Structured Rank Matrices
Lecture 1:
What are they?

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Dept. of Computer Science, K.U.Leuven, Belgium
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Contents

Outline

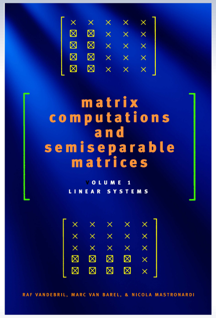
Cooperations

Many of the results described in this lecture series were derived in cooperation with:

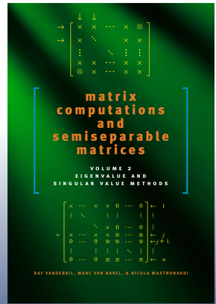
- Bernd Beckermann,
- Gianna Del Corso,
- Steven Delvaux,
- Dario Fasino,
- Katrijn Frederix,
- Luca Gemignani,
- Stefan Güttel,
- Nicola Mastronardi,
- Yvette Vanberghen,
- Ellen Van Camp,
- Paul Van Dooren,
- David Watkins,
- and many others.

Extra information,

but not all, since these lectures will contain new developments



- Vandebril R., Van Barel M. and Mastronardi N.,
  Matrix Computations & Semiseparable Matrices I: Linear Systems,
  The Johns Hopkins University Press, Baltimore, December 2007 (xviii+575 pp).



- Vandebril R., Van Barel M. and Mastronardi N.,
  Matrix Computations & Semiseparable Matrices II: Eigenvalue and Singular Value Methods,
  The Johns Hopkins University Press, Baltimore,
  December 2008 (xvi+498 pp).

Motto

Quote from Gauss

Theory attracts practice as the magnet attracts iron.

- Many examples.
- Matlab demos illustrating the theoretical results.
- If something is not clear, please ask!

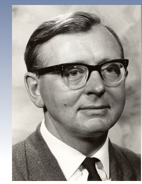
Outline

Lecture series contents

1. Introduction: structured rank matrices
2. Structure transfer via inversion and factorizations
3. The interplay of rotations and the *QR*-factorization
4. Similarity transformation to semiseparable form and convergence theory
5. Novel similarity transformations
6. The connection with orthogonal rational functions
7. Computing eigenvalues of a companion matrix
8. Orthogonal functions and matrix computations

Outline

Sparse and dense matrices



Wilkinson defined a sparse matrix as

"any matrix with enough zeros that it pays to take advantage of them."

- The other matrices are dense.
- Example of a sparse and dense matrix

$$\begin{bmatrix} 1 & 2 & & & \\ 3 & 2 & 1 & & \\ & 5 & 2 & 3 & \\ & & 3 & 8 & 10 \\ & & & 9 & 1 \end{bmatrix} \text{ versus } \begin{bmatrix} 1 & 6 & 3 & 9 & 12 \\ 4 & 2 & 1 & 3 & 4 \\ 6 & 3 & \frac{3}{2} & 3 & 6 \\ 2 & 1 & \frac{1}{2} & 8 & 10 \\ 8 & 4 & 2 & 9 & 1 \end{bmatrix}.$$

- Sparse matrices are and have been an interesting topic for many years. They are easily representable and they occur frequently in practice. (Discretization of ODE's, PDE's.)
- Structure is readily available.

Dense does not mean unstructured

Almost 'trivial' example

The red block has rank 1.

The underlined block is of rank 1.

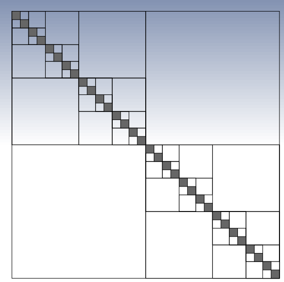
$$\begin{bmatrix} 1 & \underline{6} & \underline{3} & \underline{9} & \underline{12} \\ 4 & \underline{2} & \underline{1} & \underline{3} & \underline{4} \\ 6 & 3 & \frac{3}{2} & 3 & 6 \\ 2 & 1 & \frac{1}{2} & 8 & 10 \\ 8 & 4 & 2 & 9 & 1 \end{bmatrix}$$

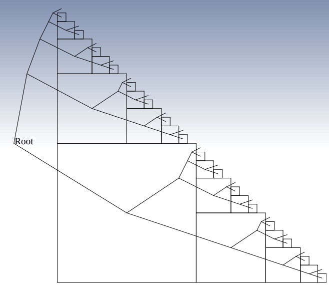
Structured rank matrix

"any matrix with enough 'low' rank blocks that it pays to take advantage of them."

- In a certain sense this is a natural extension of sparse matrices. (A block of zeros has rank 0.)
- Problem: the structure is sort of hidden in the matrix.

More sophisticated examples





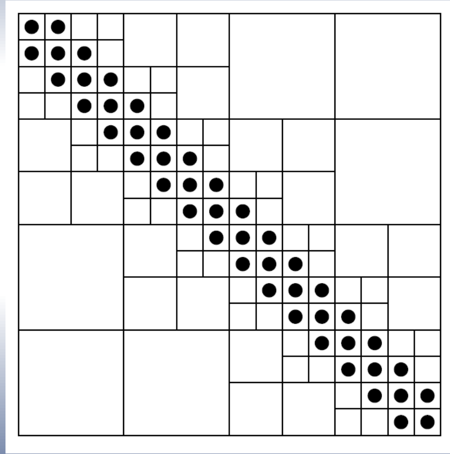
What is a hierarchical matrix? (from http://www.hlib.org)

Hierarchical matrices (or short $\mathcal{H}$-matrices) are efficient data-sparse representations of certain densely populated matrices. The basic idea is to split a given matrix into a hierarchy of rectangular blocks and approximate each of the blocks by a low-rank matrix.

- E.g., discretisation of integral equations.
- W. Hackbusch et al., Max-Planck Institute in Leipzig.

1D Hierarchical semiseparable

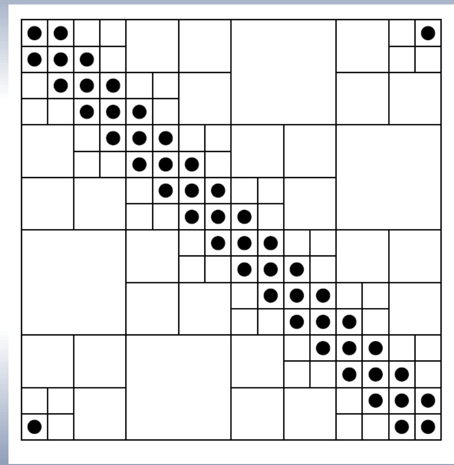
(S. Chandrasekaran, M. Gu et al.)
$A_{i,j} = \log \|x_i - x_j\|, x_i \in \mathbb{R}$.



Matrices without • have low rank, the other ones are of full rank.

1.5D Hierarchical semiseparable

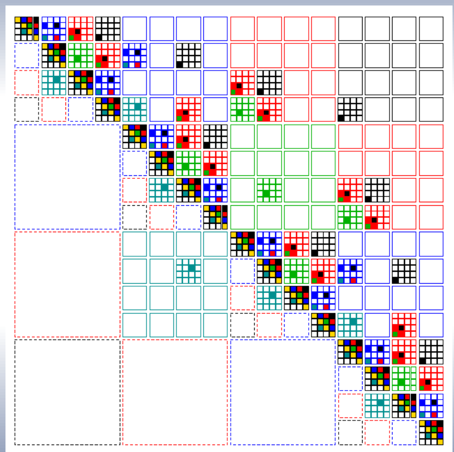
$A_{i,j} = \log \|z_i - z_j\|, z_i$ on a closed curve.



Matrices without • have low rank, the other ones are of full rank.

2D Hierarchical semiseparable

$A_{i,j} = \log \|z_i - z_j\|^{\alpha}, z_i \in \mathbb{R}^2$.



Empty submatrices are of limited rank, the colored ones of full rank.

Few remarks

What to do with and why use these matrices?

- Solving systems of equations or computing eigenvalues.
- Efficient storage leads to less memory consumption.
- Efficient algorithms lead to faster obtainable results.
- These improvements can lead to more accurate results or to increased problem sizes which one can solve.

How to find the structure?

- Quite often it is readily available, e.g., coming from discretization problems.
- Adaptive skeleton cross-approximation.
  (see, e.g., E. Tyrtyshnikov and co-workers)
  (see also, e.g., M. Bebendorf).

What are structured rank matrices?

Definition

Structured rank matrices are matrices for which a specific part in the matrix (the so-called structure), satisfies a certain rank condition.

We focus on three important –basic– classes:

- tridiagonal matrices;
- semiseparable matrices;
- quasiseparable matrices.

What are structured rank matrices?

Definition

Structured rank matrices are matrices for which a specific part in the matrix (the so-called structure), satisfies a certain rank condition.

Example

Tridiagonal (left) and semiseparable (right) matrices.
(Only the lower triangular part is shown.)

$$\begin{bmatrix} \times & \times & & & \\ \times & \times & \times & & \\ & \times & \times & \times & \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix} \quad \begin{bmatrix} \boxtimes & & & & \\ \boxtimes & \boxtimes & & & \\ \boxtimes & \boxtimes & \boxtimes & & \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix}$$

Outline

A definition

Definition

A is an $m \times n$ matrix, with

$$M = \{1, 2, \ldots, m\}, \qquad N = \{1, 2, \ldots, n\},$$
$$\alpha \subset M \quad \text{and} \quad \beta \subset N.$$

Then, $A(\alpha; \beta)$ stands for the submatrix of A with row indices in α and column indices in β.

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Definition

A structure Σ is a nonempty subset of $M \times N$.
The structured rank $\mathrm{r}(\Sigma; A)$ is defined as :

$$\mathrm{r}(\Sigma; A) = \max\{\operatorname{rank}(A(\alpha;\beta))\,|\alpha \times \beta \subseteq \Sigma\},$$

where $\alpha \times \beta$ denotes the set $\{(i,j) | i \in \alpha, j \in \beta\}$.

Some standard structures

Definition

- The subset
$$\Sigma_l = \{(i,j) | i \geq j, i \in M, j \in N\}$$
is called the lower triangular structure (including the diagonal).
- The subset
$$\Sigma_{wl} = \{(i,j) | i > j, i \in M, j \in N\}$$
is the weakly lower triangular structure (excluding the diagonal).

Some standard structures

Definition

- The subset
$$\Sigma_l^{(p)} = \{(i,j) | i > j - p, i \in M, j \in N\}$$
is the p-lower triangular structure and corresponds with all the indices of the matrix, below the pth diagonal.

  The pth diagonal refers to the pth superdiagonal (for $p > 0$);
  the $-p$th diagonal refers to the pth subdiagonal (for $p > 0$).
- Similarly: upper triangular structures (Σ_u).

Resulting equivalences

- The lower triangular structure: $\Sigma_l = \Sigma_l^{(1)}$.
- The weakly lower triangular structure: $\Sigma_{wl} = \Sigma_l^{(0)}$.
- Similar relations hold for the upper triangular structures.

A tridiagonal matrix

Example (Tridiagonal matrix)

- A tridiagonal matrix A is a structured rank matrix with:
$$\mathrm{r}(\Sigma_l^{(-1)}; A) = 0 \text{ and } \mathrm{r}(\Sigma_u^{(-1)}; A) = 0,$$
this means that all the blocks taken out of the matrix below the subdiagonal have rank equal to 0.
- All the red blocks are of rank 0.
Consider only the lower triangular part.

$$\begin{bmatrix} \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 \\ 0 & \times & \times & \times & 0 \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$$

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A semiseparable matrix

Example (Semiseparable matrix)

- A semiseparable matrix is a structured rank matrix A with:

$$r(\Sigma_l; A) \leq 1 \text{ and } r(\Sigma_u; A) \leq 1,$$

  this means that all blocks taken out of the lower triangular part have rank at most 1. (Similar for the upper triangular part.)
- All the red blocks are of rank at most 1.

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}$$

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Structured rank matrices

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Structured rank matrices

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A quasiseparable matrix

Example (Quasiseparable matrix)

- A quasiseparable matrix is a structured rank matrix A with:

$$r(\Sigma_{wl}; A) \leq 1 \text{ and } r(\Sigma_{wu}; A) \leq 1,$$

  this means that all blocks taken out of the weakly lower triangular part have rank at most 1. (Similar for the upper triangular part.)
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$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}$$

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Structured rank matrices

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Relations

Relations

- The quasiseparable class is the most general one.
- Quasiseparables include:
  - semiseparables,
  - tridiagonals.

Outline

Band matrices

Definition ($\{p,q\}$-band)

A matrix A is called a $\{p,q\}$-band matrix if

$$\mathrm{r}\left(\Sigma_l^{(-p)}; A\right) \leq 0 \quad \text{and} \quad \mathrm{r}\left(\Sigma_u^{(-q)}; A\right) \leq 0.$$

Structured rank matrices

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Example (A $\{3,2\}$-band matrix)

$$\begin{bmatrix} \times & \times & \times & & & \\ \times & \times & \times & \times & & \\ \times & \times & \times & \times & \times & \\ \times & \times & \times & \times & \times & \times \\ & \times & \times & \times & \times & \times \\ & & \times & \times & \times & \times \end{bmatrix}.$$

Generalized semiseparable matrices

Definition (Generalized semiseparable)

A matrix A is called $\{p,q\}$-semiseparable if

$$\mathrm{r}\left(\Sigma_l^{(p)}; A\right) \leq p \quad \text{and} \quad \mathrm{r}\left(\Sigma_u^{(q)}; A\right) \leq q.$$

(Note: the rank blocks cross the diagonal.)

Structured rank matrices

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(Note: the rank blocks cross the diagonal.)

Example (A $\{3,1\}$-semiseparable matrix)

All blocks taken out of the red part of rank at most 3

$$\begin{bmatrix} \color{red}\times & \color{red}\times & \color{red}\times & \color{green}\times & \color{green}\times \\ \color{red}\times & \color{red}\times & \color{red}\times & \color{red}\times & \color{green}\times \\ \color{red}\times & \color{red}\times & \color{red}\times & \color{red}\times & \color{red}\times \\ \color{red}\times & \color{red}\times & \color{red}\times & \color{red}\times & \color{red}\times \\ \color{red}\times & \color{red}\times & \color{red}\times & \color{red}\times & \color{red}\times \end{bmatrix}.$$

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(Note: the rank blocks cross the diagonal.)

Example (A $\{3,1\}$-semiseparable matrix)

All blocks taken out of the red part of rank at most 1

$$\begin{bmatrix} {\color{red}\times} & {\color{red}\times} & {\color{red}\times} & {\color{red}\times} & {\color{red}\times} \\ {\color{green}\times} & {\color{red}\times} & {\color{red}\times} & {\color{red}\times} & {\color{red}\times} \\ {\color{green}\times} & {\color{green}\times} & {\color{red}\times} & {\color{red}\times} & {\color{red}\times} \\ {\color{green}\times} & {\color{green}\times} & {\color{green}\times} & {\color{red}\times} & {\color{red}\times} \\ {\color{green}\times} & {\color{green}\times} & {\color{green}\times} & {\color{green}\times} & {\color{red}\times} \end{bmatrix}.$$

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(Note: the rank blocks do not cross the diagonal.)

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(Note: the rank blocks do not cross the diagonal.)

Example (A $\{3,1\}$-quasiseparable matrix)

All blocks taken out of the red part of rank at most 3

$$\begin{bmatrix} {\color{green}\times} & {\color{green}\times} & {\color{green}\times} & {\color{green}\times} & {\color{green}\times} \\ {\color{red}\times} & {\color{green}\times} & {\color{green}\times} & {\color{green}\times} & {\color{green}\times} \\ {\color{red}\times} & {\color{red}\times} & {\color{green}\times} & {\color{green}\times} & {\color{green}\times} \\ {\color{red}\times} & {\color{red}\times} & {\color{red}\times} & {\color{green}\times} & {\color{green}\times} \\ {\color{red}\times} & {\color{red}\times} & {\color{red}\times} & {\color{red}\times} & {\color{green}\times} \end{bmatrix}.$$

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Example (A $\{3,1\}$-quasiseparable matrix)

All blocks taken out of the red part of rank at most 1

$$\begin{bmatrix} {\color{green}\times} & {\color{red}\times} & {\color{red}\times} & {\color{red}\times} & {\color{red}\times} \\ {\color{green}\times} & {\color{green}\times} & {\color{red}\times} & {\color{red}\times} & {\color{red}\times} \\ {\color{green}\times} & {\color{green}\times} & {\color{green}\times} & {\color{red}\times} & {\color{red}\times} \\ {\color{green}\times} & {\color{green}\times} & {\color{green}\times} & {\color{green}\times} & {\color{red}\times} \\ {\color{green}\times} & {\color{green}\times} & {\color{green}\times} & {\color{green}\times} & {\color{green}\times} \end{bmatrix}.$$

Summary

- $\{p,q\}$-semiseparable (cross the diagonal).

$$r\left(\Sigma_l^{(p)};A\right)\leq p \quad \text{and} \quad r\left(\Sigma_u^{(q)};A\right)\leq q.$$

- $\{p,q\}$-band (zero outside the band).

$$r\left(\Sigma_l^{(-p)};A\right)\leq 0 \quad \text{and} \quad r\left(\Sigma_u^{(-q)};A\right)\leq 0.$$

- $\{p,q\}$-quasiseparable (do not cross the diagonal).

$$r\left(\Sigma_{wl};A\right)\leq p \quad \text{and} \quad r\left(\Sigma_{wu};A\right)\leq q.$$

- Very special structures entail summations of the above.

Relations

Relations

- The $\{p,q\}$-quasiseparable class is the most general one.
- $\{p,q\}$-quasiseparables include:
  - $\{p,q\}$-semiseparables,
  - $\{p,q\}$-band matrices.

Outline

References to structured rank matrices

All the material presented in this first part can be contributed to Fiedler.

- M. Fiedler, **Structure ranks of matrices**, Linear Algebra and Its Applications **179** (1993), 119–127.
- M. Fiedler and T. L. Markham, **Completing a matrix when certain entries of its inverse are specified**, Linear Algebra and Its Applications **74** (1986), 225–237.
- M. Fiedler and Z. Vavřín, **Generalized Hessenberg matrices**, Linear Algebra and Its Applications **380** (2004), 95–105.
- M. Fiedler **Basic matrices**, Linear Algebra and Its Applications **373** (2003), 143–151.