OPERATOR MONOTONE FUNCTIONS

(OLD AND NEW)

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Reading list

R. Bhatia : Matrix Analysis, Springer (1997)
GTM 169
esp. Ch V

R. Bhatia : Positive Definite Matrices, Princeton (2007)

X. Zhan : Matrix Inequalities, Springer (2002)
LNM 1790

F. Hiai : lecture notes (somewhere on the web)

§1. POSITIVITY

1.1. $A \in M_n(\mathbb{C})$ is POSITIVE SEMIDEFINITE (PSD), "$A \geq 0$"

$\iff \quad \forall \psi \in \mathbb{C}^n, \quad (\psi, A\psi) \geq 0$

This implies $A = A^*$

$\iff \quad \exists\, B \in M_n(\mathbb{C}) : \quad A = B^* B$

$\iff \quad \exists\, U \in U(n), \; \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ with $\lambda_i \geq 0$:

$$A = U \Lambda U^*$$

$\Rightarrow \quad \mathrm{Tr}\, A \geq 0, \quad \mathrm{Det}\, A \geq 0$

1.2. $\mathcal{P}_n = \{ A \in M_n(\mathbb{C}) : A \geq 0 \}$

- closed under $+$ and $a \cdot$ (with $a \geq 0$)
  $\Rightarrow$ $\mathcal{P}_n$ is a <u>cone</u> : "positive cone".
- $\mathcal{P}_n$ preserved under conjugations :

$$A \geq 0 \Rightarrow XAX^* \geq 0 \qquad (X \text{ invertible: } \Leftarrow)$$

1.3. PSD ORDER : $\underline{A \geq B} \iff A - B \geq 0.$ (A, B Hermitian)

- preserved under conjugations :

$$A \geq B \Rightarrow XAX^* \geq XBX^*$$

§ 2. MONOTONICITY

2.1. Key question: Which functions f ($\mathbb{R} \to \mathbb{R}$) preserve the PSD order?

$$A \geq B \implies f(A) \geq f(B) \qquad (A, B \text{ Hermitian})$$

"f is monotone"

! Depends on n (dim A,B)

- Scalars ($n=1$): f is monotone $\iff f' \geq 0$

- Matrices ($n>1$): $f' \geq 0$ is NOT enough

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \implies A - B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$$

$$A^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \implies A^2 - B^2 = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \not\geq 0$$

$\implies f(x) = x^2$ is <u>NOT</u> m.m. of order 2. since $\det(A^2 - B^2) = -1$.

- It turns out that:

a) condition for monotonicity differs for any single value of n

→ We say: f is <u>matrix monotone</u> <u>of order n</u> (or is not)

and: f is <u>operator monotone</u>

if it is matrix monotone of <u>all</u> orders.

b) It also depends on which <u>interval</u> f is monotone

→ We need to specify the interval $I = (a, b)$ and restrict $sp(A), sp(B) \subset (a, b)$.

We'll see that the only functions that are operator monotone on $\mathbb{R}$ (everywhere) are $f(x) = \alpha + \beta x$.

§3. OPERATOR CONCAVITY / CONVEXITY

a closely related notion...

- f is <u>matrix convex</u> of order n

  if $\forall A, B \in M_n(\mathbb{C})$, Hermitian

  and $\forall\ 0 \leq \lambda \leq 1$:

$$f\big((1-\lambda)A + \lambda B \big) \leq (1-\lambda) f(A) + \lambda f(B)$$

- f is operator convex if f is matrix convex of all orders.
- f is operator concave if $-f$ is operator convex.
- Again not trivial :
  - $f(x) = x^2$ is operator convex on every interval.
  - $f(x) = x^3$ is <u>NOT</u> .

§ 4. Löwner's Condition for matrix/operator monotonicity

4.1

- One first shows that every operator monotone on I is continuously differentiable.

- Define the (Fréchet) derivative of f at $A \in M_n(\mathbb{C})$ as the linear operator $Df(A)$:

$$Df(A)\ (H) = \left.\frac{d}{dt}\right|_{t=0} f(A+tH)$$

- f m.m. of order n on I

$\iff$ $Df(A)(H) \geq 0$

$\forall A \in M_n(\mathbb{C})$, Hermitian, $\sigma(A) \subset I$

$\forall H \geq 0$

4.2 • <u>Daleckii - Krein</u> : explicit formula for $Df(A)$

• consider a basés in which A is diagonal: $A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$

• let $\circ$ denote the entrywise (Schur) product

$$(A \circ B)_{ij} = A_{ij} B_{ij}$$

• let $L = f^{[1]}(A)$ denote the "matrix of divided differences"

$$L_{ij} = \begin{cases} \dfrac{f(a_i) - f(a_j)}{a_i - a_j} & ; \quad i \neq j \text{ and } a_i \neq a_j \\[2ex] f'(a_i) & ; \quad i = j \text{ or } a_i = a_j \end{cases}$$

Then $Df(A)(H) = f^{[1]}(A) \circ H$

• This implies: f m.m. of order n on I

$\Leftrightarrow \quad f^{[1]}(A) \geq 0 \qquad \forall A \in M_n(\mathbb{C})$, Hermitian, $sp(A) \subset I$.

(Löwner)

4.3 Examples:

I. $f(x) = x^2$ shouldn't work, already for $n=2$

EXERCISE

II $f(x) = x^p$

$$L_{ij} = \frac{a_i^p - a_j^p}{a_i - a_j} \; ; \quad L_{ii} = p\, a_i^p$$

$L \geq 0$???

→ Use integral (wow...)

$$0 < p < 1: \quad x^p = \int_0^\infty \frac{x}{x+t} \cdot \underbrace{\frac{\sin p\pi}{\pi} t^{p-1}\, dt}_{\text{positive measure}}$$

$\Rightarrow$ $L \geq 0$ for $f = x^p$ <u>if</u> $L \geq 0$ for $f(x) = \frac{x}{x+t}$, $t \geq 0$.

Indeed, this is the case! <u>EXERCISE</u>
(provided $x \geq 0$, thus $I = [0, \infty)$)

III This actually implies that <u>every</u> function f for which there exists a positive measure $d\mu(t)$ such that $f(x) = \int_0^\infty \frac{x}{x+t}\, d\mu(t)$

is operator monotone on $[0, \infty)$.

§ 5. Löwner's Theorems: Integral representations of operator monotones on I

5.1
- We start with a special interval $I = (-1, 1)$.
- Let K be the set of all functions f on I that are operator monotone <u>and</u> satisfy $f(0)=0$, $f'(0)=1$.
- Then:
  - K is convex
  - K is compact (in the topology of pointwise convergence)

Proof: CENSORED

This means that we can describe K by its <u>extremal points</u>.
(use Krein - Milman, Choquet theory ...)

- these extremal points are:

$$f(x) = \frac{x}{1-tx}, \quad t = \frac{1}{2} f''(0) \in [-1, 1] \quad (\text{"difficult" proof})$$

Hence, we get:

For each f in K there is a unique probability measure μ on $[-1, 1]$ s.t.

$$f(x) = \int_{-1}^{+1} \frac{x}{1-tx} \, d\mu(t).$$

- We can now remove the constraints $f(0)=0$, $f'(0)=1$ and get:

For each f operator monotone on $I=(-1,1)$

$$\boxed{f(x) = f(0) + f'(0) \int_{-1}^{+1} \frac{x}{1-tx} \, d\mu(t)} \qquad (1)$$

N.B. obviously, $f'(0) \geq 0$ is needed.

5.2 • So, in principle this allows us to get an integral representation for operator monotones on <u>any</u> interval $I = (a, b)$ via the transformation:

$$f(x) = g\left(\frac{b-a}{2} x + \frac{b+a}{2} \right) , \qquad -1 < x < 1 .$$

EXERCISE 1 : do this for $a = 0$, $b = +\infty$

EXERCISE 2 : for $f(x) = \sqrt{x}$, find $d\mu(t)$

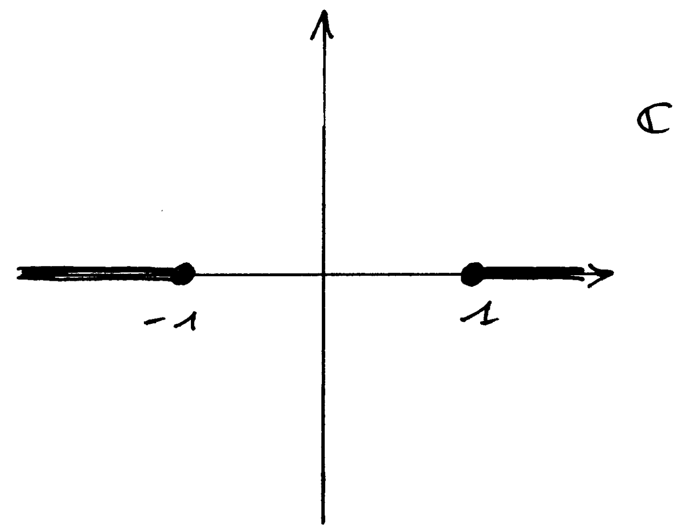
• Formula (1) is never used, as a much more convenient integral rep. can be derived from it, using Nevanlinna's theorem.

(enters complex analysis)

5.3 • By (1), f has an analytic continuation

$$f(z) = f(0) + f'(0) \int_{-1}^{1} \frac{z}{1-tz}\, d\mu(t)$$

defined on



$H_+ = \{z : \operatorname{Im} z > 0\}$

$H_- = \{z : \operatorname{Im} z < 0\}$

(or on $\mathbb{C} \setminus \mathbb{R} \cup (a,b)$ in general)

- EXERCISE Show that $z \mapsto \dfrac{z}{1-tz}$ maps H_+ (H_-) into itself $\forall\, t \in \mathbb{R}$

- Therefore, $f(z)$ maps H_+ into itself too, and H_- into itself.

Furthermore, $\overline{f(z)} = f(\bar{z})$.

$\Rightarrow$ f on H_- ~~is~~ analytic continuation of f on H_+ across $[-1,1)$ (or (a,b)) by reflection.

- One can show the converse too; thus

f is an operator monotone on (a,b)

$\iff$ f has an analytic continuation to H_+ that maps H_+ into itself] (i)

and an analytic continuation to H_- obtained by reflection across (a,b).] (ii)

By (i) we have that f is a <u>Pick function</u> (aka Herglotz function)

NEVANLINNA'S THEOREM:

$f(x)$ is a Pick function

$\Leftrightarrow$ $\exists\ \alpha \in \mathbb{R}$,
$\beta \geq 0$,
μ positive measure, $\int_{-\infty}^{+\infty} \frac{1}{1+t^2}\, d\mu(t) < \infty$

s.t. $$\boxed{f(x) = \alpha + \beta x + \int_{-\infty}^{\infty} \left(\frac{1}{t-x} - \frac{t}{1+t^2} \right) d\mu(t)} \quad (2)$$

Moreover, α, β, μ are unique:

$$\alpha = \operatorname{Re} f(i)$$

$$\beta = \lim_{y \to \infty} \frac{f(iy)}{iy}$$

$$\mu(b) - \mu(a) = \lim_{y \to 0} \frac{1}{\pi} \int_a^b \operatorname{Im} f(x+iy)\, dx,$$

for any a, b where μ is continuous.

- One can show the converse too; thus

f is an operator monotone on (a,b)

$\iff$ f has an analytic continuation to H_+ that maps H_+ into itself] (i)

and an analytic continuation to H_- obtained by reflection across (a,b).] (ii)

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$$f(x) = \alpha + \beta x + \int_{-\infty}^{\infty} \left(\frac{1}{t-x} - \frac{t}{1+t^2} \right) d\mu(t) \qquad (2)$$

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By (i) we have that f is a Pick function (aka Herglotz function)

thus $$f(x) = \alpha + \beta x + \int_{-\infty}^{\infty} \left(\frac{1}{t-x} - \frac{t}{1+t^2} \right) d\mu(t) \qquad (2)$$

Moreover, (ii) implies that μ has zero mass on (a, b);

i.e. the interval (a, b) can be excluded from the integral (2).

Note: (a, b) now enters in $d\mu(t)$, not in the integrand, as in (1)

5.4 Special cases

- $I = (a, b) = (0, \infty)$

$$\to \quad f(x) = \alpha + \beta x + \int_{-\infty}^{0} \left(\frac{1}{t-x} - \frac{t}{t^2+1} \right) d\mu(t) \qquad (3)$$

- An equivalent form of (3) is $\quad (t \to -t)$

$$f(x) = \alpha + \beta x + \int_{0}^{\infty} \left(\frac{t}{t^2+1} - \frac{1}{t+x} \right) d\mu(t) \qquad (3')$$

with $\int_0^\infty \frac{1}{t^2+1} \, d\mu(t) < \infty$

- Suppose, in addition, that

$$f(0) := \lim_{x \to 0} f(x) > -\infty, \qquad \text{i.e. } I = [0, \infty),$$

then
$$f(0) = \alpha + \underbrace{\int_0^\infty \left(\frac{t}{t^2+1} - \frac{1}{t} \right) d\mu(t)}_{\text{converges, therefore.}} > -\infty$$

and
$$f(x) - f(0) = \beta x + \int_0^\infty \underbrace{\left(\frac{1}{t} - \frac{1}{t+x} \right)}_{= \frac{x}{t(t+x)}} d\mu(t)$$

$$\Rightarrow \quad f(x) = \underbrace{\gamma}_{f(0)} + \beta x + \int_0^\infty \frac{tx}{t+x} \cdot \underbrace{\frac{1}{t^2} d\mu(t)}_{\equiv d\omega(t)} \qquad (4)$$

must satisfy $\int_0^\infty \frac{t}{1+t} d\omega(t) < \infty$

5.5 Examples

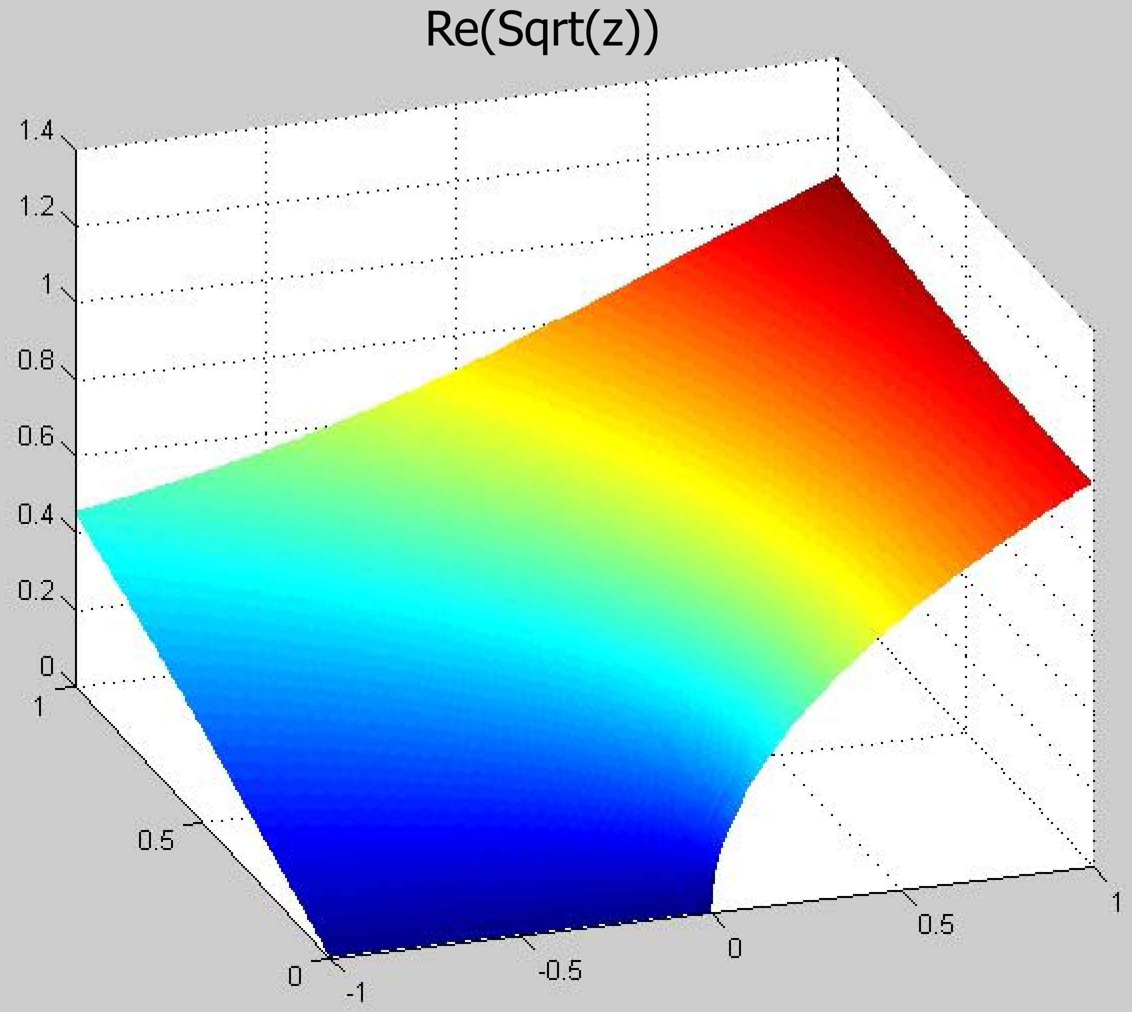
- f operator monotone on $I = \mathbb{R}$

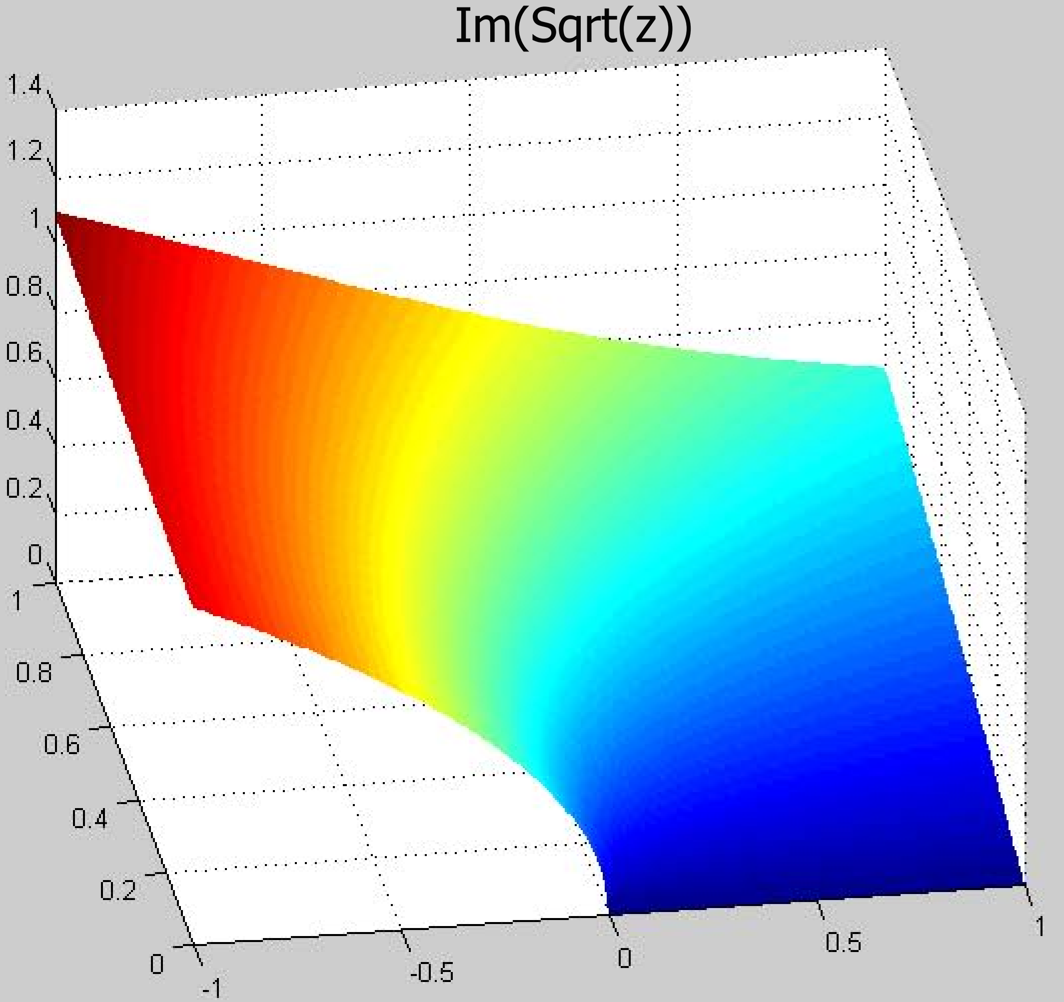
  then $\mu = 0$, hence $f(x) = \gamma + \beta x$.

  (Quite boring)

- $$\log x = \int_0^\infty \left(\frac{t}{t^2+1} - \frac{1}{t+x} \right) dt$$

- $$\underset{\substack{0<p<1 \\ 0<x<\infty}}{x^p} = \cos \frac{p\pi}{2} + \sin \frac{p\pi}{\pi} \int_{-\infty}^{0} \left(\frac{1}{t-x} - \frac{t}{t^2+1} \right) |t|^p \, dt .$$


Re(Sqrt(z))
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
0.5
0
-1
-0.5
0
0.5
1


Im(Sqrt(z))
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1

§ 6. OPERATOR CONVEXITY / CONCAVITY

(Some results; w/o proof)

6.1 Let $f(x)$ map $[0,\infty)$ into itself.

f is operator monotone on $[0,\infty)$ $\iff$ f is operator concave on $[0,\infty)$

6.2 Let $f(x)$ be a continuous function from $(0,\infty)$ into itself.

If $f(x)$ is operator monotone, then $1/f(x)$ is operator convex.

6.3 Let $f(x)$ be a real-valued function on $[0,\alpha)$ TFAE:

i) f is operator convex and $f(0) \leq 0$

ii) $f(x)/x$ is operator monotone on $(0,\alpha)$

6.4 If $f(x)$ is operator monotone on I, then

$F(x) = \int_0^x f(s)\,ds$ is operator convex.

6.5 examples

- $f(x) = -x \log x$ is operator concave on $(0, \infty)$

- $f(x) = x^p$ on $(0, \infty)$

$$\begin{cases} -1 \le p < 0: & f \text{ operator convex} \\ 0 < p \le 1: & f \text{ operator monotone, operator concave} \\ 1 \le p \le 2: & f \text{ operator convex} \end{cases}$$

Underful \hbox (badness 10000) in paragraph at lines 345–347
[]

§7 Applications

7.1 "Simple" consequences

- Let $C(A)$ be a "pinching" of A.

  If f is operator convex on I, then

  - $f(C(A)) \leq C(f(A)) \qquad \forall A : \operatorname{sp} A \subset I$
  - $f(V^*AV) \leq V^* f(A) V \qquad , V$ isometry
  - if $f(0) \leq 0$, $\quad f(PAP) \leq P f(A) P \qquad , P$ projection

- Let $\lambda_1(A)$ denote the largest eigenvalue of A when A has positive eigenvalues.

Let $A, B > 0$.

For $0 \le r \le 1$, $\lambda_1(A^r B^r) \le (\lambda_1(AB))^r$.

Proof. Let $\lambda_1(AB) = a$.

$$\lambda_1(AB) = \lambda_1(A^{1/2} B A^{1/2}) = a$$

$$\Rightarrow \quad A^{1/2} B A^{1/2} \le a I$$

$$\Rightarrow \quad B \le a A^{-1}$$ (conjugate with $A^{-1/2} \cdot A^{-1/2}$)

$$\Rightarrow \quad B^r \le a^r A^{-r}$$

$$\Rightarrow \quad A^{r/2} B^r A^{r/2} \le a^r I$$

$$\Rightarrow \quad \lambda_1(A^{r/2} B^r A^{r/2}) \le a^r$$

$$= \lambda_1(A^r B^r)$$ ∎

(This eventually leads to the Lieb-Thirring inequality)

7.2 "Quantum Chernoff Bound"

Here's a problem in quantum state discrimination that has been open for a very long time and got recently solved in 2006:

CENSORED
explicit physics content!

(KA et al, Phys. Rev. Lett. 98, 160501 (2007))

(KA et al, Commun. Math. Phys. 279, 251 (2008))

Thus, to prove this, we needed to show:

$\forall\, A, B > 0, \ \forall\, s \in [0,1]$:

$$\frac{1}{2}\left(\operatorname{Tr} A + \operatorname{Tr} B - \operatorname{Tr}|A-B|\right) \le \operatorname{Tr} A^{1-s} B^{s} .$$

A very short proof was found by N. Ozawa (unpublished, but reported in Jakšić et al arxiv: 1109.3804)

Interlude: The positive part

3st Eier
5TL Süßstoff
7EL Wasser
150g Roggenvollkorn-mehl
1Pk Backpulver
4TL Zimt
700g Pflaumen

Let A be Hermitian.

- In a basis in which A is diagonal:

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$$

suppose $a_1 \cdots a_k > 0$
$a_{k+1} \cdots a_n < 0$, then

the <u>positive part</u> of A, denoted A_+, is

$$A_+ = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

- In a general basis: $A_+ = \frac{1}{2}(A + |A|)$, where $|A|$ is the modulus of A
$|A| = \sqrt{(A^2)}$

- Likewise: <u>negative part</u>: $A_- = \frac{1}{2}(|A| - A)$

diagonal: $$A_- = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & -a_{k+1} & & \\ & & & & \ddots & \\ & & & & & -a_n \end{pmatrix}$$

Thus : A_+ , A_- are both ≥ 0 ! (by convention)

$$\begin{cases} A = A_+ - A_- \\ |A| = A_+ + A_- \end{cases}$$

(the so-called Jordan decomposition)

(Note: $A_+ A_- = 0$)

$$\Rightarrow \quad A_+ \geq A \geq - A_-$$

OZAWA'S PROOF

Write $\Delta := A - B$

Jordan decomposition $\Delta = \Delta_+ - \Delta_-$

Define $C = B + \Delta_+ = A + \Delta_-$

clearly $B \leq C$ and $A \leq C$

for $0 \leq s \leq 1$: x^s, x^{1-s} op. monotone, hence $\quad A^s \leq C^s \quad B^s \leq C^s$

$\quad A^{1-s} \leq C^{1-s} \quad B^{1-s} \leq C^{1-s}$

Then:
$$
\begin{aligned}
\mathrm{Tr}\left(A - B^s A^{1-s}\right) &= \mathrm{Tr}\,(A^s - B^s)\,\underbrace{A^{1-s}}_{\geq 0} \\
&\leq \mathrm{Tr}\,\underbrace{(C^s - B^s)}_{\geq 0}\,A^{1-s} \\
&\leq \mathrm{Tr}\,(C^s - B^s)\,C^{1-s} \quad = \mathrm{Tr}\left(C - B^s C^{1-s}\right) \\
&= \mathrm{Tr}\,\Delta_+ + \mathrm{Tr}\left(B - B^s C^{1-s}\right) \\
&= \mathrm{Tr}\,\Delta_+ + \mathrm{Tr}\,\underbrace{B^s}_{\geq 0}\,\underbrace{\left(B^{1-s} - C^{1-s}\right)}_{\leq 0} \\
&\leq \mathrm{Tr}\,\Delta_+ .
\end{aligned}
$$

$$\Rightarrow \frac{1}{2}\mathrm{Tr}\left(A + B - |A - B|\right) = \mathrm{Tr}\,(A - \Delta_+) \leq \mathrm{Tr}\,A^{1-s}B^s \qquad \blacksquare$$

7.3 NORM INEQUALITIES FOR OPERATOR MONOTONE FUNCTIONS

(T. ANDO & X. ZHAN, Math. Ann. 315, 771–780 (1999))

Let $A, B \geq 0$.

(I) For every <u>non-negative</u> <u>operator monotone</u> f <u>on $[0,\infty)$</u> and every UI norm $|||\cdot|||$

$$|||f(A+B)||| \leq |||f(A)+f(B)||| \qquad (\text{"subadditive"})$$

(II) For every non-negative increasing function g on $[0,\infty)$ where $g(0)=0$, $g(\infty)=\infty$ and g^{-1} is operator monotone,

$$|||g(A+B)||| \geq |||g(A)+g(B)|||$$

<u>Proof</u> (main idea) prove this for $f(x) = \frac{x}{x+1}$ and then use integral rep on p. 9

something related

Proof for operator norm

$\equiv$ for largest eigenvalue (when arguments are PSD)

$A \geq 0: \quad \|A\| = \lambda_1^{\downarrow}(A) = \max_{\psi} (\psi, A\psi)$ (ψ normalised vector)

- Now let ψ be the eigenvector of $A+B$ pertaining to its largest eigenvalue.

  Core of the proof: shows that

$$(\psi, f(A+B)\psi) \leq (\psi, (f(A)+f(B))\psi) \qquad (*)$$

$$\text{for } f(x) = \frac{x}{x+1}$$

- Then it easily follows that $(*)$ also holds for $f_s(x) = \dfrac{x}{x+s}$, $s > 0$. (by homogeneity)

- Finally, integrating $(*)$ yields

$$\left(\psi, \int_0^\infty d\mu(s)\, s\, f_s(A+B)\, \psi\right) \leq \left(\psi, \left(\int_0^\infty d\mu(s)\, s\, f_s(A) + \int_0^\infty d\mu(s)\, s\, f_s(B)\right)\psi\right)$$

- ... and $(*)$ obviously holds for $f(x) = \alpha + \beta x$ when $\alpha, \beta \geq 0$

Thus, exploiting the integral rep. for non-negative operator monotone functions

$$f(x) = \alpha + \beta x + \int_0^\infty d\mu(s) \, \frac{s x}{s + x} \qquad \alpha, \beta \geq 0, \quad d\mu \geq 0$$

yields (*) for all such f

Hence $\| f(A+B) \| = (\psi, f(A+B) \psi) \leq (\psi, (f(A)+f(B)) \psi)$

by choice of ψ = optimality of ψ for $f(A+B)$

$\leq \| f(A)+f(B) \|$ $\blacksquare$

by sub-optimality of this ψ for $f(A)+f(B)$

7.4 Zhan's Conjecture for singular values of Heinz means

Let $A, B \geq 0$. and $0 \leq s \leq 1$.

X. Zhan conjectured: $\sigma_j(A^s B^{1-s} + A^{1-s} B^s) \leq \sigma_j(A+B)$

(σ_j = j-th largest singular value)

The proof (KA, Lin. Alg. Appl. 422, 279–283 (2007)) relies on the operator monotonicity of $x \to x^s, x^{1-s}$.

R. Bhatia and F. Kittaneh then simplified the original proof (Bhatia & Kittaneh, Lin. Alg. Appl. 428, 2177–2191 (2008))

<u>Proof</u> Let $f(x)$ be operator monotone on $[0,\infty)$, and non-negative.

We've seen that this implies that

f is also <u>operator concave</u> $\qquad (6.1)$

and $x f(x)$ is <u>operator convex</u>. $\qquad (6.3)$

Let $A, B \geq 0$. Then

$$\frac{1}{2}\left(A f(A) + B f(B)\right) \geq \frac{A+B}{2}\, f\!\left(\frac{A+B}{2}\right)$$

$$= \frac{1}{2}\,(A+B)^{1/2}\, \underbrace{f\!\left(\frac{A+B}{2}\right)}_{\geq \frac{f(A)+f(B)}{2}}\, (A+B)^{1/2}$$

Thus $\quad A f(A) + B f(B) \geq \frac{1}{2}\,(A+B)^{1/2}\left(f(A) + f(B)\right)(A+B)^{1/2}$

In particular, for $f = x^r$, $0 \le r \le 1$:

$$A^{r+1} + B^{r+1} \geq \frac{1}{2} (A+B)^{1/2} (A^r + B^r) (A+B)^{1/2}$$

$$\Rightarrow \lambda_j^{\downarrow}(A^{r+1}+B^{r+1}) \geq \frac{1}{2} \lambda_j^{\downarrow}\big((A+B)(A^r+B^r)\big)$$

$$\underset{(\cdots)}{\geq} \sigma_j \left(A^{1/2} (A^r + B^r) B^{1/2} \right)$$

$$= \sigma_j \left(A^{\frac{1}{2}+r} B^{\frac{1}{2}} + A^{\frac{1}{2}} B^{\frac{1}{2}+r} \right)$$

Now substitute $a = A^{r+1}$, $b = B^{r+1}$

$$\Rightarrow \lambda_j^{\downarrow}(a+b) \geq \sigma_j \left(a^{\overset{=\vartheta}{\frac{2r+1}{2r+2}}} b^{\overset{=1-\vartheta}{\frac{1}{2r+2}}} + a^{\frac{1}{2r+2}} b^{\frac{2r+1}{2r+2}} \right)$$

$$\lambda_j^{\downarrow}(a+b) = \sigma_j(a+b)$$

↪ Covers case $\frac{1}{2} \leq \vartheta \leq \frac{3}{4}$. Other ranges: similar ∎

$(\cdots)$: Except for zeros, the eigenvalues of $(A+B)(A^r+B^r)$ are the same as those of $\begin{pmatrix} A^{1/2} & B^{1/2} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{pmatrix}\begin{pmatrix} A^r+B^r & 0 \\ 0 & 0 \end{pmatrix}$

$=$ as those of $\begin{pmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{pmatrix}\begin{pmatrix} A^r+B^r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A^{1/2} & B^{1/2} \\ 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} A^{1/2}(A^r+B^r)A^{1/2} & A^{1/2}(A^r+B^r)B^{1/2} \\ B^{1/2}(A^r+B^r)A^{1/2} & B^{1/2}(A^r+B^r)B^{1/2} \end{pmatrix}$$, which is ≥ 0

<u>Theorem</u>: (Y. Tao) If $\begin{pmatrix} G & H \\ H^* & K \end{pmatrix} \geq 0$ then $\sigma_j\begin{pmatrix} G & H \\ H^* & K \end{pmatrix} \geq 2\,\sigma_j(H)$

Hence $\lambda_j^{\downarrow}\big((A+B)(A^r+B^r)\big) \geq 2\,\sigma_j\big(A^{1/2}(A^r+B^r)B^{1/2}\big)$. Indeed!

7.5 On a question by Kwong regarding Lyapunov-type equations.

Kwong asked: (M.K. Kwong, Lin. Alg. Appl. 118, 129–153 (1989))

For which functions g is the solution X

of $AX + XA = g(A)B + Bg(A)$

always PSD, for $A, B \geq 0$?

Call this set G, or G_n for $A, B \in M_n(\mathbb{C})$

Initial reduction: it suffices to consider

diagonal $A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$

$$B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} = [1]$$

This question was widely advertised by R. Bhatia.

The equation then simplifies to

$$\forall i,j: \quad a_i X_{ij} + X_{ij} a_j = g(a_i) + g(a_j)$$

hence $$X_{ij} = \frac{g(a_i) + g(a_j)}{a_i + a_j} \qquad (\text{we assume } A > 0)$$

Thus the question is related very much to Löwner's condition for operator monotone functions

Löwner: f matrix monotone order n on $(0, \infty)$

$$\Leftrightarrow \forall a_1 \ldots a_n > 0: \underbrace{\left(\frac{f(a_i) - f(a_j)}{a_i - a_j} \right)_{i,j=1}^{n}}_{L_f} \geq 0$$

Kwong: $\underbrace{g \in G_n}_{\text{unknown set}} \Leftrightarrow \forall a_1 \ldots a_n > 0: \underbrace{\left(\frac{g(a_i) + g(a_j)}{a_i + a_j} \right)_{i,j=1}^{n}}_{K_g} \geq 0$

It is easy to show that $\forall n$,

G_n contains the non-negative <u>operator</u> monotones f

again via the integral

$$f(x) = \alpha + \beta x + \int_0^\infty \frac{x}{x+t}\, d\mu(t).$$

Indeed, consider $g(x) = \dfrac{x}{x+t}$, $t \geq 0$

then $\dfrac{g(a_i) + g(a_j)}{a_i + a_j} =$ <u>EXERCISE</u>

Fortunately (and amazingly) Kwong's problem can be solved by a reduction to Löwner's theory.

Theorem I. (KA, Proc AMS 139, 4217–4223 (2011))

$$g \in G = \bigcap_n G_n \quad \text{on } (0,\infty)$$

$$\Longleftrightarrow \quad g(\sqrt{x})\sqrt{x} \quad \text{non-negative operator monotone on } (0,\infty)$$

↪ integral rep:

$$g(x) = \frac{\alpha}{x} + \beta x + \int_0^\infty \frac{x}{x^2+t}\, d\mu(t)$$

Consequence of:

Theorem II. $g \in G_{2N}$ on (a,b)

$$\Rightarrow \quad g(\sqrt{x})\sqrt{x} \quad \text{non-negative m.m. order } N \text{ on } (a^2, b^2)$$

(Note: "2" missing in paper)

- Main idea of proof:

$$K_g + L_g = \left(\frac{g(x_i)+g(x_j)}{x_i + x_j} + \frac{g(x_i)-g(x_j)}{x_i - x_j} \right)_{i,j=1}^{N}$$

$$= 2 \left(\frac{x_i g(x_i) - x_j g(x_j)}{x_i^2 - x_j^2} \right)_{i,j=1}^{N}$$

$\hookrightarrow$ (2 x) Löwner matrix of $g(\sqrt{x})\sqrt{x}$ in the points x_i^2.

- Show that positivity of K_g of order $2N$ implies positivity of $K_g + L_g$ of order N.

(Requires powerful undergraduate mathematics!)

Consider K_g in the $2N$ points $(x_1, \dots, x_N, x_1+\varepsilon, \dots, x_N+\varepsilon)$

then we get

$$K' = \begin{pmatrix} \dfrac{g(x_i) + g(x_j)}{x_i + x_j} & \dfrac{g(x_i) + g(x_j+\varepsilon)}{x_i + (x_j+\varepsilon)} \\[3ex] \dfrac{g(x_i+\varepsilon) + g(x_j)}{(x_i+\varepsilon) + x_j} & \dfrac{g(x_i+\varepsilon) + g(x_j+\varepsilon)}{(x_i+\varepsilon) + (x_j+\varepsilon)} \end{pmatrix}$$

and $K' \geq 0$ by assumption.

Let K'' be the matrix obtained from K' by putting in minus signs

$$K'' = \begin{pmatrix} \dfrac{g(x_i) + g(x_j)}{x_i + x_j} & \dfrac{g(x_i) - g(x_j+\varepsilon)}{x_i - (x_j+\varepsilon)} \\[3ex] \dfrac{-g(x_i+\varepsilon) + g(x_j)}{-(x_i+\varepsilon) + x_j} & \dfrac{-g(x_i+\varepsilon) - g(x_j+\varepsilon)}{-(x_i+\varepsilon) - (x_j+\varepsilon)} \end{pmatrix}$$

Proposition Let $s_i = \pm 1$, $g_i \geq 0$, $x_i > 0$ (all distinct)

$$Z := \left(\frac{s_i g_i + s_j g_j}{s_i x_i + s_j x_j} \right)_{i,j=1}^{N} .$$

Det Z does not change sign when the s_i change sign.

$\hookrightarrow$ Apply this to any principal submatrix of K' and K'': $(g_i := g(x_i))$

This proposition shows they all have the same sign.

Thus, if $K' \geq 0$, then $K'' \geq 0$!

- In the limit $\varepsilon \to 0$

κ' becomes $\begin{pmatrix} \kappa_g & \kappa_g \\ \kappa_g & \kappa_g \end{pmatrix}$ with κ_g in the points $(x_1, \dots x_n)$

while κ'' becomes $\begin{pmatrix} \kappa_g & L_g \\ L_g & \kappa_g \end{pmatrix}$.

The latter is unitarily equivalent to

$$\begin{pmatrix} \kappa_g + L_g & \\ & \kappa_g - L_g \end{pmatrix}$$

Hence we've shown $\kappa_g + L_g \geq 0$ in the points $(x_1, \dots x_n)$ ∎