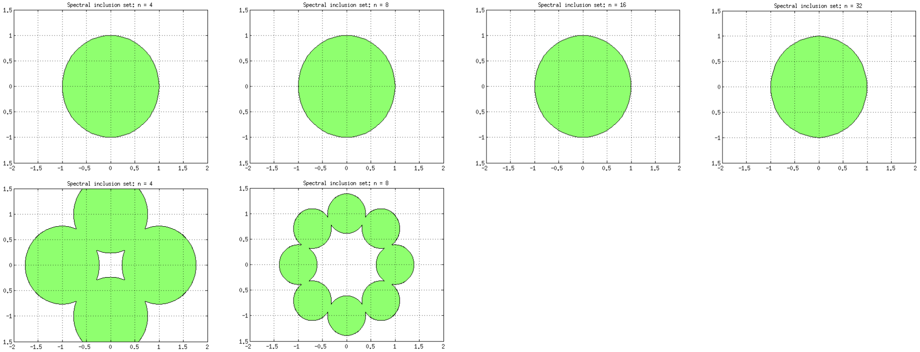




Method 1 vs. Method 2: An Example

Look at the shift operator

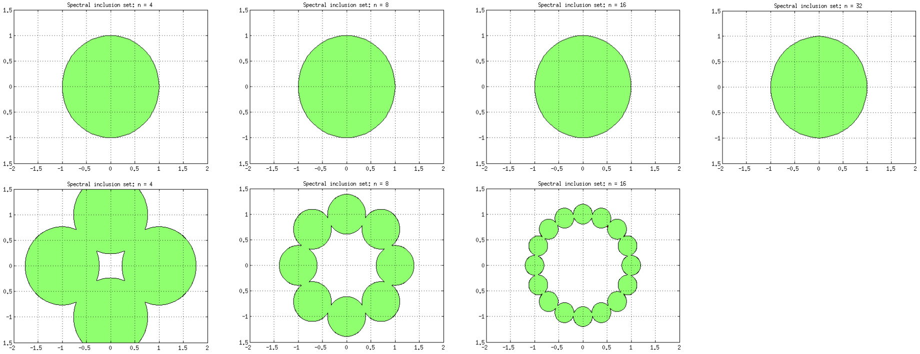
$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$



Method 1 vs. Method 2: An Example

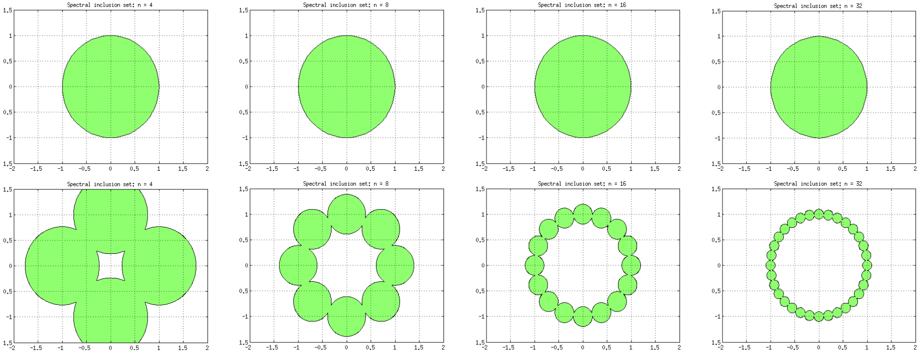
Look at the shift operator

$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$



Look at the shift operator

$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$



Method 1 vs. Method 2

Summary on Methods 1 & 2

- Both methods give **upper bounds** on $\operatorname{spec} A$ and $\operatorname{spec}_\varepsilon A$.

Method 1 vs. Method 2

Summary on Methods 1 & 2

- Both methods give **upper bounds** on $\operatorname{spec} A$ and $\operatorname{spec}_\varepsilon A$.
- The bound from Method 2 always appears to be **sharper**.

Method 1 vs. Method 2

Summary on Methods 1 & 2

- Both methods give **upper bounds** on $\operatorname{spec} A$ and $\operatorname{spec}_\varepsilon A$.
- The bound from Method 2 always appears to be **sharper**.
- Conjecture: Method 2 **converges** to $\operatorname{spec}_\varepsilon A$ as $n \to \infty$.

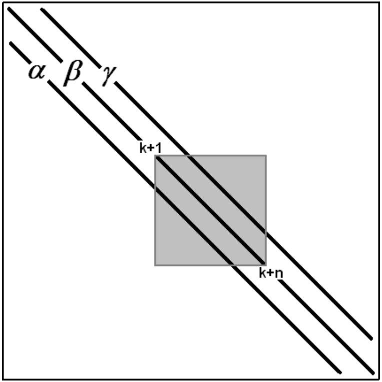
Method 1 vs. Method 2

Summary on Methods 1 & 2

- Both methods give **upper bounds** on $\operatorname{spec} A$ and $\operatorname{spec}_\varepsilon A$.
- The bound from Method 2 always appears to be **sharper**.
- Conjecture: Method 2 **converges** to $\operatorname{spec}_\varepsilon A$ as $n \to \infty$.
- Method 1 also works for **semi-infinite** and **finite** matrices A!

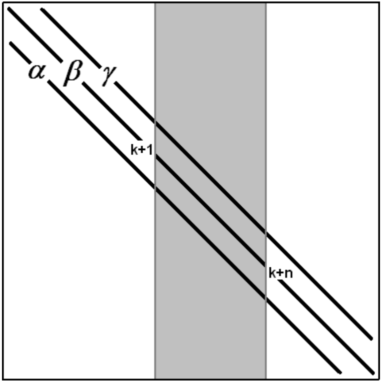
Here is another idea: Method 3

Instead of



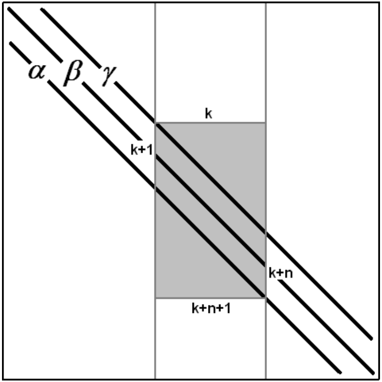
Here is another idea: Method 3

We do a **"one-sided"** truncation.



Here is another idea: Method 3

We do a **“one-sided”** truncation.



In a sense, we work with **rectangular** finite submatrices.

This is motivated by work of Davies 1998 and Hansen 2008.
(Also see Heinemeyer/ML/Potthast [SIAM Num. Anal. 2007].)

Method 3: Projection Operator

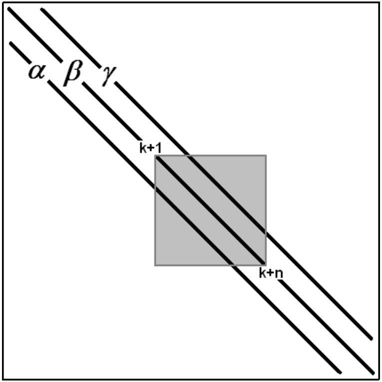
For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let $P_{n,k} : \ell^2 \to \ell^2$ denote the projection

$$(P_{n,k}x)(i) \;:=\; \begin{cases} x(i), & i \in \{k+1, ..., k+n\}, \\ \quad 0 & \text{otherwise.} \end{cases}$$

Further, we put $X_{n,k} := \operatorname{im} P_{n,k}$ and identify it with $\mathbb{C}^n$ in the obvious way.

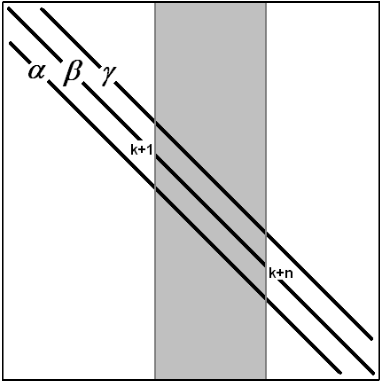
Method 3: Truncations

Method 1:



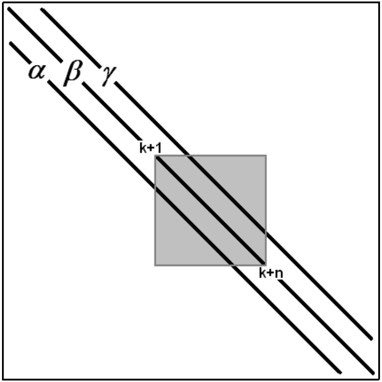
$$P_{n,k}(A-\lambda I)P_{n,k}|x_{n,k}$$

Method 3:



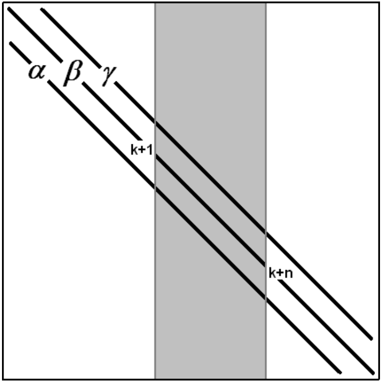
$$(A-\lambda I)P_{n,k}|x_{n,k}$$

Method 1:



$$\mathbf{P_{n,k}}(A-\lambda I)\mathbf{P_{n,k}}|_{X_{n,k}}$$

Method 3:



$$(A-\lambda I)\mathbf{P_{n,k}}|_{X_{n,k}}$$

Method 1:

$$\lambda \in \mathrm{spec}_\varepsilon(A) \quad \Longrightarrow \quad \text{For some } k \in \mathbb{Z}:$$

$$\lambda \in \mathrm{spec}_{\varepsilon+\varepsilon_n}\left(P_{n,k}AP_{n,k}|_{X_{n,k}}\right)$$

Method 1:

$$\lambda \in \operatorname{spec}_\varepsilon(A) \quad \Longrightarrow \quad \text{For some } k \in \mathbb{Z}:$$

$$\lambda \in \operatorname{spec}_{\varepsilon+\varepsilon_n}\big(P_{n,k}AP_{n,k}|_{X_{n,k}}\big)$$

$$\text{i.e.} \quad s_{\min}\big(P_{n,k}(A-\lambda I)P_{n,k}\big) < \varepsilon + \varepsilon_n$$

Method 1:

$$\lambda \in \mathrm{spec}_\varepsilon(A) \quad \Longrightarrow \quad \text{For some } k \in \mathbb{Z}:$$

$$\lambda \in \mathrm{spec}_{\varepsilon+\varepsilon_n}\big(P_{n,k}AP_{n,k}|_{X_{n,k}}\big)$$

$$\text{i.e.} \quad s_{\min}\big(P_{n,k}(A-\lambda I)P_{n,k}\big) < \varepsilon+\varepsilon_n$$

$$\min \mathrm{spec}\Big(\big(P_{n,k}(A-\lambda I)P_{n,k}\big)^*\big(P_{n,k}(A-\lambda I)P_{n,k}\big)\Big) < \big(\varepsilon+\varepsilon_n\big)^2$$

Method 1:

$$\lambda \;\in\; \mathrm{spec}_\varepsilon(A) \quad\Longrightarrow\quad \text{For some } k \in \mathbb{Z}:$$

$$\lambda \;\in\; \mathrm{spec}_{\varepsilon+\varepsilon_n}\left(P_{n,k}AP_{n,k}|_{X_{n,k}}\right)$$

$$\text{i.e.}\quad s_{\min}\left(P_{n,k}(A-\lambda I)P_{n,k}\right) \;<\; \varepsilon+\varepsilon_n$$

$$\min\mathrm{spec}\left(\left(P_{n,k}(A-\lambda I)^*P_{n,k}\right)\left(P_{n,k}(A-\lambda I)P_{n,k}\right)\right) \;<\; \left(\varepsilon+\varepsilon_n\right)^2$$

Method 1:

$\lambda \in \text{spec}_\varepsilon(A) \quad \Longrightarrow \quad$ For some $k \in \mathbb{Z}$:

$$\lambda \in \text{spec}_{\varepsilon+\varepsilon_n}\left(P_{n,k}AP_{n,k}|_{X_{n,k}}\right)$$

i.e. $$s_{\min}\left(P_{n,k}(A-\lambda I)P_{n,k}\right) < \varepsilon + \varepsilon_n$$

$$\min \text{spec}\left(P_{n,k}(A-\lambda I)^* P_{n,k} P_{n,k}(A-\lambda I)P_{n,k}\right) < (\varepsilon+\varepsilon_n)^2$$

Method 1:

$$\lambda \in \mathrm{spec}_\varepsilon(A) \quad \Longrightarrow \quad \text{For some } k \in \mathbb{Z}:$$

$$\lambda \in \mathrm{spec}_{\varepsilon+\varepsilon_n}\big(P_{n,k}AP_{n,k}|_{X_{n,k}}\big)$$

$$\text{i.e.} \quad s_{\min}\big(P_{n,k}(A-\lambda I)P_{n,k}\big) < \varepsilon + \varepsilon_n$$

$$\min\mathrm{spec}\Big(P_{n,k}(A-\lambda I)^* P_{n,k}(A-\lambda I)P_{n,k}\Big) < (\varepsilon+\varepsilon_n)^2$$

Method 1:

$$\lambda \in \mathrm{spec}_\varepsilon(A) \implies \text{For some } k \in \mathbb{Z}:$$

$$\lambda \in \mathrm{spec}_{\varepsilon+\varepsilon_n}\left(P_{n,k}AP_{n,k}|_{X_{n,k}}\right)$$

$$\text{i.e.} \quad s_{\min}\left(P_{n,k}(A-\lambda I)P_{n,k}\right) < \varepsilon + \varepsilon_n$$

Idea: $\min \mathrm{spec}\left(P_{n,k}(A-\lambda I)^* \cancel{P_{n,k}}(A-\lambda I)P_{n,k}\right) < (\varepsilon+\varepsilon_n)^2$

Method 1:

$$\lambda \in \operatorname{spec}_\varepsilon(A) \quad \Longrightarrow \quad \text{For some } k \in \mathbb{Z}:$$

$$\lambda \in \operatorname{spec}_{\varepsilon+\varepsilon_n}(P_{n,k}AP_{n,k}|_{X_{n,k}})$$

$$\text{i.e.} \quad s_{\min}\left(\cancel{P_{n,k}}(A-\lambda I)P_{n,k}\right) < \varepsilon + \varepsilon_n$$

Idea: $$\min\operatorname{spec}\left(P_{n,k}(A-\lambda I)^*\cancel{P_{n,k}}(A-\lambda I)P_{n,k}\right) < (\varepsilon+\varepsilon_n)^2$$

Method 3: Method 1 revisited

Method 1:

$$\lambda \in \text{spec}_\varepsilon(A) \quad \Longrightarrow \quad \text{For some } k \in \mathbb{Z}:$$

$$\lambda \in \text{spec}_{\varepsilon+\varepsilon_n}\left(\cancel{P_{n,k}} A P_{n,k}|_{X_{n,k}}\right)$$

$$\text{i.e.} \quad s_{\min}\left(\cancel{P_{n,k}}(A-\lambda I)P_{n,k}\right) < \varepsilon + \varepsilon_n$$

Idea: $$\min \text{spec}\left(P_{n,k}(A-\lambda I)^* \cancel{P_{n,k}}(A-\lambda I)P_{n,k}\right) < (\varepsilon+\varepsilon_n)^2$$

Method 1:

Idea: $\min \operatorname{spec}\Big(P_{n,k}(A-\lambda I)^*\cancel{P_{n,k}}(A-\lambda I)P_{n,k}\Big) \;<\; \big(\varepsilon+\varepsilon_n\big)^2$

Method 1:

Idea: $\min\operatorname{spec}\Big(P_{n,k}(A-\lambda I)^*\cancel{P_{n,k}}(A-\lambda I)P_{n,k}\Big) \;<\; \big(\varepsilon+\varepsilon_n\big)^2$

Method 3

Let $\gamma_\varepsilon^{n,k}(A)$ be the set of all $\lambda\in\mathbb{C}$, for which

$$\min\operatorname{spec}\Big(P_{n,k}(A-\lambda I)^*(A-\lambda I)P_{n,k}\Big) \;<\; \big(\varepsilon+\varepsilon_n\big)^2$$

Method 3: Method 1 revisited

Method 1:

Idea: $$\min\operatorname{spec}\Big(P_{n,k}(A-\lambda I)^*\cancel{P_{n,k}}(A-\lambda I)P_{n,k}\Big) \;<\; (\varepsilon+\varepsilon_n)^2$$

Method 3

Let $\gamma_\varepsilon^{n,k}(A)$ be the set of all $\lambda \in \mathbb{C}$, for which

$$\min\operatorname{spec}\Big(P_{n,k}(A-\lambda I)^*(A-\lambda I)P_{n,k}\Big) \;<\; (\varepsilon+\varepsilon_n)^2$$

and $$\min\operatorname{spec}\Big(P_{n,k}(A-\lambda I)(A-\lambda I)^*P_{n,k}\Big) \;<\; (\varepsilon+\varepsilon_n)^2.$$

Method 3: Method 1 revisited

Method 1:

Idea: $$\min\operatorname{spec}\Big(P_{n,k}(A-\lambda I)^*\cancel{P_{n,k}}(A-\lambda I)P_{n,k}\Big) \;<\; (\varepsilon+\varepsilon_n)^2$$

Method 3

Let $\gamma_\varepsilon^{n,k}(A)$ be the set of all $\lambda \in \mathbb{C}$, for which

$$\min\operatorname{spec}\Big(P_{n,k}(A-\lambda I)^*(A-\lambda I)P_{n,k}\Big) \;<\; (\varepsilon+\varepsilon_n)^2$$

and $$\min\operatorname{spec}\Big(P_{n,k}(A-\lambda I)(A-\lambda I)^*P_{n,k}\Big) \;<\; (\varepsilon+\varepsilon_n)^2.$$

Then put

$$\Gamma_\varepsilon^n(A) \;:=\; \bigcup_{k\in\mathbb{Z}} \gamma_\varepsilon^{n,k}(A).$$

Method 3: Spectral bounds

Again we get (as in Methods 1 & 2)

Upper Bound

$$\mathrm{spec}_\varepsilon(A) \quad \subset \quad \bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon+\varepsilon_n}(A) \quad = \quad \Gamma^n_{\varepsilon+\varepsilon_n}(A)$$

$$\text{with} \qquad \varepsilon_n \; < \; \frac{\pi}{n}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

and this time the upper bound looks even sharper than before.

Again we get (as in Methods 1 & 2)

Upper Bound

$$\mathrm{spec}_\varepsilon(A) \quad \subset \quad \bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon+\varepsilon_n}(A) \quad = \quad \Gamma^n_{\varepsilon+\varepsilon_n}(A)$$

$$\text{with} \qquad \varepsilon_n \;<\; \frac{\pi}{n}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

and this time the upper bound looks even sharper than before.
But now we also have

Lower Bound

$$\Gamma^n_\varepsilon(A) \quad \subset \quad \mathrm{spec}_\varepsilon(A).$$

From the lower and upper bound

$$\Gamma^n_\varepsilon(A) \subset \mathrm{spec}_\varepsilon(A) \quad \text{and} \quad \mathrm{spec}_\varepsilon(A) \subset \Gamma^n_{\varepsilon+\varepsilon_n}(A)$$

we get

Sandwich 1

$$\Gamma^n_\varepsilon(A) \quad \subset \quad \mathrm{spec}_\varepsilon(A) \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon_n}(A)$$

Method 3: Spectral bounds

From the lower and upper bound

$$\Gamma^n_\varepsilon(A) \subset \mathrm{spec}_\varepsilon(A) \quad \text{and} \quad \mathrm{spec}_\varepsilon(A) \subset \Gamma^n_{\varepsilon+\varepsilon_n}(A)$$

we get

Sandwich 1

$$\Gamma^n_\varepsilon(A) \quad \subset \quad \mathrm{spec}_\varepsilon(A) \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon_n}(A)$$

Sandwich 2

$$\mathrm{spec}_\varepsilon(A) \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon_n}(A) \quad \subset \quad \mathrm{spec}_{\varepsilon+\varepsilon_n}(A).$$

Method 3: Spectral bounds

From the lower and upper bound

$$\Gamma^n_\varepsilon(A) \;\subset\; \mathrm{spec}_\varepsilon(A) \qquad \text{and} \qquad \mathrm{spec}_\varepsilon(A) \;\subset\; \Gamma^n_{\varepsilon+\varepsilon_n}(A)$$

we get

Sandwich 1

$$\Gamma^n_\varepsilon(A) \quad\subset\quad \mathrm{spec}_\varepsilon(A) \quad\subset\quad \Gamma^n_{\varepsilon+\varepsilon_n}(A)$$

Sandwich 2

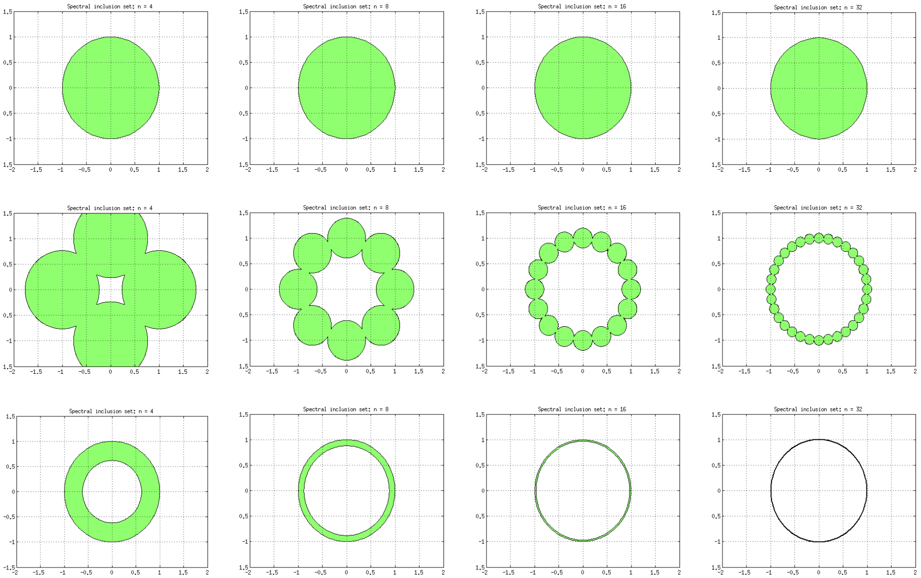
$$\mathrm{spec}_\varepsilon(A) \quad\subset\quad \Gamma^n_{\varepsilon+\varepsilon_n}(A) \quad\subset\quad \mathrm{spec}_{\varepsilon+\varepsilon_n}(A).$$

In particular, it follows that

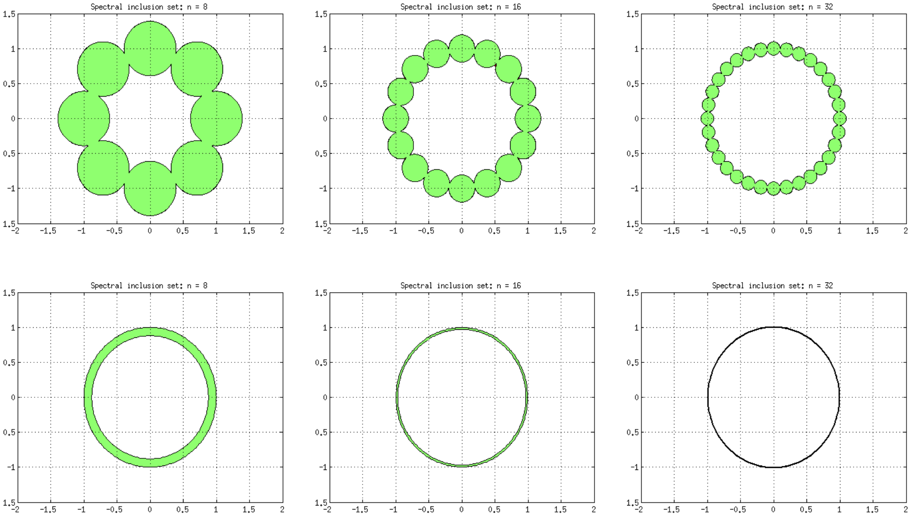
$$\Gamma^n_{\varepsilon+\varepsilon_n}(A) \quad\to\quad \mathrm{spec}_\varepsilon(A), \qquad n\to\infty$$

in the Hausdorff metric.

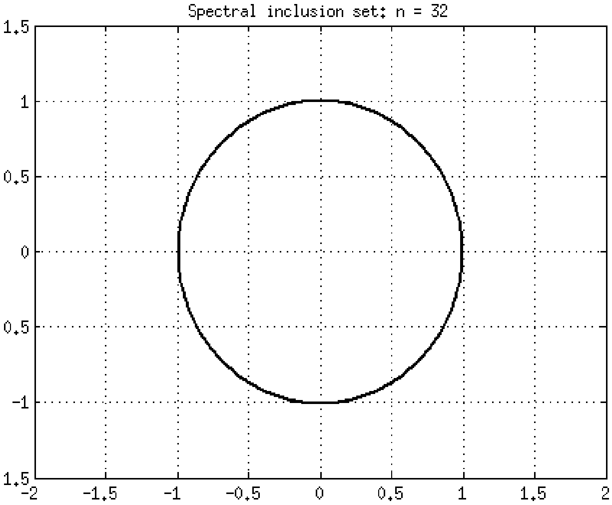
Methods 1, 2 & 3: The Shift Operator



Methods 2 & 3: The Shift Operator



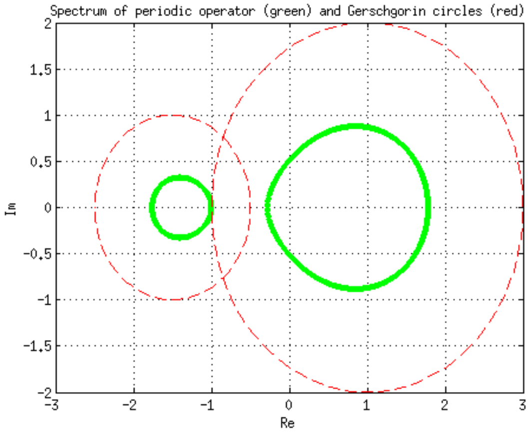
Method 3: The Shift Operator



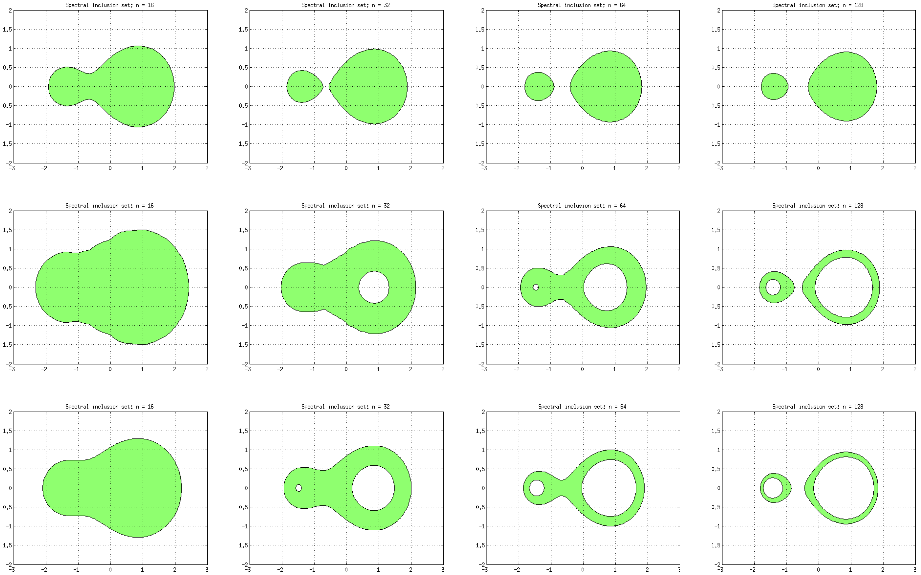
Methods 1, 2 & 3: Second Example

We now look at a matrix A with 3-periodic diagonals:

main diagonal: $\cdots, \ -\frac{3}{2}, \ 1, \ 1, \ \cdots$
super-diagonal: $\cdots, \ 1, \ 2, \ 1, \ \cdots$

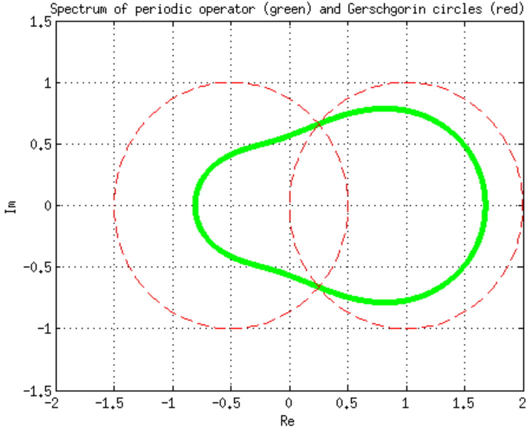


Methods 1, 2 & 3: Second Example

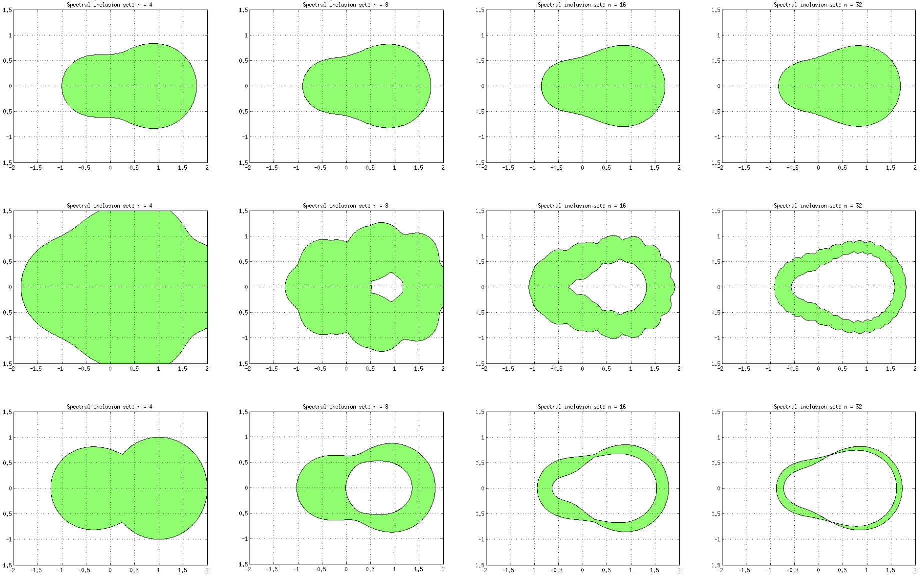


We now look at a matrix A with 3-periodic diagonals:

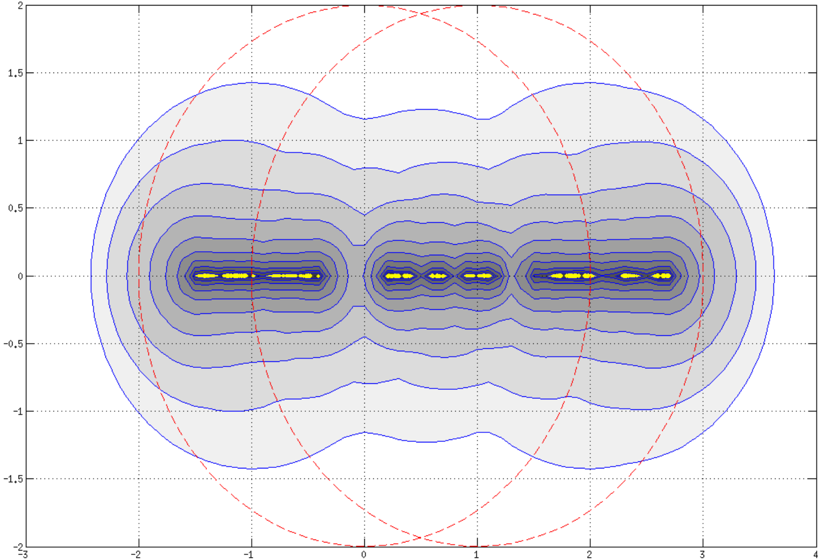
main diagonal: $\cdots,\ -\frac{1}{2},\ 1,\ 1,\ \cdots$
super-diagonal: $\cdots,\ 1,\ 1,\ 1,\ \cdots$



Methods 1, 2 & 3: Third Example



Method 3: Schrödinger operator with Cantor spectrum



Spectral Bounds: An Example

From a talk of Anthony Zee (MSRI Berkeley, 1999)

Exploring Non-Hermitean Random Matrices 16

largely terra incognita!

Consider

$$H = \begin{pmatrix} 0 & \pm1 & & & & 1 \\ 1 & 0 & \pm1 & & & \\ & 1 & 0 & \pm1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & \ddots & \pm1 \\ \pm1 & & & & 1 & 0 \end{pmatrix}$$

+1 or −1 with probability ½

If H regarded as Hamiltonian of quantum particle $\sum_j H_{ij}\psi_j = E\psi_i$

$$\Rightarrow \quad \psi_{i-1} + t_i\,\psi_{i+1} = E\,\psi_i \,, \qquad t_i = \pm 1$$

Feinberg & Zee. Nucl. Phys

$$\bullet \quad \bullet \quad \underset{i-1}{\bullet} \overset{+1}{\curvearrowleft} \underset{i}{\bullet} \overset{\pm1}{\curvearrowright} \underset{i+1}{\bullet} \quad \bullet \quad \bullet \quad \bullet$$

What does the spectrum look like?

Example [Feinberg/Zee 1999]

Look at the **bi-infinite** matrix

$$A^b = \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 0 & 1 & & & \\ & b_{-1} & 0 & 1 & & \\ & & b_0 & 0 & 1 & \\ & & & b_1 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix},$$

where $b = (\cdots, b_{-1}, b_0, b_1, \cdots) \in \{\pm 1\}^{\mathbb{Z}}$ is a **pseudoergodic** sequence

Example [Feinberg/Zee 1999]

Look at the **bi-infinite** matrix

$$A^b = \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 0 & 1 & & & \\ & b_{-1} & 0 & 1 & & \\ & & b_0 & 0 & 1 & \\ & & & b_1 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix},$$

where $b = (\cdots, b_{-1}, b_0, b_1, \cdots) \in \{\pm 1\}^{\mathbb{Z}}$ is a **pseudoergodic** sequence; that means:

every finite pattern of ± 1's can be found somewhere in the infinite sequence b.

Example [Feinberg/Zee 1999]

Spectral Formula Chandler-Wilde, ML 2007

If b is pseudoergodic then

$$\operatorname{spec} A^b \;=\; \operatorname{spec}_{\mathrm{ess}} A^b \;=\; \bigcup_{c\in\{\pm 1\}^{\mathbb{Z}}} \operatorname{spec}^{\infty}_{\mathrm{point}} A^c.$$

Example [Feinberg/Zee 1999]

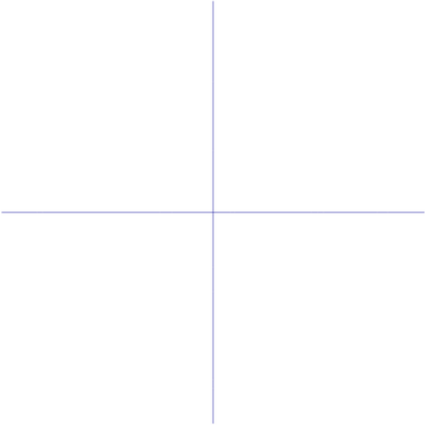
Spectral Formula Chandler-Wilde, ML 2007

If b is pseudoergodic then

$$\operatorname{spec} A^b \;=\; \operatorname{spec}_{\mathrm{ess}} A^b \;=\; \bigcup_{c\in\{\pm 1\}^{\mathbb{Z}}} \operatorname{spec}^{\infty}_{\mathrm{point}} A^c.$$

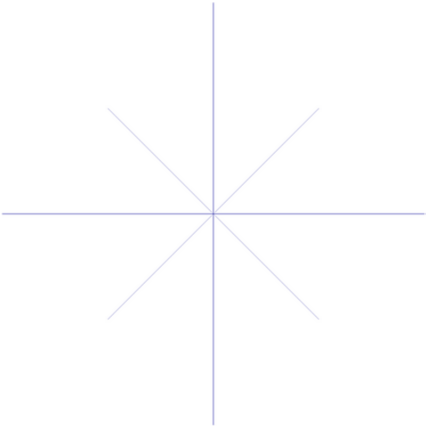
Idea: Try to "exhaust" the RHS by running through all **periodic** ± 1 sequences c.

Example [Feinberg/Zee 1999]

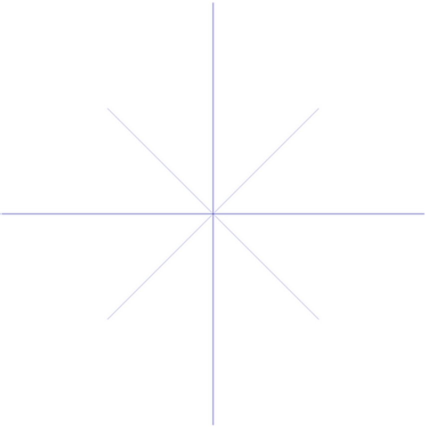


Period 1

Example [Feinberg/Zee 1999]

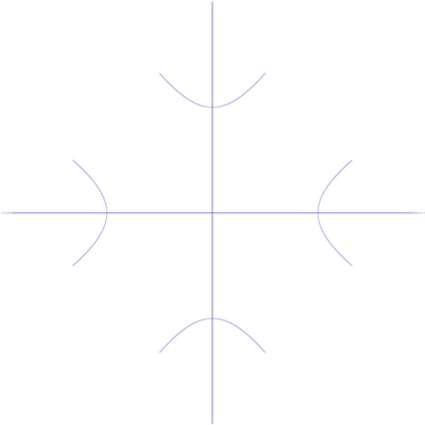


Period 2

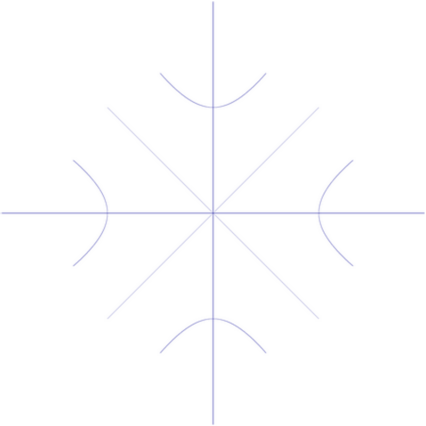


Periods $1, 2$

Example [Feinberg/Zee 1999]

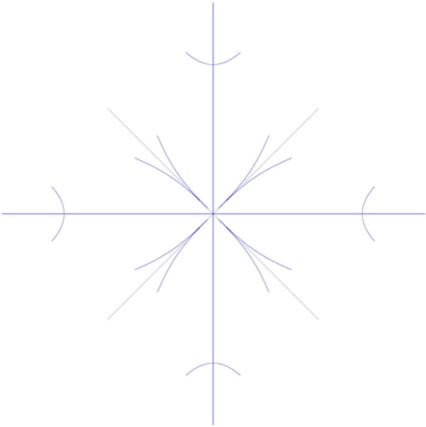


Period 3

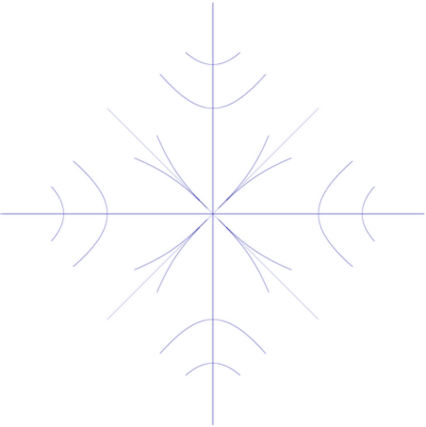


Periods $1, \ldots, 3$

Example [Feinberg/Zee 1999]

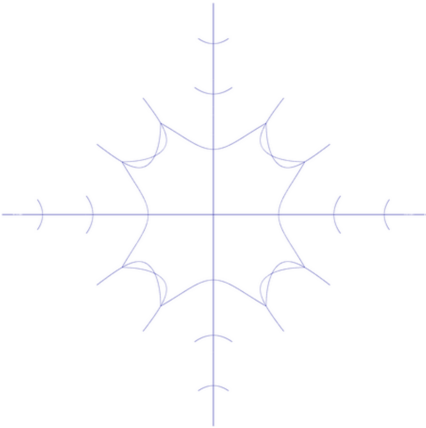


Period 4

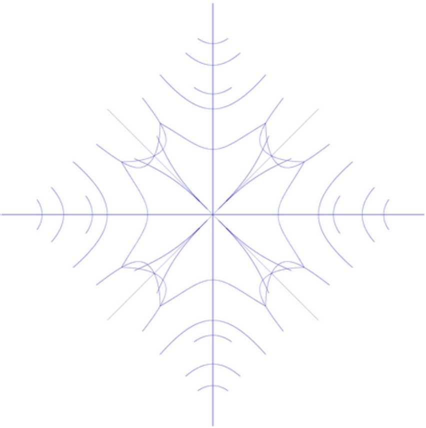


Periods 1, ..., 4

Example [Feinberg/Zee 1999]

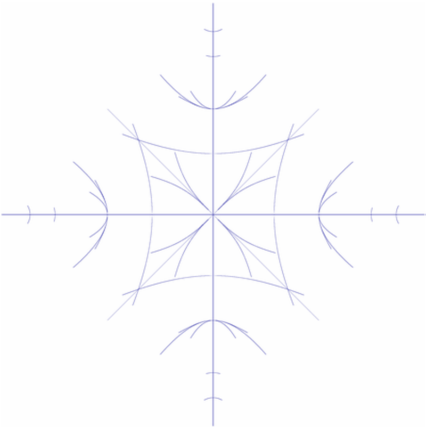


Period 5

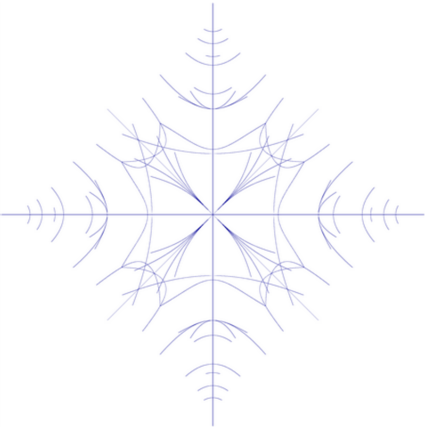


Periods 1, ..., 5

Example [Feinberg/Zee 1999]

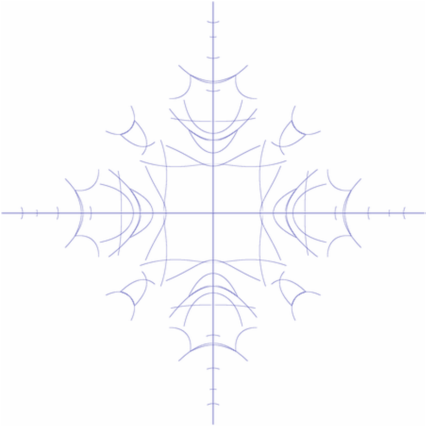


Period 6

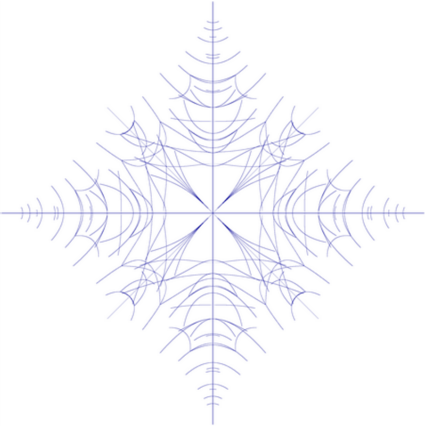


Periods 1, ..., 6

Example [Feinberg/Zee 1999]

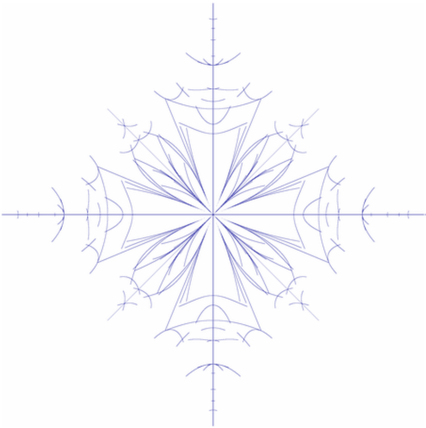


Period 7

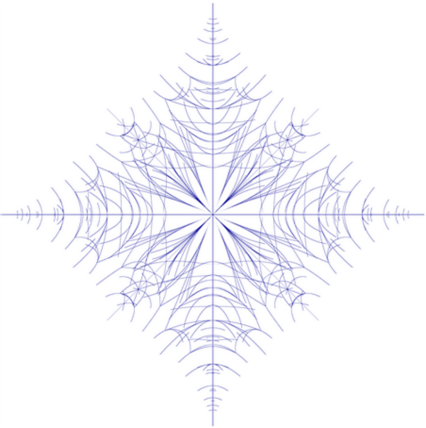


Periods $1, \ldots, 7$

Example [Feinberg/Zee 1999]

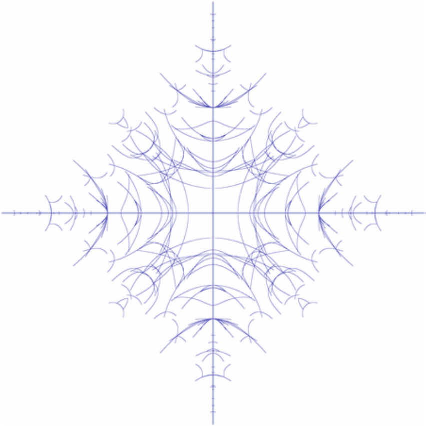


Period 8

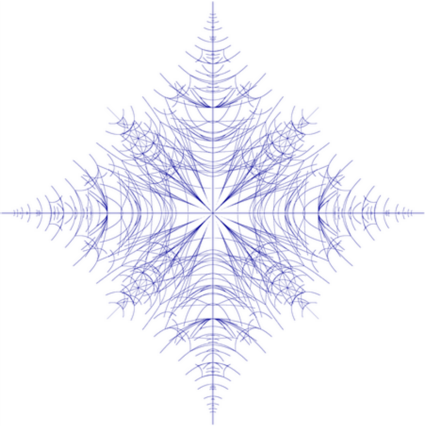


Periods $1, \ldots, 8$

Example [Feinberg/Zee 1999]

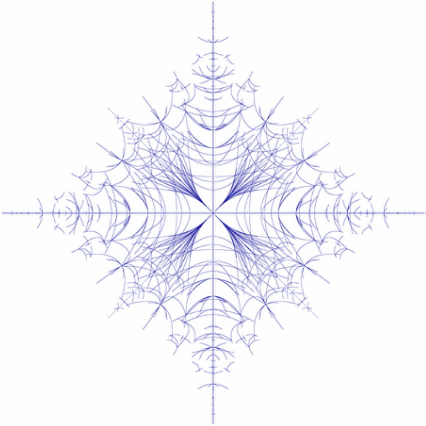


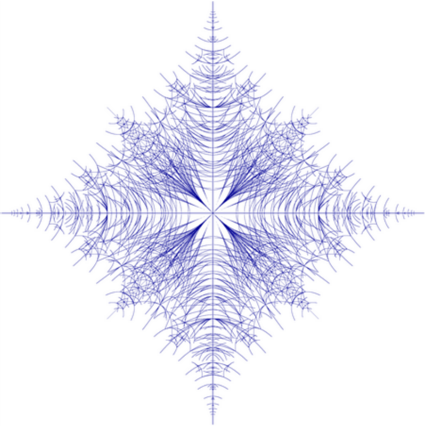
Period 9



Periods $1, ..., 9$

Example [Feinberg/Zee 1999]

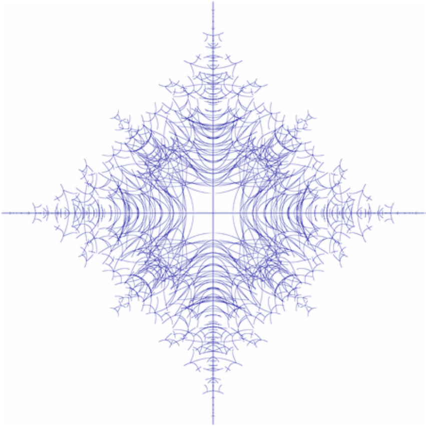




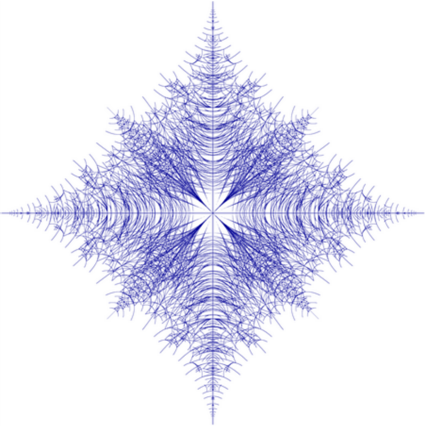
Period 10

Periods 1, ..., 10

Example [Feinberg/Zee 1999]

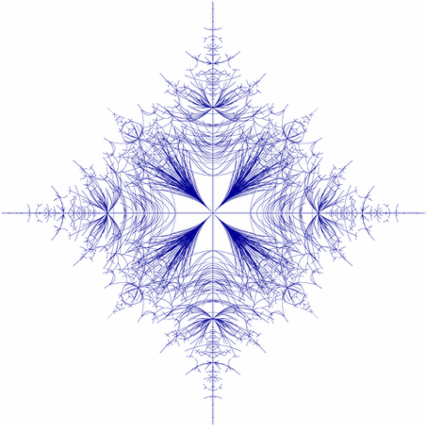


Period 11

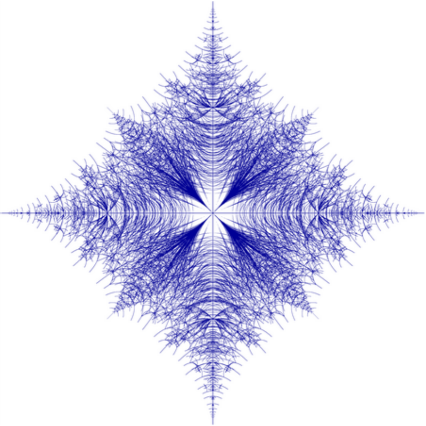


Periods 1, ..., 11

Example [Feinberg/Zee 1999]

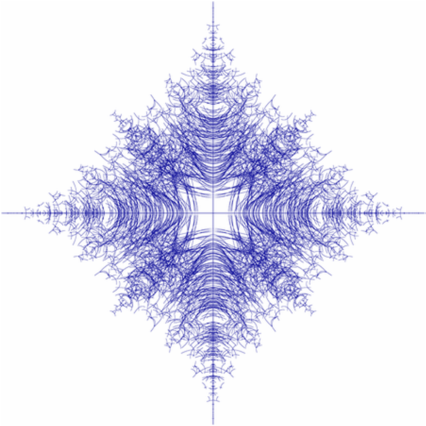


Period 12

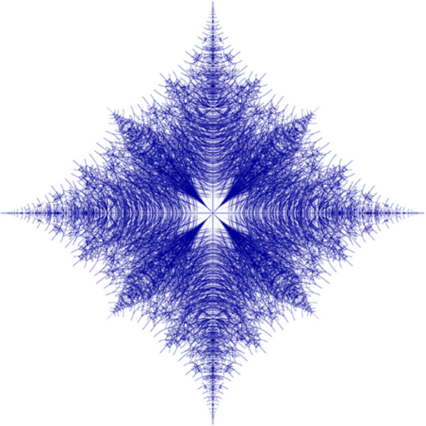


Periods 1, ..., 12

Example [Feinberg/Zee 1999]

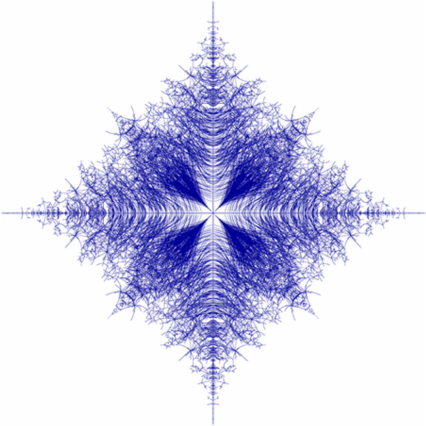


Period 13

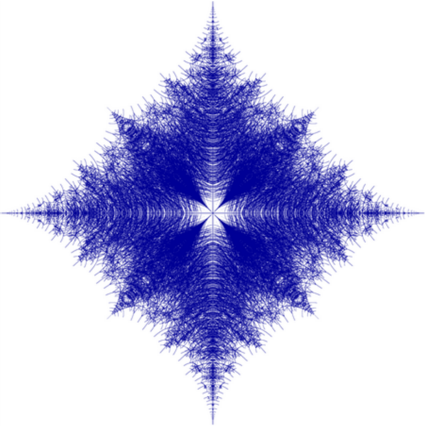


Periods 1, ..., 13

Example [Feinberg/Zee 1999]

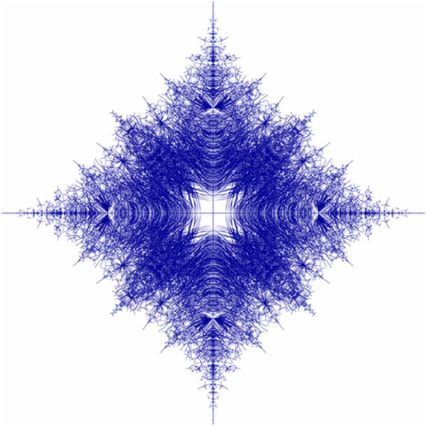


Period 14

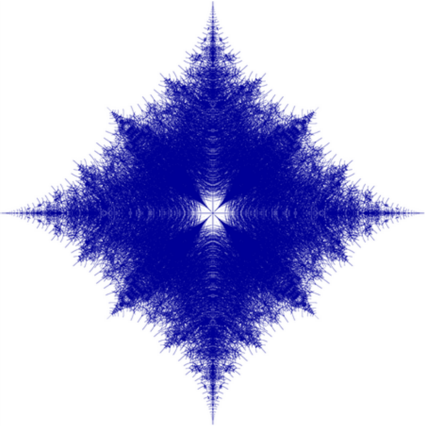


Periods 1, ..., 14

Example [Feinberg/Zee 1999]

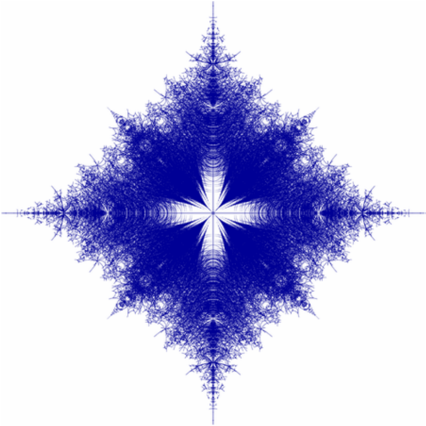


Period 15

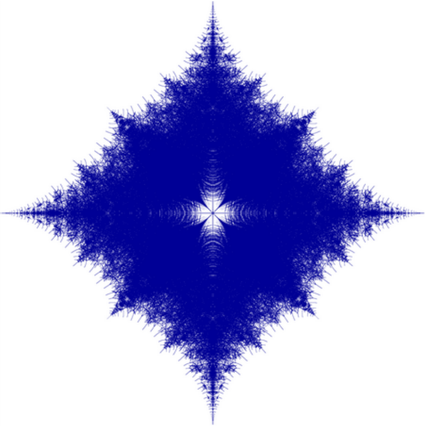


Periods $1, \ldots, 15$

Example [Feinberg/Zee 1999]

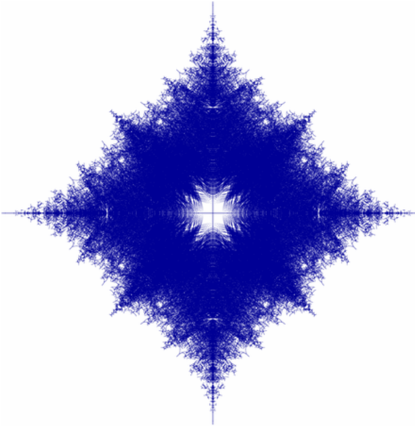


Period 16

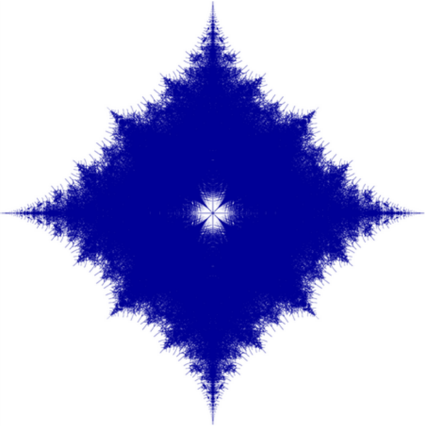


Periods 1, ..., 16

Example [Feinberg/Zee 1999]

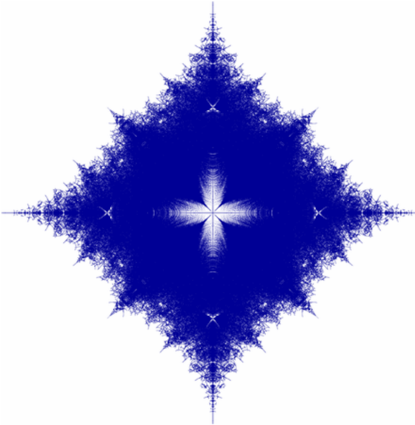


Period 17

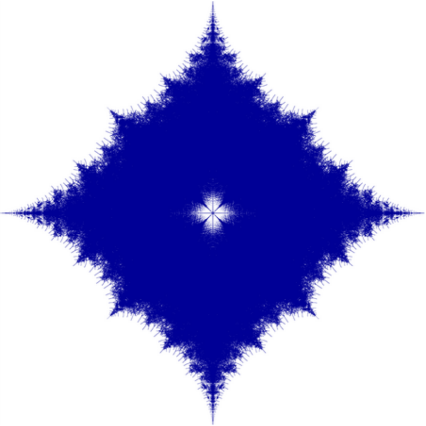


Periods 1, ..., 17

Example [Feinberg/Zee 1999]

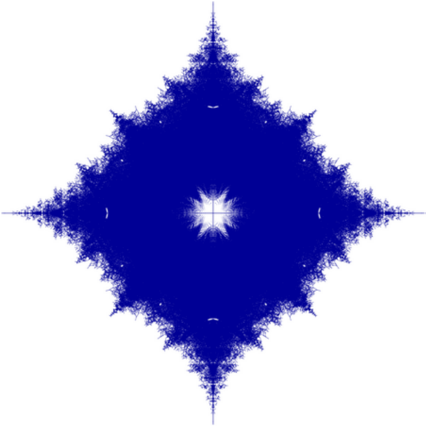


Period 18

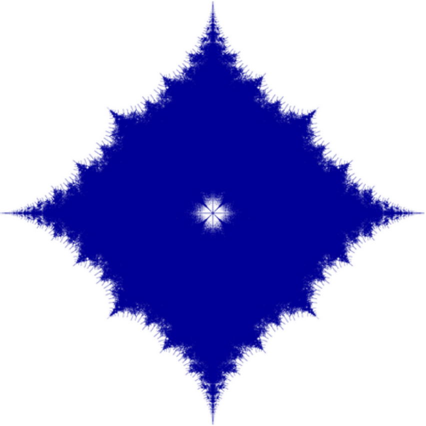


Periods 1, ..., 18

Example [Feinberg/Zee 1999]

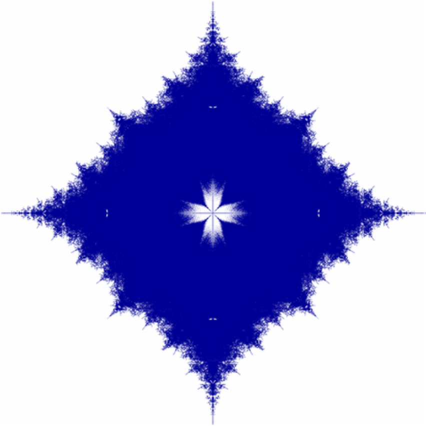


Period 19

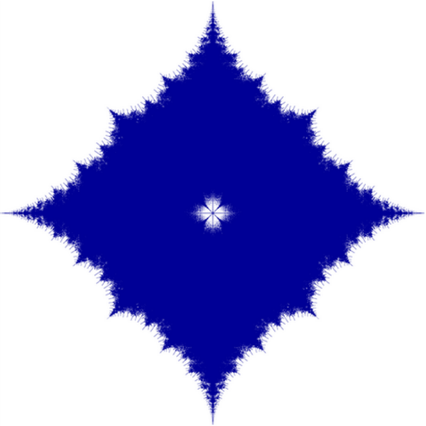


Periods 1, ..., 19

Example [Feinberg/Zee 1999]

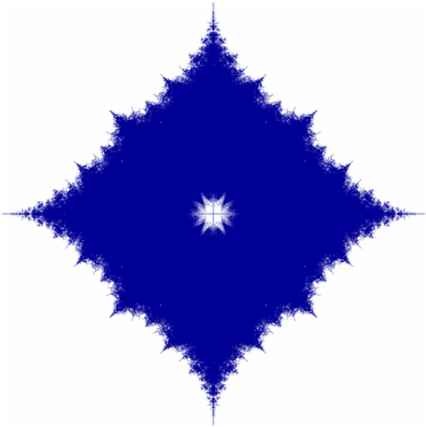


Period 20

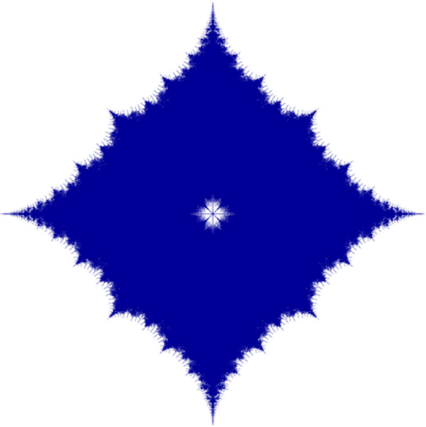


Periods 1, ..., 20

Example [Feinberg/Zee 1999]

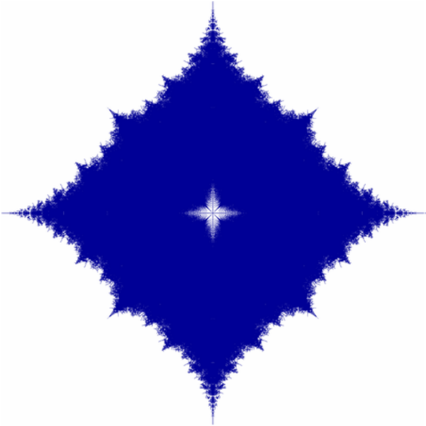


Period 21

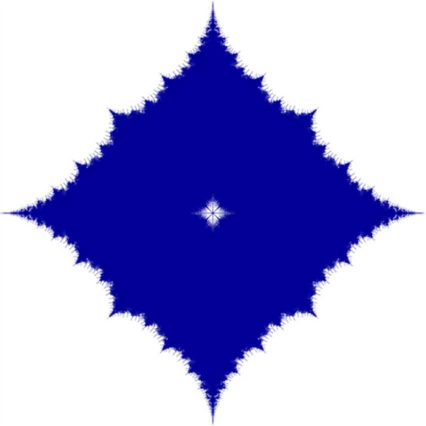


Periods 1, ..., 21

Example [Feinberg/Zee 1999]

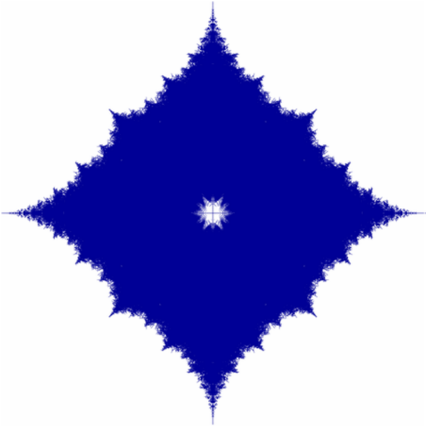


Period 22

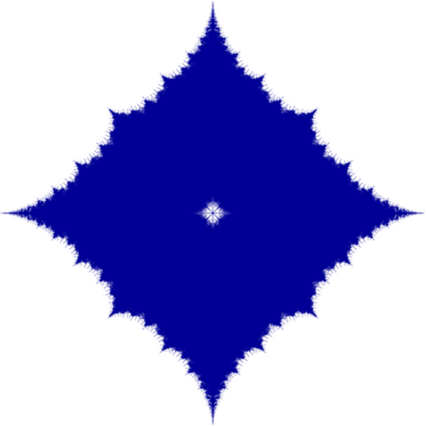


Periods 1, ..., 22

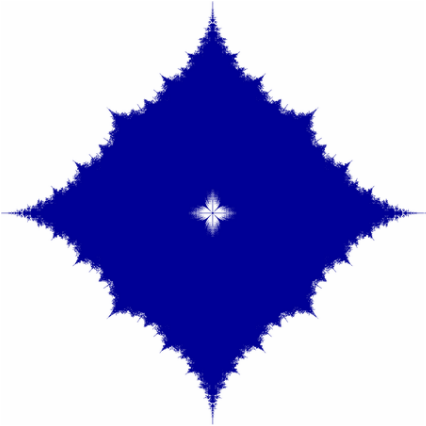
Example [Feinberg/Zee 1999]



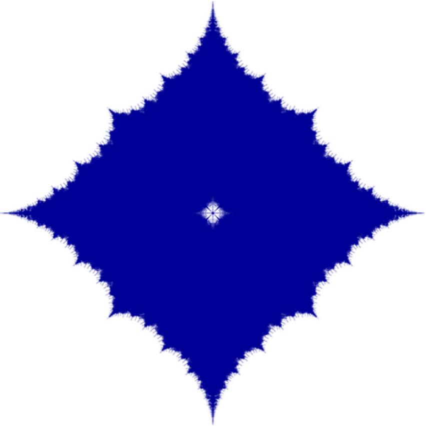
Period 23



Periods 1, ..., 23



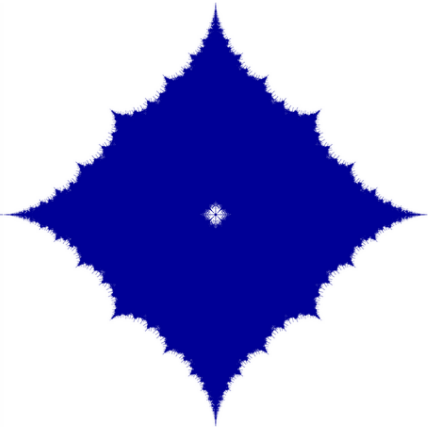
Period 24



Periods 1, ..., 24

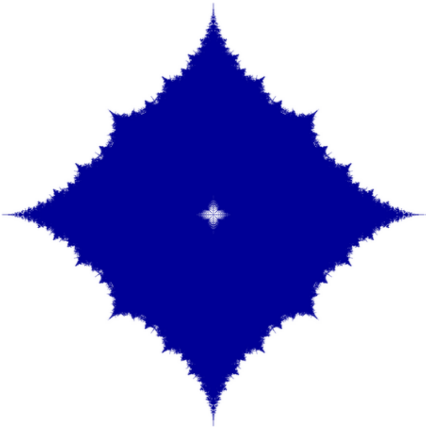
Example [Feinberg/Zee 1999]

Period 25

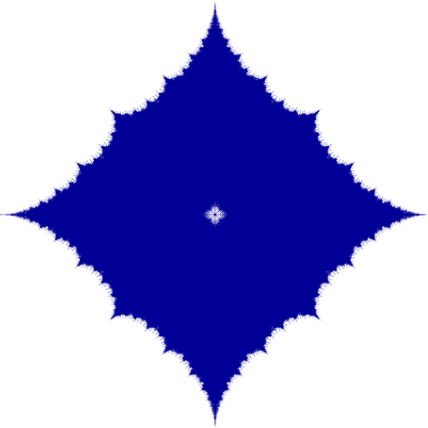


Periods 1, ..., 25

Example [Feinberg/Zee 1999]

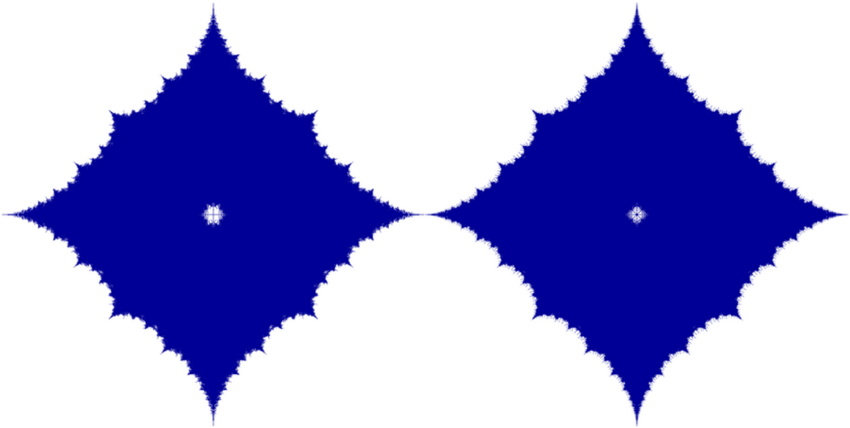


Period 26



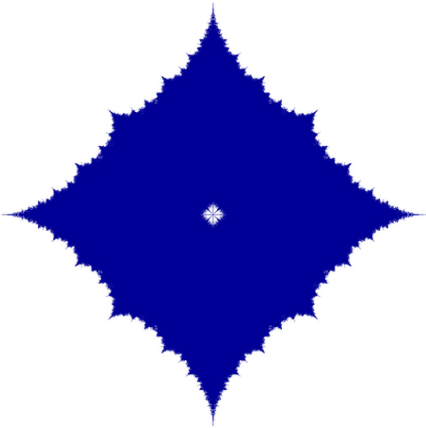
Periods 1, ..., 26

Example [Feinberg/Zee 1999]

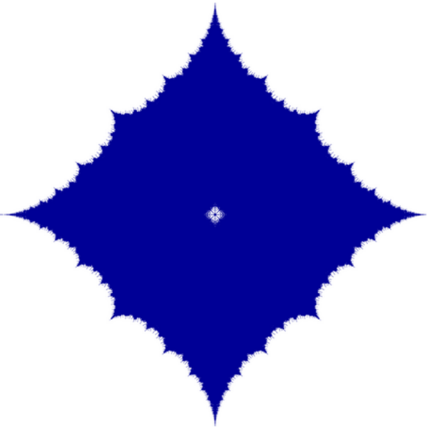


Period 27 Periods 1, ..., 27

Example [Feinberg/Zee 1999]

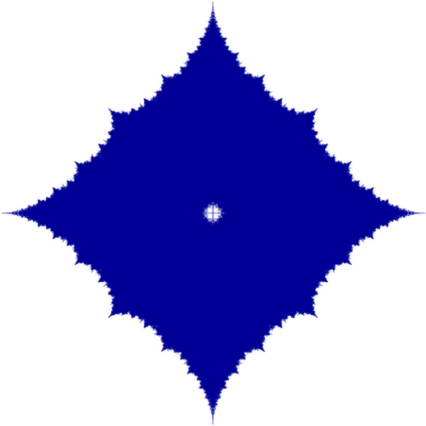


Period 28

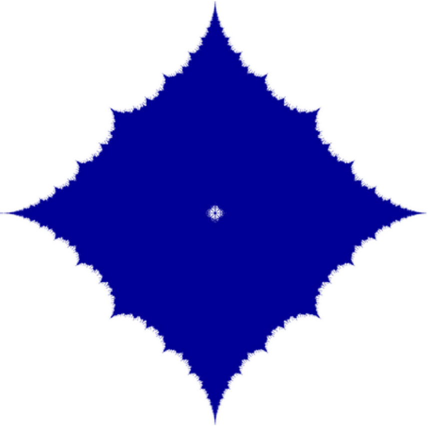


Periods 1, ..., 28

Example [Feinberg/Zee 1999]

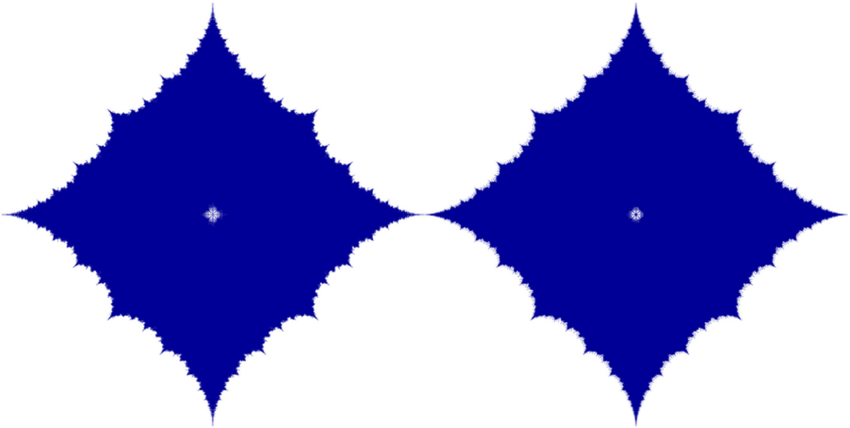


Period 29



Periods 1, ..., 29

Example [Feinberg/Zee 1999]



Period 30 Periods 1, ..., 30

Example [Feinberg/Zee 1999]

Recall our "Sandwich 1": In this example, one has

$$\underbrace{\bigcup_{k\in\mathbb{Z}} \mathrm{spec}_\varepsilon(P_{n,k}A^b P_{n,k})}_{=:\ \sigma_n^\varepsilon} \ \subset\ \mathrm{spec}_\varepsilon(A^b)\ \subset\ \underbrace{\bigcup_{k\in\mathbb{Z}} \mathrm{spec}_{\varepsilon+\varepsilon_n}(P_{n,k}A^b P_{n,k})}_{\Sigma_n^\varepsilon\ :=\ \sigma_n^{\varepsilon+\varepsilon_n}}$$

for all $n \in \mathbb{N}$, so let's look at σ_n^ε for $\varepsilon = 0$.

Here are the $n \times n$ matrix eigenvalues

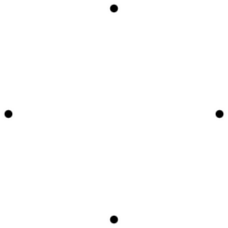
$$\sigma_n^0 \ = \ \bigcup_{k\in\mathbb{Z}} \mathrm{spec}\,(P_{n,k}A^b P_{n,k})$$

for $n = 1, ..., 30$:

Example [Feinberg/Zee 1999]

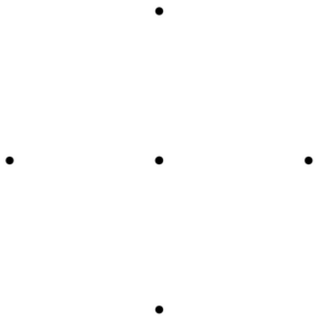
Size 1

Example [Feinberg/Zee 1999]



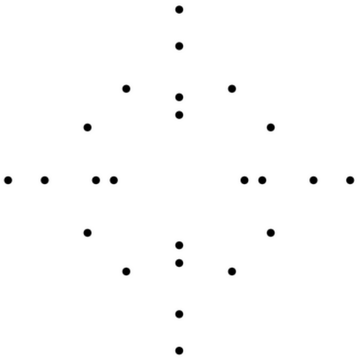
Size 2

Example [Feinberg/Zee 1999]



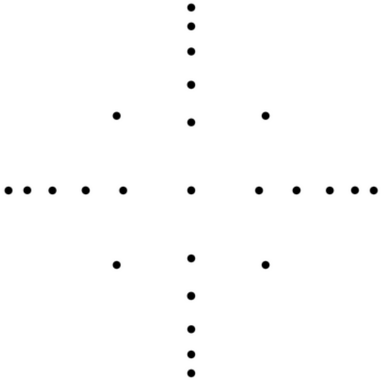
Size 3

Example [Feinberg/Zee 1999]



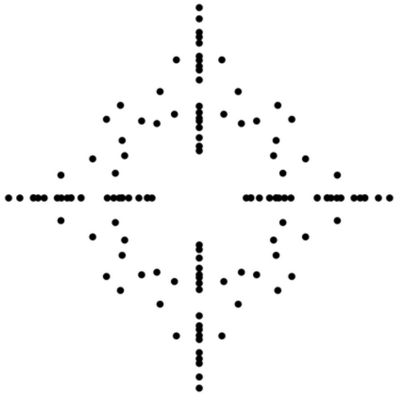
Size 4

Example [Feinberg/Zee 1999]



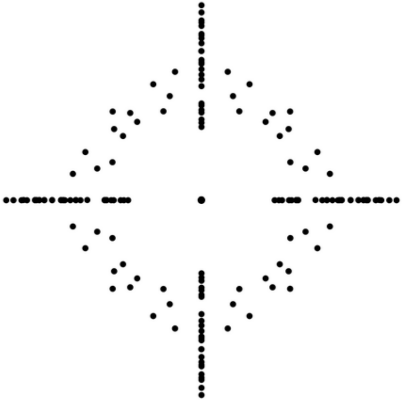
Size 5

Example [Feinberg/Zee 1999]



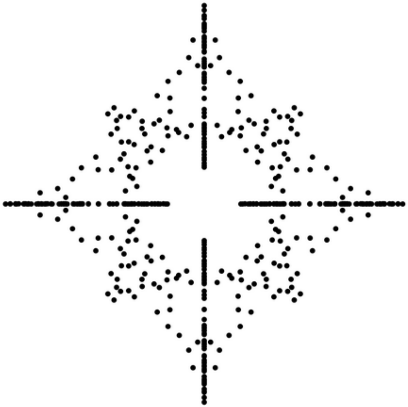
Size 6

Example [Feinberg/Zee 1999]



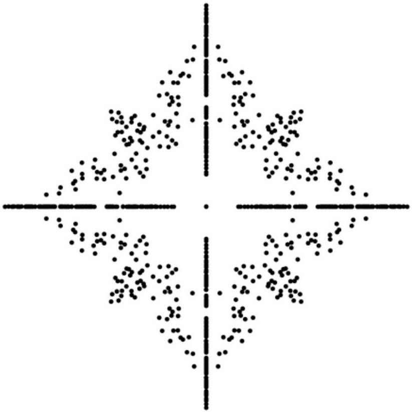
Size 7

Example [Feinberg/Zee 1999]



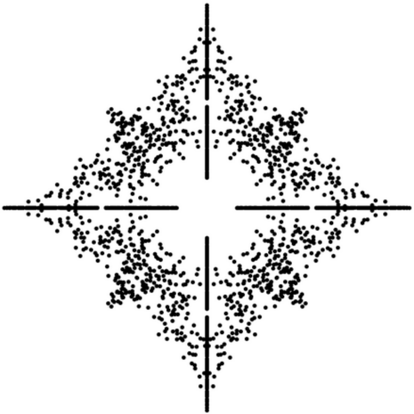
Size 8

Example [Feinberg/Zee 1999]



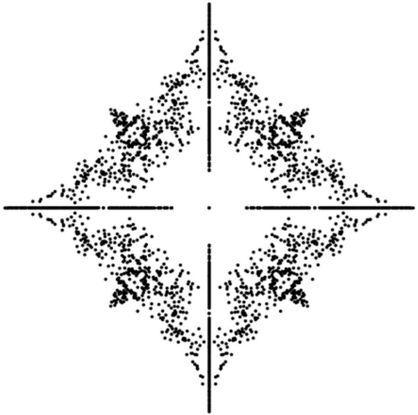
Size 9

Example [Feinberg/Zee 1999]



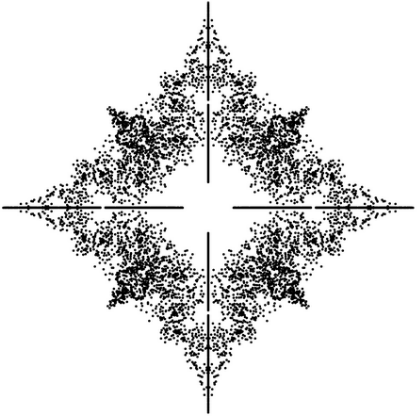
Size 10

Example [Feinberg/Zee 1999]



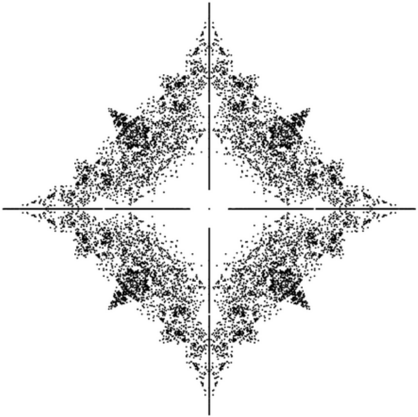
Size 11

Example [Feinberg/Zee 1999]



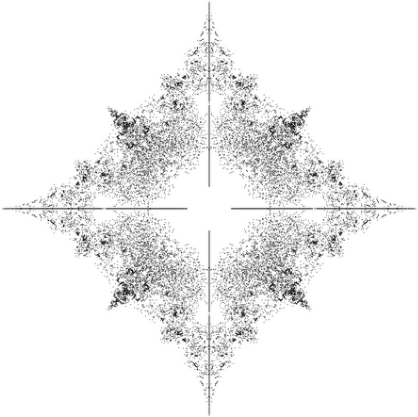
Size 12

Example [Feinberg/Zee 1999]



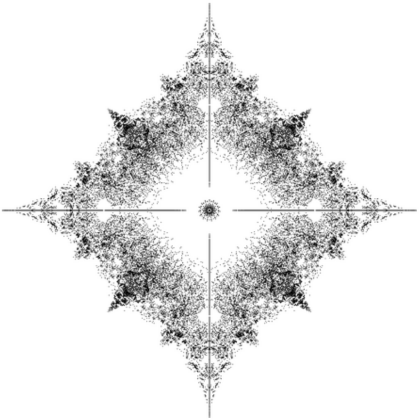
Size 13

Example [Feinberg/Zee 1999]



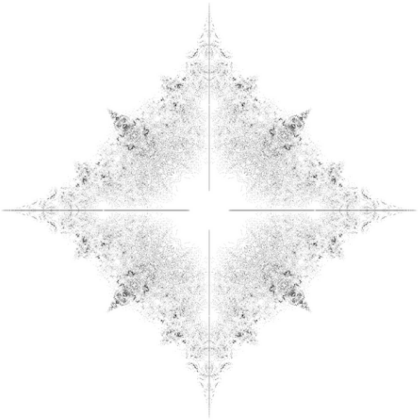
Size 14

Example [Feinberg/Zee 1999]



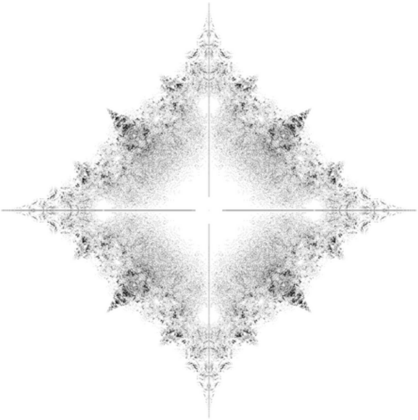
Size 15

Example [Feinberg/Zee 1999]



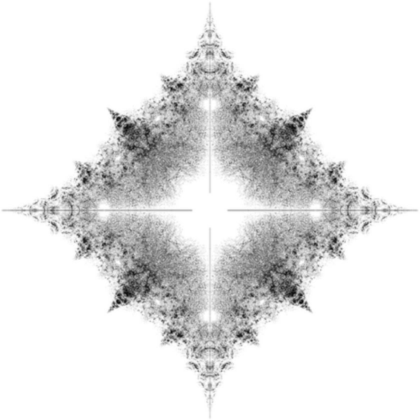
Size 16

Example [Feinberg/Zee 1999]



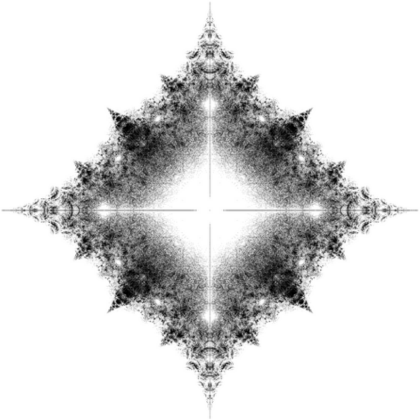
Size 17

Example [Feinberg/Zee 1999]



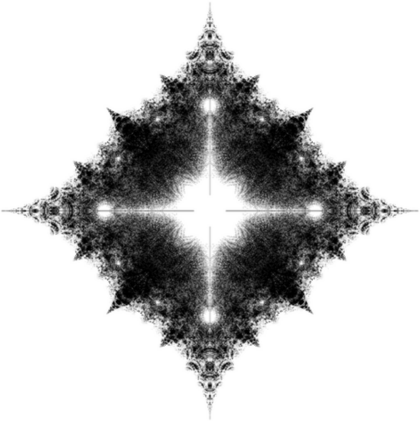
Size 18

Example [Feinberg/Zee 1999]



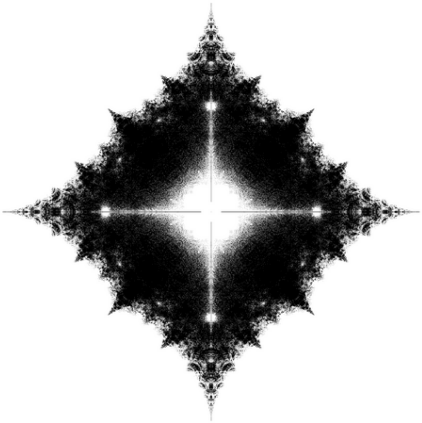
Size 19

Example [Feinberg/Zee 1999]



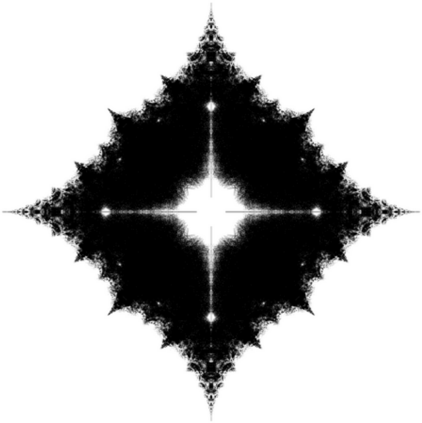
Size 20

Example [Feinberg/Zee 1999]



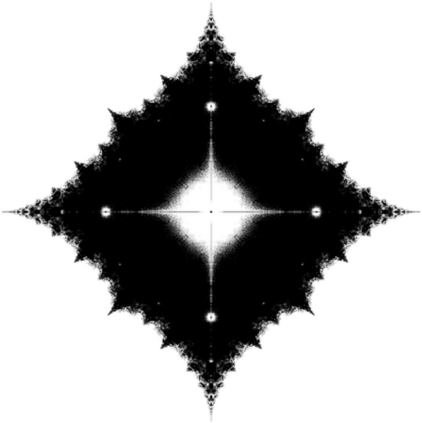
Size 21

Example [Feinberg/Zee 1999]



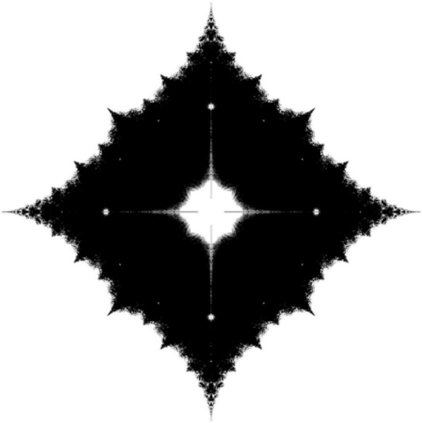
Size 22

Example [Feinberg/Zee 1999]



Size 23

Example [Feinberg/Zee 1999]



Size 24

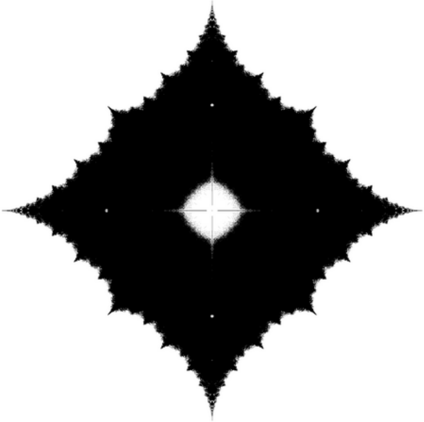
Example [Feinberg/Zee 1999]

Size 25

Example [Feinberg/Zee 1999]

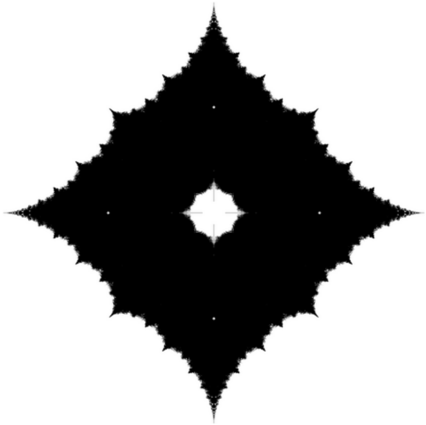
Size 26

Example [Feinberg/Zee 1999]



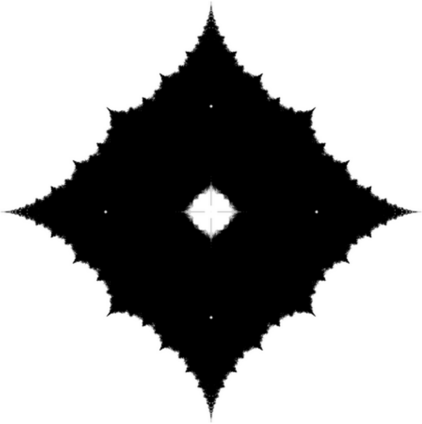
Size 27

Example [Feinberg/Zee 1999]



Size 28

Example [Feinberg/Zee 1999]

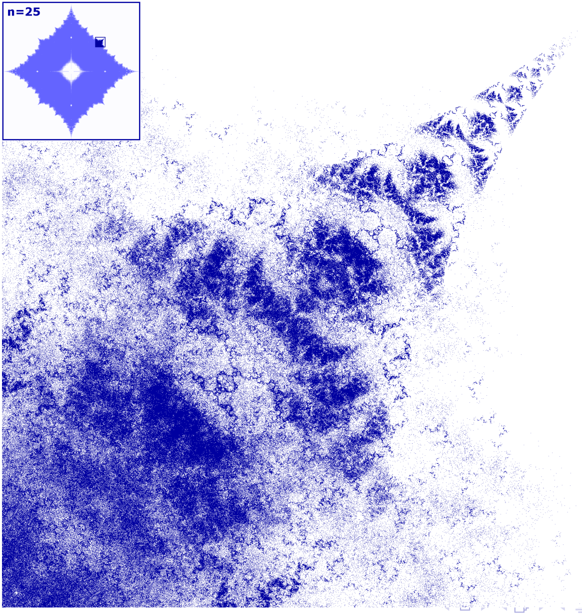


Size 29

Example [Feinberg/Zee 1999]

Size 30

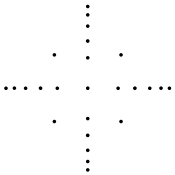
Zoom into Region $1 + i$ of σ_{25}^{0}



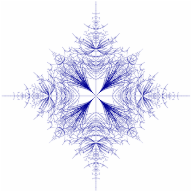
Example [Feinberg/Zee 1999]

The **finite** matrix spectra σ_n^0 are even **contained** in the **periodic** (infinite) matrix spectra shown before.

More precisely, the spectra of all $n \times n$ principal submatrices are **contained** in the set of all (2n+2)-periodic matrices:



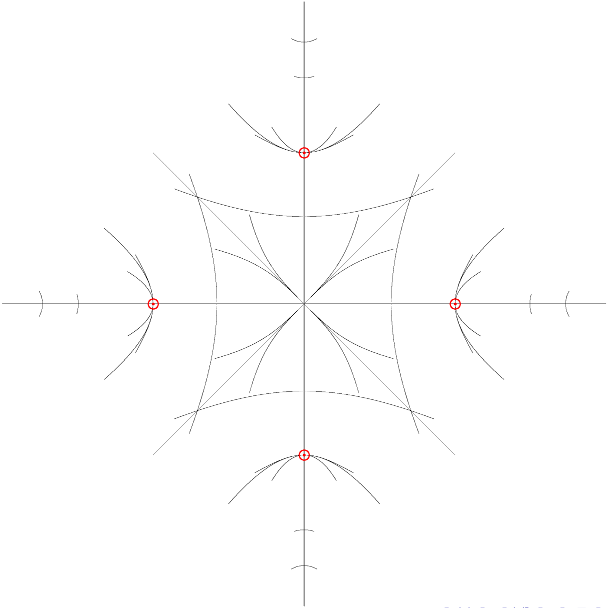
$\subset$



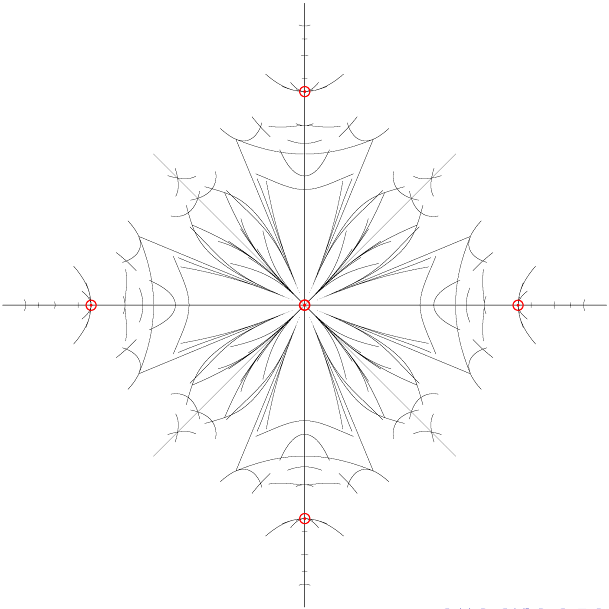
Size $n = 5$ Period $2n + 2 = 12$

Here we demonstrate this inclusion for some values of n.

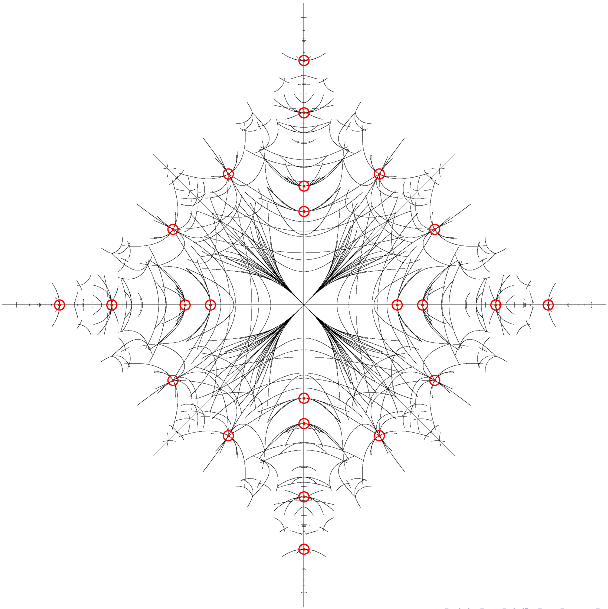
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 2$



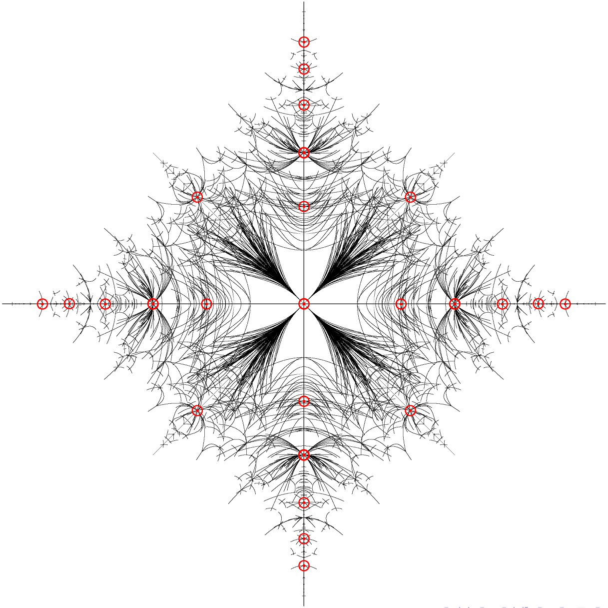
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 3$



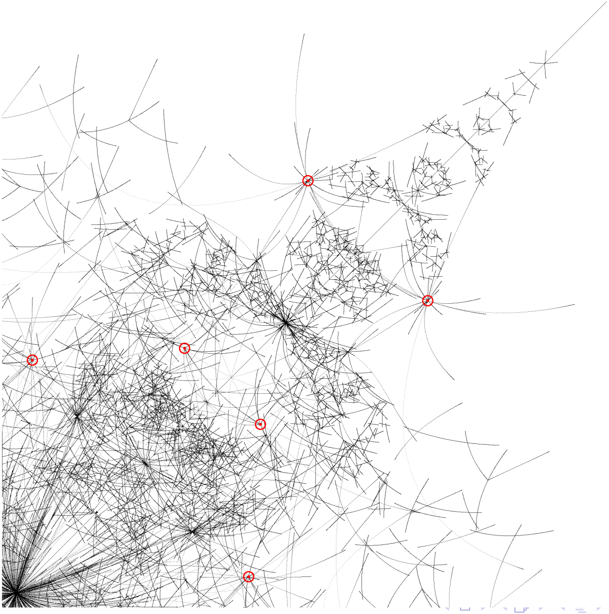
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 4$



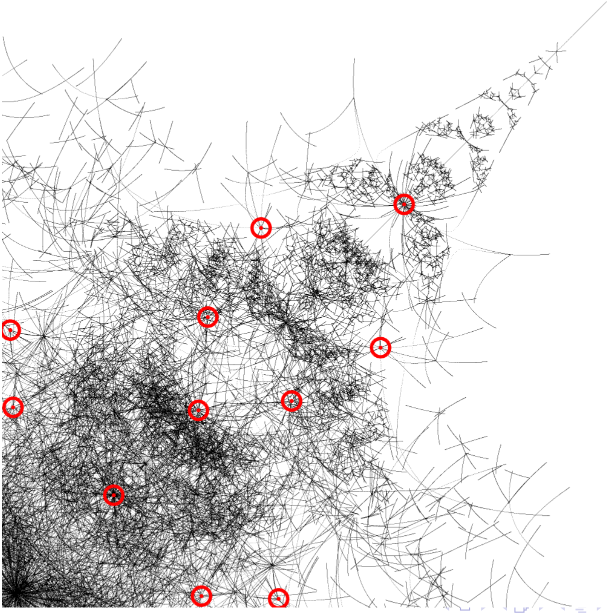
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 5$



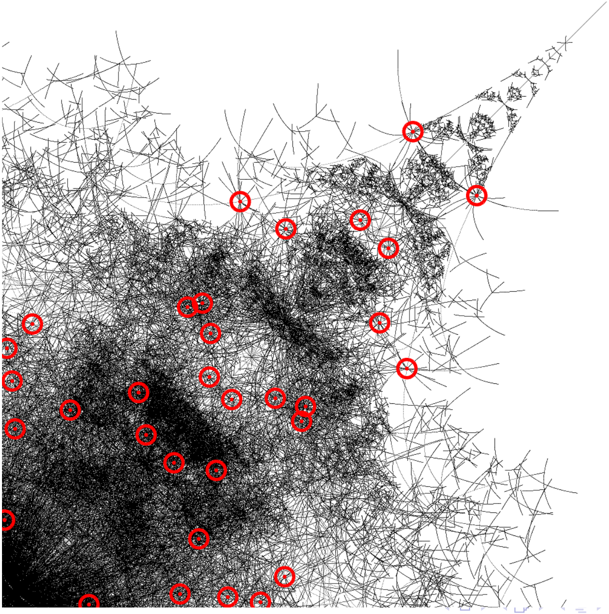
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 8$



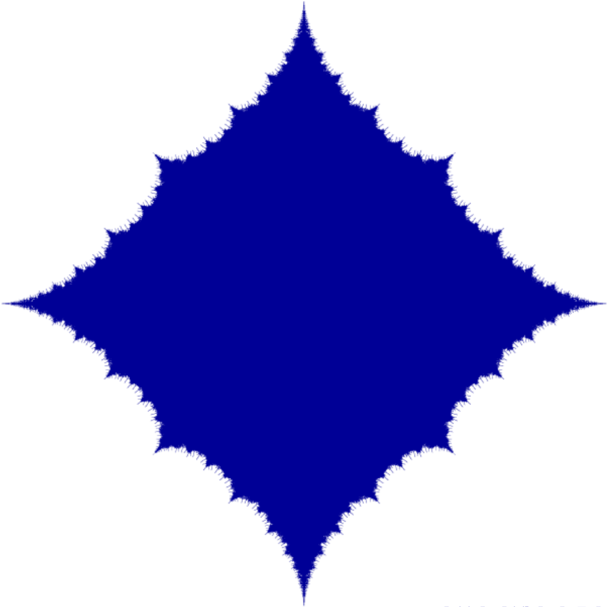
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 9$



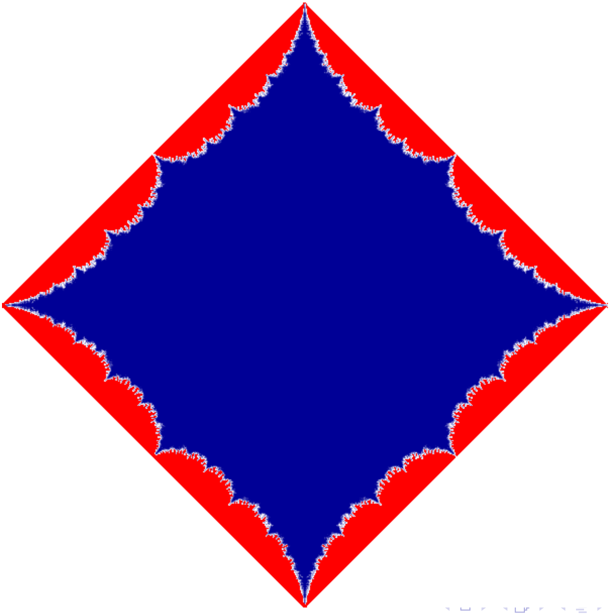
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 10$



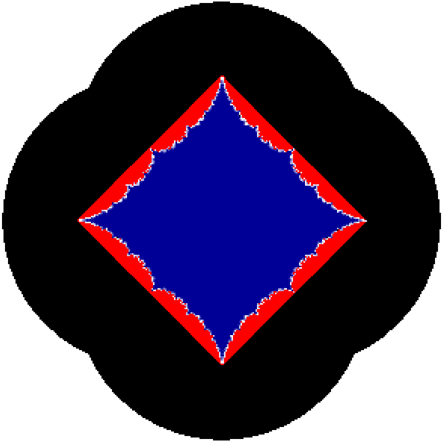
Conjecture: spec A^b if b is pseudoergodic



Upper bound on spec A^b by the closed numerical range

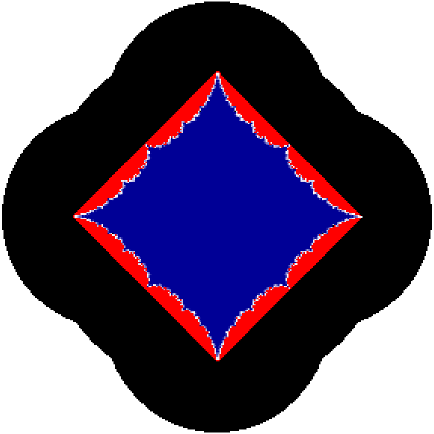


...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$



$n = 2$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

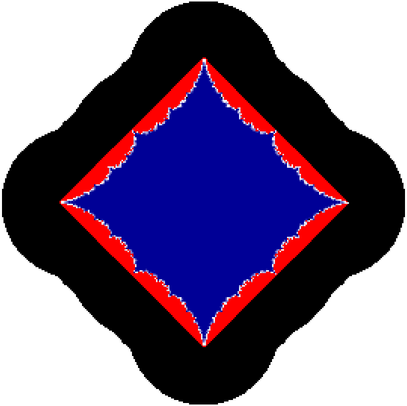


$n = 3$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

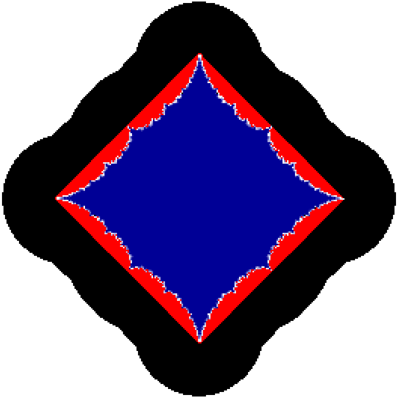
$n = 4$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$



$n = 5$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$



$n = 6$

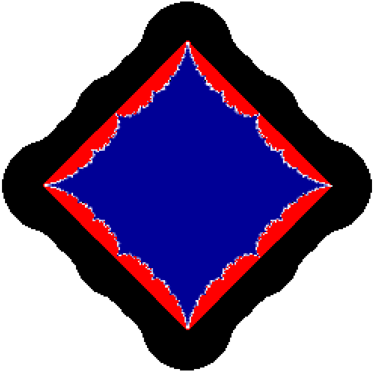
...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 7$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

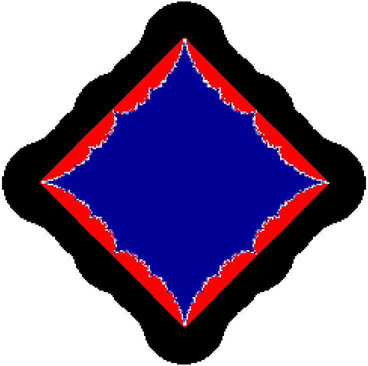
$n = 8$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$



$n = 9$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$



$n = 10$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 11$

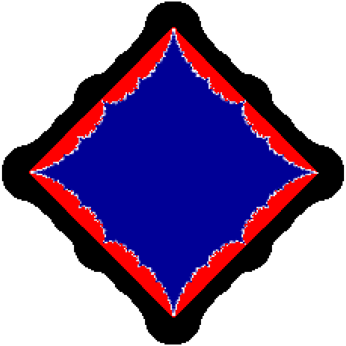
$n = 12$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 13$

$n = 14$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$



$n = 15$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 16$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 17$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 18$

Where does Σ_n^0 go as $n \to \infty$?

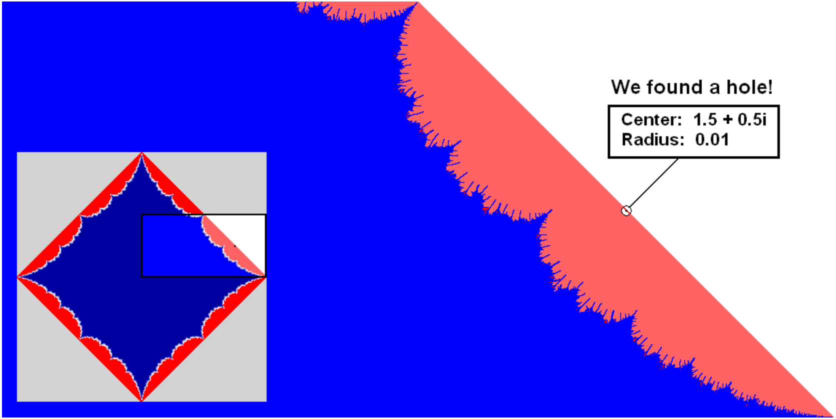
Computational cost for these pics: $n \cdot 2^{n-1} \times$ number of pixels.

Where does Σ_n^0 go as $n \to \infty$?

Computational cost for these pics: $n \cdot 2^{n-1} \times$ number of pixels.
So let us focus on just **one** point (pixel) λ:

Where does Σ_n^0 go as $n \to \infty$?

Computational cost for these pics: $n \cdot 2^{n-1} \times$ number of pixels.
So let us focus on just **one** point (pixel) λ:



$$\lambda = 1.5 + 0.5i \;\notin\; \Sigma_{36}^0 \;\supset\; \operatorname{spec} A^b$$

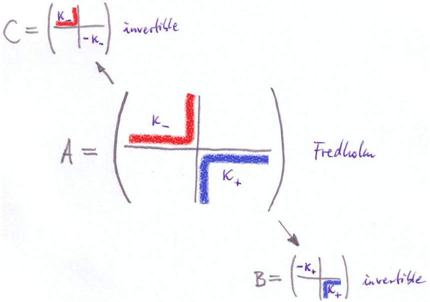
Now we come back to the FSM

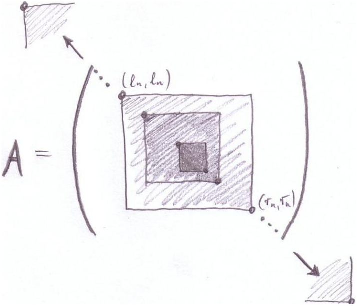
and bring in our knowledge

on Fredholm indices.

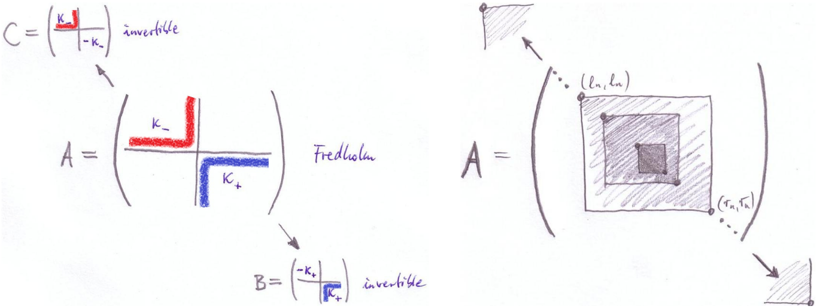
The FSM and the Fredholm index

Recall the following two facts in the **bi-infinite case**:





Recall the following two facts in the **bi-infinite case**:



So ind $A_+ = 0 =$ ind A_- is **necessary** for applicability of the FSM!

The FSM and the Fredholm index

Abbreviate $\operatorname{ind} A_+ =: \kappa_+$ and $\operatorname{ind} A_- =: \kappa_-$.

$\kappa_+ = 0 = \kappa_-$ is **necessary** for applicability of the FSM!

Otherwise: **Shift** the system up or down accordingly, i.e. place the corners of your finite sections A_n on another (the κ_-th) diagonal!

The FSM and the Fredholm index

Abbreviate ind $A_+ =: \kappa_+$ and ind $A_- =: \kappa_-$.

$\kappa_+ = 0 = \kappa_-$ is **necessary** for applicability of the FSM!

Otherwise: **Shift** the system up or down accordingly, i.e. place the corners of your finite sections A_n on another (the $\kappa_-{}^{\text{th}}$) diagonal!

This means: Replace $Ax = b$ by $V_{\kappa_+} Ax = V_{\kappa_+} b$.
Reason: The new system has plus-index zero:

$$\text{ind}\,(V_{\kappa_+} A)_+ \;=\; \text{ind}\,(V_{\kappa_+})_+ + \text{ind}\, A_+ \;=\; -\kappa_+ + \kappa_+ \;=\; 0$$

This process is called **index cancellation**.

$\Rightarrow$ We have found the (from the FSM perspective) "proper" main diagonal of A!

Example: FSM for slowly oscillating operators

Suppose $A \in BDO(E)$ has slowly oscillating diagonals.
We want to solve $Ax = b$ by the FSM.

Assumption (minimal): A be invertible.

Step 1: Compute the plus-index $\kappa_+ := \operatorname{ind} A_+$.

Example: FSM for slowly oscillating operators

Suppose $A \in BDO(E)$ has slowly oscillating diagonals.
We want to solve $Ax = b$ by the FSM.

Assumption (minimal): A be invertible.

Step 1: Compute the plus-index $\kappa_+ := \operatorname{ind} A_+$.

Therefore, take an arbitrary limop B of A at $+\infty$ and recall that $\operatorname{ind} B_+ = \operatorname{ind} A_+$.

A is slowly oscillating $\Rightarrow$ B_+ is Toeplitz $\Rightarrow$ $\kappa_+ = -\operatorname{wind}(a(\mathbb{T}), 0)$

Example: FSM for slowly oscillating operators

Suppose $A \in BDO(E)$ has slowly oscillating diagonals.
We want to solve $Ax = b$ by the FSM.

Assumption (minimal): A be invertible.

Step 1: Compute the plus-index $\kappa_+ := \operatorname{ind} A_+$.

Step 2: Perform index cancellation (i.e. shift down by κ_+ rows).

Example: FSM for slowly oscillating operators

Suppose $A \in BDO(E)$ has slowly oscillating diagonals.
We want to solve $Ax = b$ by the FSM.

Assumption (minimal): A be invertible.

Step 1: Compute the plus-index $\kappa_+ := \operatorname{ind} A_+$.

Step 2: Perform index cancellation (i.e. shift down by κ_+ rows).

Step 3: Truncate.
Remarkable fact: We can truncate at **arbitrary** points l_n and r_n!
Reason: All limops B and C (w.r.t. subsequences of r and l, resp) are Laurent operators. So all B_- and all C_+ are Toeplitz operators that are Fredholm of index zero (after index cancellation).

Suppose $A \in BDO(E)$ has slowly oscillating diagonals.
We want to solve $Ax = b$ by the FSM.

Assumption (minimal): A be invertible.

Step 1: Compute the plus-index $\kappa_+ := \operatorname{ind} A_+$.

Step 2: Perform index cancellation (i.e. shift down by κ_+ rows).

Step 3: Truncate.
Remarkable fact: We can truncate at **arbitrary** points l_n and r_n!
Reason: All limops B and C (w.r.t. subsequences of r and l, resp) are Laurent operators. So all B_- and all C_+ are Toeplitz operators that are Fredholm of index zero (after index cancellation).

Coburn's lemma $\Rightarrow$ all B_- and all C_+ are invertible, as well as A.

FSM theorem $\Rightarrow$ The FSM applies.

Final example: Back to our random Jacobi operator

$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & v_{-2} & w_{-1} & & & & \\ & u_{-2} & v_{-1} & w_0 & & & \\ & & u_{-1} & v_0 & w_1 & & \\ & & & u_0 & v_1 & w_2 & \\ & & & & u_1 & v_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}$$

with iid entries $u_i \in U$, $v_i \in V$ and $w_i \in W$.

Assumption (minimal): A is invertible.

How do we truncate A to get an applicable FSM?

Assumption (minimal): $A \in \Psi E(U, V, W)$ is invertible.
How do we truncate A to get an applicable FSM?

Step 1. Compute $\kappa_+ := \text{ind}\, A_+$

Assumption (minimal): $A \in \Psi E(U, V, W)$ is invertible.
How do we truncate A to get an applicable FSM?

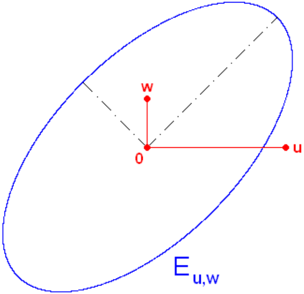
Step 1. Compute $\kappa_+ := \operatorname{ind} A_+$

We know that $\operatorname{ind} A_+ = \operatorname{ind} B_+$ for all limops B of A at $+\infty$.
Let's take a Laurent operator B. So pick **arbitrary** $u \in U$, $v \in V$ and $w \in W$. Then $\kappa_+ = \operatorname{ind} A_+ = \operatorname{ind} B_+ = -\operatorname{wind}(E_{u,w}, v)$.

Assumption (minimal): $A \in \Psi E(U, V, W)$ is invertible.
How do we truncate A to get an applicable FSM?

Step 1. Compute $\kappa_+ := \text{ind}\, A_+$

We know that $\text{ind}\, A_+ = \text{ind}\, B_+$ for all limops B of A at $+\infty$. Let's take a Laurent operator B. So pick **arbitrary** $u \in U$, $v \in V$ and $w \in W$. Then $\kappa_+ = \text{ind}\, A_+ = \text{ind}\, B_+ = -\text{wind}(E_{u,w}, v)$.



It's very simple:

- if v is outside $E_{u,w}$: $\kappa_+ = 0$
- if v is inside $E_{u,w}$: $\kappa_+ = \pm 1$
    - if $|u| > |w|$: $\kappa_+ = -1$
    - if $|u| < |w|$: $\kappa_+ = +1$

Note: The result κ_+ is independent of $u \in U$, $v \in V$, $w \in W$!

Step 1. Compute $\kappa_+ := \text{ind}\, A_+$

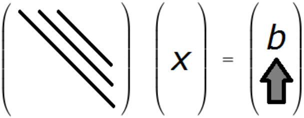
Step 2. Perform index cancellation.

Finite Sections of Random Jacobi Operators

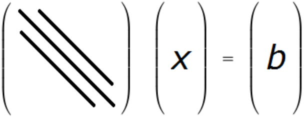
Step 1. Compute $\kappa_+ := \operatorname{ind} A_+$

Step 2. Perform index cancellation.

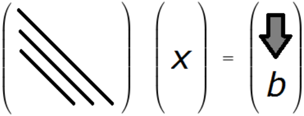
- $\kappa_+ = -1$: shift **up**



- $\kappa_+ = 0$: leave as it is



- $\kappa_+ = +1$: shift **down**



In either case, the new system $\tilde{A}x = \tilde{b}$ has $\operatorname{ind} \tilde{A}_+ = 0 = \operatorname{ind} \tilde{A}_-$.

Step 3. Do the truncations.

Choose the truncation points $\cdots < l_2 < l_1 < r_1 < r_2 < \cdots$ so that

$$\begin{pmatrix} v_{l_n} & w_{l_n+1} & \\ u_{l_n} & \ddots & \ddots \\ & \ddots & \ddots \end{pmatrix} \longrightarrow \begin{pmatrix} v & w & \\ u & \ddots & \ddots \\ & \ddots & \ddots \end{pmatrix} =: C_+$$

and
$$\begin{pmatrix} \ddots & \ddots & \\ \ddots & \ddots & w_{r_n} \\ & u_{r_n-1} & v_{r_n} \end{pmatrix} \longrightarrow \begin{pmatrix} \ddots & \ddots & \\ \ddots & \ddots & w \\ & u & v \end{pmatrix} =: B_-$$

as $n \to \infty$, for some fixed $u \in U$, $v \in V$ and $w \in W$.

Step 3. Do the truncations.

Choose the truncation points $\cdots < l_2 < l_1 < r_1 < r_2 < \cdots$ so that

$$\begin{pmatrix} v_{l_n} & w_{l_n+1} & \\ u_{l_n} & \ddots & \ddots \\ & \ddots & \ddots \end{pmatrix} \longrightarrow \begin{pmatrix} v & w & \\ u & \ddots & \ddots \\ & \ddots & \ddots \end{pmatrix} =: \; C_+$$

and
$$\begin{pmatrix} \ddots & \ddots & \\ \ddots & \ddots & w_{r_n} \\ & u_{r_n-1} & v_{r_n} \end{pmatrix} \longrightarrow \begin{pmatrix} \ddots & \ddots & \\ \ddots & \ddots & w \\ & u & v \end{pmatrix} =: \; B_-$$

as $n \to \infty$, for some fixed $u \in U$, $v \in V$ and $w \in W$.

Both **Toeplitz** operators C_+ and B_- are Fredholm of index 0 (because $\operatorname{ind} A_+ = 0 = \operatorname{ind} A_-$) so they are **invertible** (Coburn).

But how about the 'full' FSM?

The previous truncation pattern was specially adapted to the operator $A \in \Psi E(U, V, W)$ at hand.

The **full (or usual) FSM** uses the cut-off sequences $l = (-1, -2, ...)$ and $r = (1, 2, ...)$.

Theorem ML, Roch 2011

For $A \in \Psi E(U, V, W)$ we have the following results

But how about the 'full' FSM?

The previous truncation pattern was specially adapted to the operator $A \in \Psi E(U, V, W)$ at hand.

The **full (or usual) FSM** uses the cut-off sequences $l = (-1, -2, ...)$ and $r = (1, 2, ...)$.

Theorem ML, Roch 2011

For $A \in \Psi E(U, V, W)$ we have the following results

1. The following are equivalent:
   - the full FSM is applicable to A,
   - the full FSM is applicable to **all** $B \in J(U, V, W)$,
   - **all** operators B_+ with $B \in J(U, V, W)$ are invertible.

But how about the 'full' FSM?

The previous truncation pattern was specially adapted to the operator $A \in \Psi E(U, V, W)$ at hand.

The **full (or usual) FSM** uses the cut-off sequences $l = (-1, -2, ...)$ and $r = (1, 2, ...)$.

Theorem ML, Roch 2011

For $A \in \Psi E(U, V, W)$ we have the following results

1. The following are equivalent:
   - the full FSM is applicable to A,
   - the full FSM is applicable to **all** $B \in J(U, V, W)$,
   - **all** operators B_+ with $B \in J(U, V, W)$ are invertible.
2. If $0 \in U, W$ and A is invertible then the full FSM is applicable.

But how about the 'full' FSM?

The previous truncation pattern was specially adapted to the operator $A \in \Psi E(U, V, W)$ at hand.

The **full (or usual) FSM** uses the cut-off sequences $l = (-1, -2, \ldots)$ and $r = (1, 2, \ldots)$.

Theorem ML, Roch 2011

For $A \in \Psi E(U, V, W)$ we have the following results

1. The following are equivalent:
   - the full FSM is applicable to A,
   - the full FSM is applicable to **all** $B \in J(U, V, W)$,
   - **all** operators B_+ with $B \in J(U, V, W)$ are invertible.
2. If $0 \in U, W$ and A is invertible then the full FSM is applicable.
3. If $\kappa_+ := \operatorname{ind} A_+ = \pm 1$ and A is invertible then, after index cancellation, the full FSM is applicable.

Thank you!

Literature

V.S. Rabinovich, S. Roch and B. Silbermann:
Limit Operators and Applications in Operator Theory,
Birkhäuser 2004.

M. Lindner:
Infinite Matrices and their Finite Sections,
Birkhäuser 2006.

S. Roch:
Finite Sections of Band-Dominated Operators,
Volume 191 (Number 895) of Memoirs of the AMS, 2008.

S.N. Chandler-Wilde and M. Lindner:
Limit Operators, Collective Compactness & Spectral Theory,
Volume 210 (Number 989) of Memoirs of the AMS, 2011.

M. Lindner and S. Roch:
Finite sections of random Jacobi operators,
to appear in *SIAM J Numerical Analysis*