Status Report on Rationally Generated Block Toeplitz and Wiener-Hopf Determinants

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The subject of this paper is determinants of truncated Toeplitz and Wiener-Hopf integral operators generated by rational matrix functions. We assort a few central results of this large field, reveal the relationships between them, and outline some ideas underlying their proofs. The paper aims at giving an introduction to this research topic as well as at providing the reader a little formulary containing both well known results and certain formulas obtained only recently.

1 Introduction

This is an extended version of the report I gave at the Oberwolfach conference in December 1989. I started working on Toeplitz determinants jointly with Silbermann more than ten years ago, and at that time we focussed our efforts on the smoothness conditions needed to ensure the validity of the Szegö-Widom limit theorem and did not pay much attention to such "supersmooth" symbols as rational functions. My serious interest in rational generating functions arose in the early eighties, when I became a PhD student at Rostov-on-Don University and there met Misha Gorodetsky, who told me about his fresh results [17] on rationally generated block Toeplitz determinants. Since then I have become acquainted with various interesting and (seemingly or really) different formulas for Toeplitz and Wiener-Hopf determinants with rational symbols, which awakened my desire to collect all these things in a single survey. Thus, the present article primarily attempts at summarizing what I know about this topic and at offering the reader a few guiding data in this very extensive field.

We start with some definitions. Let f be a rational matrix function of the size $r \times r$. If f has no poles on the complex unit circle $\mathbf{T}$, we let $f(z) = \sum f_k z^k$ be the Laurent series of f in a sufficiently thin annulus $1 - \varepsilon < |z| < 1 + \varepsilon$ and define the (block) Toeplitz determinants $D_n(f)$ for $n = 1, 2, \ldots$ by

$$D_n(f) = \det(f_{i-j})_{i,j=1}^{n}.$$

Notice that $D_n(f)$ is an $nr \times nr$ determinant. In case f does not have poles on the real line $\mathbf{R}$ and $f(\infty)$ is the identity matrix I, there is a matrix function $k \in L^2_{r\times r}(\mathbf{R})$ whose Fourier-Plancherel transform equals $I - f$ on $\mathbf{R}$, i.e.

$$f(x) = I - \hat{k}(x) := I - \int_{\mathbf{R}} k(t)e^{ixt}dt \quad (x \in \mathbf{R}).$$

We remark that the rationality of f forces k to be continuous on $\mathbf{R} \setminus \{0\}$. For $\tau \in (0, \infty)$, the truncated Wiener-Hopf operator $W_\tau(k)$ on $L^2_r(0, \tau)$ defined by

$$(W_\tau(k)u)(t) = \int_0^\tau k(t-s)u(s)ds \quad (0 < t < \tau)$$

is Hilbert-Schmidt, and so the second regularized Fredholm determinant

$$\widetilde{D}_\tau(f) = \det{}_2\,(I - W_\tau(k))$$

is well-defined. Recall that $\det_2(I - W_\tau(k))$ may be defined as $\prod(1-\lambda_j)\exp\lambda_j$ where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of $W_\tau(k)$ counted up to algebraic multiplicity (see e.g. [15], [25]). The operator $W_\tau(k)$ is of trace class if and only if k is continuous at the origin. In that case one can also consider the usual Fredholm determinants

$$D_\tau(f) = \det(I - W_\tau(k)) := \prod(1 - \lambda_j).$$

The two determinants $D_\tau(f)$ and $\widetilde{D}_\tau(f)$ are related to one another by the equalities

$$D_\tau(f) = \widetilde{D}_\tau(f)e^{-\operatorname{tr} W_\tau(k)} = \widetilde{D}_\tau(f)e^{-\tau \operatorname{tr} k(0)},$$

$\operatorname{tr} F$ referring to the trace of F.

The problem we shall be concerned with is to find "exact" or asymptotic formulas for the determinants $D_n(f)$ and $\widetilde{D}_\tau(f)$. By an "exact" formula we mean an expression for the determinants whose complexity is independent of n or τ, that is, we require that n or τ enters the formula as a parameter only and that the formula does not become more complicated as n or τ increases.

In the following the Toeplitz and Wiener-Hopf cases are treated separately. In the Toeplitz case, we first survey some results of Bart, Gohberg, Kaashoek, and van Schagen (Section 2) and then turn over to the two sets of the Gorodetsky formulas (Sections 3 and 4). I am embarrassed to report that I have unfortunately not been able to deduce the formulas of Section 3 from the results of Section 2, although I feel that this should be possible without undue effort. Once the formulas of Gorodetsky are available, it is a relatively easy matter to obtain several (but not all!) other expressions for $D_n(f)$, which were established partly before and independently of Gorodetsky and mostly by different methods by a series of authors, including Baxter, Schmidt, Day, Trench, and Tismenetsky (Sections 4 and 5).

In the Wiener-Hopf case we also start with the BGKvS theory (Section 6). Then we discuss what we know about Wiener-Hopf analogues of the Gorodetsky formulas, thus entering a topic which contains still many open questions. I have not yet lost my trust in solving all these open problems at one blow by understanding how in the Toeplitz case the first set of Gorodetsky's formulas can be derived from the BGKvS formula. Fortunately we have a round theory in the scalar case (Wiener-Hopf analogues of Day's formula), whose results are illustrated in Section 7.

2 The BGKvS formula for Toeplitz determinants

2.1. Realizations. If C, A, B are any matrices of the sizes $r \times m$, $m \times m$, $m \times r$, respectively, then $C(zI - A)^{-1}B$ is a rational matrix function of the size $r \times r$ which equals the zero matrix at infinity. It is well known from linear systems theory that the converse is also true: if g is a rational matrix function of the size $r \times r$ and $g(\infty)$ is the zero matrix, then there exist $r \times m$, $m \times m$, $m \times r$ matrices C, A, B such that $g(z)$ is equal to $C(zI - A)^{-1}B$. The latter representation is usually called a realization of g. We remark that each rational matrix function has many different realizations. The poles of g are clearly just the eigenvalues of A.

Hence, given a rational matrix function f for which $f(\infty)$ is a finite and invertible matrix, we have a realization

$$f(z) = f(\infty)[I + C(zI - A)^{-1}B]. \tag{1}$$

In case $f(0)$ is finite and invertible, we can realize $f(1/z)$ in the preceding form to obtain that

$$f(z) = f(0)[I + C((1/z)I - A)^{-1}B] = f(0)[I + zC(I - zA)^{-1}B]. \tag{2}$$

Whenever f is realized in the form (1) or (2) we denote by $A^\times$ the matrix $A - BC$; note that $A^\times$ depends not only on A but also on B and C. Also notice that if (1) or (2) is in force, then

$$f^{-1}(z) = [I - C(zI - A^\times)^{-1}B]f^{-1}(\infty)$$

or

$$f^{-1}(z) = [I - zC(I - zA^\times)^{-1}B]f^{-1}(0),$$

respectively.

2.2. Riesz projections. Throughout what follows we make no distinction between an $m \times m$ matrix T and the linear operator on $\mathbf{C}^m$ whose matrix representation with respect to the standard basis of $\mathbf{C}^m$ is T.

Let A be any $m \times m$ matrix and G be any open, bounded, and connected subset of $\mathbf{C}$ with a smooth boundary ∂G, and assume no eigenvalue of A lies on ∂G. The Riesz projection $\chi_G(A)$ is then defined as

$$\chi_G(A) = \frac{1}{2\pi i} \int_{\partial G} (zI - A)^{-1} dz,$$

where ∂G is given the orientation that leaves G on the left.

Given an $m \times m$ matrix A and a projection P on $\mathbf{C}^m$, we denote by $PAP|\operatorname{Im} P$ the compression of (the operator on $\mathbf{C}^m$ induced by) A to the image $\operatorname{Im} P$ of P:

$$PAP|\operatorname{Im} P : \operatorname{Im} P \to \operatorname{Im} P, \quad x \mapsto PAx.$$

It is easily seen that

$$\det(PAP|\operatorname{Im} P) = \det(I - P + PAP) = \det(I - P + PA). \tag{3}$$

Now suppose P is the Riesz projection $\chi_G(A)$. From identity (3) we infer that $PAP|\operatorname{Im} P$ is invertible if and only if so is $I - P + PA$, and hence the spectral theorem can be applied to conclude that $PAP|\operatorname{Im} P$ is invertible if and only if G does not contain the origin.

2.3. The BGKvS formula. Let f be a rational matrix function of the size $r \times r$ which has no poles on $\mathbf{T}$, suppose $f(0)$ is finite and invertible, and assume we are given a realization of the form (2). Put $P = I - \chi_{\mathbf{D}}(A)$ where $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ is the open unit disk. By what was said in the last paragraph of 2.2, $PAP|\operatorname{Im} P$ is invertible.

Theorem. *We have*

$$D_n(f) = (\det f(0))^n \frac{\det(P(A^\times)^n P|\operatorname{Im} P)}{(\det(PAP|\operatorname{Im} P))^n} \quad \textit{for all} \quad n \geq 1.$$

Alternatively,

$$D_n(f) = (\det f(0))^n \frac{\det(I - P + P(A^\times)^n)}{(\det(I - P + PA))^n} \quad \textit{for all} \quad n \geq 1.$$

This theorem grew out of the work of Bart, Gohberg, and Kaashoek (e.g. [1], [2], [3]) and is contained in the form presented here in Gohberg, Kaashoek, and van Schagen's paper [14].

2.4. Comments. Thus, the BGKvS formula is an "exact" formula which reduces the computation of the $nr \times nr$ determinant $D_n(f)$ to the evaluation of the determinant $\det(P(A^\times)^n P|\operatorname{Im} P)$, whose order is $\dim \operatorname{Im} P$ (and independent of n). If the Jordan canonical form of the projection P is $S = \operatorname{diag}(1, \ldots, 1, 0, \ldots, 0)$ (m units) and $P = R^{-1}SR$, then

$$\begin{aligned} \det(P(A^\times)^n P|\operatorname{Im} P) &= \det(I - P + P(A^\times)^n P) \\ &= \det(I - R^{-1}SR + R^{-1}SR(A^\times)^n R^{-1}SR) \\ &= \det(I - S + S(RA^\times R^{-1})^n S) = \det(S(RA^\times R^{-1})^n S|\operatorname{Im} S), \end{aligned}$$

and consequently, we are left with finding the determinant of the $m \times m$ north-west corner of $(RA^\times R^{-1})^n$. Analogously, if J denotes the Jordan canonical form of $A^\times$ and if $A^\times = Q^{-1}JQ$, we have

$$\det(P(A^\times)^n P|\operatorname{Im} P) = \det(QPQ^{-1}J^n QPQ^{-1}|\operatorname{Im} QPQ^{-1}).$$

The BGKvS formula requires that $f(0)$ be finite and invertible. If 0 is a pole of f or if $f(0)$ is not invertible, we may pass over to the matrix function $\widetilde{f}^{\top}$ defined by

$$\widetilde{f}^{\top}(z) = \text{transposed of } f(1/z), \tag{4}$$

because clearly $D_n(f) = D_n(\widetilde{f}^{\top})$. The matrix $\widetilde{f}^{\top}(0)$ is finite and invertible if and only if so is $f(\infty)$, and thus the BGKvS formula is applicable whenever

$$f(0) \ \text{ is finite and invertible} \tag{5}$$

or

$$f(\infty) \ \text{ is finite and invertible.} \tag{6}$$

Although (5) and (6) are "generically" satisfied, the hypothesis that (5) and (6) be in force is in fact a serious restriction: it excludes all functions f with a Laurent series of the form

$$f(z) = \sum_{k=-p}^{q} f_k z^k \quad (1 - \varepsilon < |z| < 1 + \varepsilon)$$

with $p, q \geq 1$ and thus rules out Toeplitz band matrices!

2.5. Proof of the BGKvS formula. ([1], [2], [14].) The philosophy behind the proof is a simple identity: if X, Y, D, E are $q \times p$, $p \times q$, $p \times p$, $q \times q$ matrices, respectively, and if D and E are invertible, then

$$\det(D + YE^{-1}X) = (\det D)(\det E)^{-1} \det(E + XD^{-1}Y). \tag{7}$$

This identity is easily verified. It is prompted by the fact that $E + XD^{-1}Y$ is a so-called indicator of $D + YE^{-1}X$ (see e.g. [1], [2]), viz that $D + YE^{-1}X$ and $E + XD^{-1}Y$ are matricially coupled by the relation

$$\begin{pmatrix} D + YE^{-1}X & -YE^{-1} \\ -E^{-1}X & E^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} D^{-1} & D^{-1}Y \\ XD^{-1} & E + XD^{-1}Y \end{pmatrix}.$$

Thus, the strategy of the proof is to represent the (block) Toeplitz matrix $T_n(f) := (f_{i-j})_{i,j=1}^n$ in the form $D + YE^{-1}X$ so that $E + XD^{-1}Y$ becomes as simple as possible. In [14], this is done as follows.

Assume without loss of generality that $f(0) = I$. Then we have the Laurent series

$$\begin{aligned} f(z) &= I + zC(I - zA)^{-1}B \\ &= -\sum_{k=-\infty}^{0} CP(PAP)^{-k-1}PB\, z^k + \sum_{k=1}^{\infty} CA^{k-1}(I-P)B\, z^k \end{aligned}$$

in the annulus $1 - \varepsilon < |z| < 1 + \varepsilon$ and hence

$$T_n(f) = I + \begin{pmatrix} -CP(PAP)^{-1}PB & \dots & -CP(PAP)^{-n}PB \\ C(I-P)B & & -CP(PAP)^{-n+1}PB \\ CA(I-P)B & & -CP(PAP)^{-n+2}PB \\ \vdots & \ddots & \vdots \\ CA^{n-2}(I-P)B & & -CP(PAP)^{-1}PB \end{pmatrix} = I + H - RS$$

where

$$S = (P(PAP)^{-1}PB, P(PAP)^{-2}PB, \ldots, P(PAP)^{-n}PB),$$

$$H = \begin{pmatrix} 0 & & & & \\ CB & 0 & & & \\ CAB & CB & 0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ CA^{n-2}B & CA^{n-3}B & \ldots & CB & 0 \end{pmatrix}, \quad R = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$

Put $D = I + H$, $E = I$, $X = -S$, $Y = R$. It is easily seen that

$$(I+H)^{-1} = \begin{pmatrix} I & & & & \\ -CB & I & & & \\ -CA^{\times}B & -CB & I & & \\ \vdots & \vdots & \ddots & \ddots & \\ -C(A^{\times})^{n-2}B & -C(A^{\times})^{n-3}B & \ldots & -CB & I \end{pmatrix}$$

and consequently,

$$(I+H)^{-1}R = \begin{pmatrix} C \\ CA^{\times} \\ \vdots \\ C(A^{\times})^{n-1} \end{pmatrix},$$

which yields that

$$\begin{aligned} E + XD^{-1}Y &= I - S(I+H)^{-1}R = I - \sum_{j=1}^{n} P(PAP)^{-j}PBC(A^{\times})^{j-1} \\ &= I - \sum_{j=1}^{n} P(PAP)^{-j}P(A - A^{\times})(A^{\times})^{j-1} \\ &= I + P(PAP)^{-n}P(A^{\times})^{n} - P. \end{aligned}$$

Because $\det(I+H) = 1$, formula (7) finally gives that

$$\begin{aligned} D_n(f) &= \det(I - P + P(PAP)^{-n}P(A^{\times})^{n}) \\ &= \det(P(A^{\times})^{n}P|\operatorname{Im} P)/(\det(PAP|\operatorname{Im} P))^{n}. \quad \blacksquare \end{aligned}$$

2.6. The Szegö-Widom theorem. Let f be as in 2.3. In addition, assume now that

$$\det f(z) \neq 0 \text{ for all } z \in \mathbf{T}, \quad \arg\det f(e^{i\theta})\Big|_{\theta=-\pi}^{\pi} = 0. \tag{8}$$

The Szegö-Widom theorem (see [30], [11] and the references given there) says that then

$$\lim_{n\to\infty} \frac{D_n(f)}{G(f)^n} = E(f),$$

where $G(f) \neq 0$ and $E(f)$ are certain well-defined constants. Our assumptions ensure that $\det f$ has an analytic logarithm $\log\det f$ in some annulus $1-\varepsilon < |z| < 1+\varepsilon$. Let

$$\log\det f(z) = \sum_{k=-\infty}^{\infty} (\log\det f)_k z^k$$

be the Laurent series in that annulus. One can show that

$$G(f) = \exp(\log\det f)_0. \tag{9}$$

Widom [30] discovered that $E(f)$ may be given in the form

$$E(f) = \det T(f)T(f^{-1}), \tag{10}$$

where $T(f)$ and $T(f^{-1})$ stand for the semi-infinite (block) Toeplitz operators on $\ell_r^2(\mathbf{Z}_+)$ with the symbols f and f^{-1}. Because $T(f)T(f^{-1})-I$ can be shown to be the product of two Hilbert-Schmidt Hankel operators, the right-hand side of (10) is well-defined. In the scalar case ($r=1$ and $\log\det = \log$) one has

$$\det T(f)T(f^{-1}) = \exp\sum_{k=1}^{\infty} k(\log f)_k(\log f)_{-k}.$$

2.7. The GKvS version of the Szegö-Widom theorem. Again let f be a rational matrix function of the size $r\times r$, suppose f has no poles on $\mathbf{T}$, and assume (8) is satisfied. Let $f(z) = f(0)[I + zC(I-zA)^{-1}B]$ be a realization of the form (2).

The condition that $\det f$ does not vanish on $\mathbf{T}$ is equivalent to the requirement that $A^\times = A - BC$ has no eigenvalues on the unit circle $\mathbf{T}$. Hence, the Riesz projection $P^\times = I - \chi_{\mathbf{D}}(A^\times)$ is well-defined. Let $P = I - \chi_{\mathbf{D}}(A)$ be as in 2.3. Under the above hypotheses, Gohberg, Kaashoek, and van Schagen established the following result in [14].

Theorem. *We have* $\lim D_n(f)/G(f)^n = E(f)$ *as* $n\to\infty$ *where*

$$G(f) = (\det f(0))\frac{\det(P^\times A^\times P^\times|\operatorname{Im} P^\times)}{\det(PAP|\operatorname{Im} P)} = (\det f(0))\frac{\det(I - P^\times + P^\times A^\times)}{\det(I-P+PA)} \tag{11}$$

and

$$E(f) = \det[(I-P)(I-P^\times) + PP^\times] = \det(I - P - P^\times). \tag{12}$$

Once the formula cited in 2.3 is available, the theorem can be derived from it almost straightforwardly [14]. Indeed, we have

$$D_n(f)/G(f)^n = \det[(I - P + P(A^\times)^n)(I - P^\times + P^\times A^\times)^{-n}]$$

and

$$\begin{aligned}
&[I - P + P(A^\times)^n][I - P^\times + P^\times A^\times]^{-n} \\
&= (I-P)[I-P^\times + P^\times A^\times]^{-n} + P[(I-P^\times)A^\times + P^\times]^n \\
&= (I-P)[I-P^\times + (P^\times A^\times)^n]^{-1} + P[((I-P^\times)A^\times)^n + P^\times] \\
&\to (I-P)(I-P^\times) + PP^\times = I - P - P^\times;
\end{aligned}$$

the convergence follows because the eigenvalues of

$$P^{\times}A^{\times} = A^{\times} - \chi_{\mathbf{D}}(A^{\times})A^{\times} \quad \text{and} \quad (I - P^{\times})A^{\times} = \chi_{\mathbf{D}}(A^{\times})A^{\times}$$

are located in $\mathbf{C} \setminus \overline{\mathbf{D}}$ and $\mathbf{D}$, respectively. ■

A direct proof of the fact that the right-hand sides of (9) and (11) coincide is in [14]. In [31], it is shown in a direct way that the right-hand sides of (10) and (12) are equal to one another.

3 Towards more explicity

3.1. Prologue. The price one has to pay when applying the very compact BGKvS formula 2.3 to a concrete situation is protracted computations to evaluate the determinant $\det(P(A^{\times})^n P|\operatorname{Im} P)$. Therefore the search for more "explicit" formulas is of course desirable.

The formulas cited in this section meet in my opinion the requirement for explicity in a satisfactory manner. They were found by Gorodetsky [17], [18] in the early eighties, but due to his slack publication practice they have remained widely unknown.

3.2. The Cramer-Jacobi rule. In the next subsection, we shall make heavy use of the following result (the "Cramer-Jacobi rule"). Let $AX = Y$ where A, X, Y are $m \times m$, $m \times r$, $m \times r$ matrices, respectively, and let U denote the $r \times r$ matrix that is constituted by the rows $k_1, \ldots, k_r$ of X. Then $(\det A)(\det U)$ equals the determinant of the matrix that is obtained from A by replacing the k_1st$, \ldots, k_r$th columns with the 1st$, \ldots, r$th columns of Y. For example, if

$$\begin{pmatrix} a_{11} & \ldots & a_{14} \\ \vdots & & \vdots \\ a_{41} & \ldots & a_{44} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ \vdots & \vdots \\ x_{41} & x_{42} \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ \vdots & \vdots \\ y_{41} & y_{42} \end{pmatrix},$$

then

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & 0 & 0 \\ x_{21} & x_{22} & 1 & 0 \\ x_{31} & x_{32} & 0 & 0 \\ x_{41} & x_{42} & 0 & 1 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} & a_{12} & a_{14} \\ y_{21} & y_{22} & a_{22} & a_{24} \\ y_{31} & y_{32} & a_{32} & a_{34} \\ y_{41} & y_{42} & a_{42} & a_{44} \end{pmatrix},$$

whence

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix} = \begin{vmatrix} y_{11} & a_{12} & y_{12} & a_{14} \\ y_{21} & a_{22} & y_{22} & a_{24} \\ y_{31} & a_{32} & y_{32} & a_{34} \\ y_{41} & a_{42} & y_{42} & a_{44} \end{vmatrix},$$

and now it is clear how to prove what is asserted above in the general case.

3.3. The simple zeros case. Before quoting Gorodetsky's formula in full generality, let us explicate his result in the case where the determinant of the numerator matrix polynomial has only simple zeros.

Let $a(z) = a_s z^s + \ldots + a_0$ (with $a_k \in \mathbf{C}^{r\times r}$) be a matrix polynomial such that $\det a_s \neq 0$ and $\det a$ has rs pairwise distinct zeros $t_1, \ldots, t_{rs}$. Also let g and h be scalar polynomials of the form

$$h(z) = \prod_{j=1}^{q}(1 - z/c_j) \;\; (|c_j| > 1), \quad g(z) = \prod_{j=1}^{p}(z - d_j) \;\; (|d_j| < 1).$$

Assume that $q \geq 1$ and $p \geq 1$. Put $s = p+q$ and consider the rational matrix function $f = g^{-1}h^{-1}a$.

For $j = 1, \ldots, rs$, define the rows H_j and G_j by

$$H_j = h(t_j)(1, t_j, \ldots, t_j^{p-1}), \quad G_j = g(t_j)(1, t_j, \ldots, t_j^{q-1})$$

let $y_j = (y_{j1}, \ldots, y_{jr}) \in \mathbf{C}^r$ be any nonzero row-vector such that $y_j a(t_j) = 0$, and denote by $M_n(f)$ $(n \geq 0)$ the $rs \times rs$ matrix whose jth row is

$$(y_{j1}H_j, \ldots, y_{jr}H_j, y_{j1}t_j^n G_j, \ldots, y_{jr}t_j^n G_j). \tag{13}$$

The following theorem was stated in [17], [18].

Theorem. *We have* $\det M_0(f) \neq 0$ *and*

$$D_n(f) = (-1)^{qrn}(\det a_s)^n \frac{\det M_n(f)}{\det M_0(f)} \quad \textit{for all} \quad n \geq 1.$$

Proof outline. Fix $n \geq 1$ and suppose for the time being that $D_n(f) \neq 0$. Then the system

$$\begin{pmatrix} f_0 & \ldots & f_{-n+1} \\ \vdots & \ddots & \vdots \\ f_{n-1} & \ldots & f_0 \end{pmatrix} \begin{pmatrix} w_0 \\ \vdots \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (w_k \in \mathbf{C}^{r\times r}) \tag{14}$$

has a unique solution, and from the Cramer-Jacobi rule we infer that

$$\det w_0 = D_{n-1}(f)/D_n(f), \tag{15}$$

where $D_0(f) := 1$. Let $w(z) := \sum w_k z^k$. From (14) we obtain that

$$f(z)w(z) = I + c_-(z) + z^n c_+(z), \tag{16}$$

where

$$c_-(z) = \sum_{k\geq 1} c_k^- z^{-k} \quad \text{and} \quad c_+(z) = \sum_{k\geq 0} c_k^+ z^k$$

for all z in some annulus $1-\varepsilon < |z| < 1+\varepsilon$. Multiplying (16) by $g(z)$ we get

$$h^{-1}(z)a(z)w(z) = g(z)I + g(z)c_-(z) + z^n g(z)c_+(z), \tag{17}$$

and since $h^{-1}aw$, g, and c_+ are analytic in $\mathbf{D}$, so also is gc_-. Hence, gc_- is an analytic matrix polynomial of a degree not exceeding $p-1$. Multiplication of (17) by $h(z)$ gives

$$a(z)w(z) = h(z)g(z)I + h(z)g(z)c_-(z) + z^n g(z)h(z)c_+(z), \tag{18}$$

from which it is readily seen that hc_+ is an analytic matrix polynomial whose degree is at most $q-1$. Letting

$$u(z) := g(z)I - z^p I + g(z)c_-(z), \quad v(z) := h(z)c_+(z)$$

we can rewrite (18) in the form

$$a(z)w(z) = h(z)z^p I + h(z)u(z) + z^n g(z)v(z); \tag{19}$$

recall that u and v are analytic polynomials of a degree at most $p-1$ and $q-1$, respectively.

Now let $w_k(z)$, $u_k(z)$, $v_k(z)$ denote the kth column of $w(z)$, $u(z)$, $v(z)$, respectively. Putting $z = t_j$ in (19) and multiplying the result from the left by y_j we obtain that

$$0 = h(t_j)t_j^p y_{jk} + h(t_j)y_j u_k(t_j) + t_j^n g(t_j)y_j v_k(t_j). \tag{20}$$

In order to rewrite (20), let $M_n(f)$ denote the matrix whose jth row is (13) and let $w_{\ell k}^m$, $u_{\ell k}^m$, $v_{\ell k}^m$ stand for the coefficient of z^m of the ℓk entry of $w(z)$, $u(z)$, $v(z)$, respectively. So (20) assumes the form

$$M_n(f)(\, U_k \;\; V_k \,)^\top = Y_k^\top,$$

where

$$\begin{aligned}
U_k &= (u_{1k}^0, \dots, u_{1k}^{p-1}, \dots, u_{rk}^0, \dots, u_{rk}^{p-1}),\\
V_k &= (v_{1k}^0, \dots, v_{1k}^{q-1}, \dots, v_{rk}^0, \dots, v_{rk}^{q-1}),\\
Y_k &= -(h(t_1)t_1^p y_{1k}, \dots, h(t_{rs})t_{rs}^p y_{rs,k}).
\end{aligned}$$

Since, by the Cramer-Jacobi rule, $\det M_n(f)$ times

$$\det \begin{pmatrix} u_{11}^0 & \dots & u_{1r}^0 \\ \vdots & & \vdots \\ u_{r1}^0 & \dots & u_{rr}^0 \end{pmatrix} = \det u(0)$$

is equal to the determinant of the matrix that results from $M_n(f)$ by replacing the columns $1, p+1, \dots, (r-1)p+1$ by $Y_1^\top, Y_2^\top, \dots, Y_r^\top$, respectively, and since the latter determinant is obviously

$$(-1)^{pr}\Big(\prod_{j=1}^{rs} t_j\Big) \det M_{n-1}(f),$$

it follows that

$$(\det M_n(f))(\det u(0)) = (-1)^{pr}\Big(\prod_{j=1}^{rs} t_j\Big) \det M_{n-1}(f). \tag{21}$$

On the other hand, setting $z = 0$ in (19) we see that

$$a(0)w_0 = u_0, \tag{22}$$

and since

$$\det a(0) = (\det a_s)(-1)^{sr}\Big(\prod_{j=1}^{rs} t_j\Big),$$

we deduce from (21) and (22) that

$$\frac{\det M_{n-1}(f)}{M_n(f)} = (-1)^{qr}(\det a_s)(\det w_0). \tag{23}$$

Combining (23) and (15) finally implies that

$$\frac{\det M_{n-1}(f)}{M_n(f)} = (-1)^{qr}(\det a_s)\frac{D_{n-1}(f)}{D_n(f)}. \tag{24}$$

Thus, if $D_k(f) \neq 0$ for all $k \leq n$ then (24) yields that

$$\frac{D_n(f)}{D_0(f)} = (-1)^{qrn}(\det a_s)^n \frac{\det M_n(f)}{\det M_0(f)}, \tag{25}$$

and since $D_0(f) = 1$, we arrive at the asserted formula.

By continuity, formula (25) also holds without our working hypothesis that $D_k(f)$ be nonzero for all $k \leq n$. ■

3.4. Day's formula. An "exact" formula for Toeplitz determinants generated by scalar-valued rational functions with simple zeros was first given by Day [12] in the middle of the seventies. To state this formula, let $a(z) = a_s(z - t_1) \dots (z - t_s)$, where $t_1, \dots, t_s$ are pairwise distinct complex numbers, and let $h(z)$ and $g(z)$ be as in 3.3. Assume $p \geq 1$, $q \geq 1$, $s \geq p + q$. Also as above, put $f = g^{-1}h^{-1}a$.

Theorem. *We have*

$$D_n(f) = \sum_M C_M r_M^n \quad \textit{for all} \quad n \geq 1,$$

the sum taken over all $\binom{s}{p}$ subsets $M \subset \{1, \dots, s\}$ of cardinality $|M| = p$. Letting $\overline{M} := \{1, \dots, s\} \setminus M$, $P := \{1, \dots, p\}$, $Q := \{1, \dots, q\}$, the constants C_M and r_M can be given by

$$r_M = (-1)^{s-p} a_s \prod_{k \in \overline{M}} t_k,$$

$$C_M = \prod_{k \in \overline{M}, m \in P} (t_k - d_m) \prod_{\ell \in Q, j \in M} (c_\ell - t_j) \prod_{\ell \in Q, m \in P} (c_\ell - d_m)^{-1} \prod_{k \in \overline{M}, j \in M} (t_k - t_j)^{-1}.$$

This theorem was established by Day in [12]. For Toeplitz determinants generated by Laurent polynomials (which is the case $q = 0$ and $d_j = 0$ for all j), the theorem was obtained by Widom [29] in the late fifties (see also [24]). In the $s = p+q$ case, the theorem results from the theorem in 3.3 (for $r = 1$) by expanding $\det M_n(f)$ according to Laplace's theorem with respect to the first p columns and computing the arising Vandermonde determinants. For $s > p + q$, the theorem can be derived in the same way from the theorem that will follow in 3.7. Independent proofs are in [12], [22], [9].

3.5. Assumptions. We now describe the class of rational matrix functions which are tractable by Gorodetsky's formula in the general case. Let $a(z) = a_s z^s + \ldots + a_0$ ($a_k \in \mathbf{C}^{r\times r}$) be a matrix polynomial and suppose that a_s is invertible and that

$$\det a(z) = (\det a_s) \prod_{j=1}^{R} (z - t_j)^{m_j},$$

where $t_1, \ldots, t_R$ are pairwise distinct and $m_1 + \ldots + m_R = rs$. Again let

$$g(z) = \prod_{j=1}^{p} (z - d_j) \quad (|d_j| < 1), \qquad h(z) = \prod_{j=1}^{q} (1 - z/c_j) \quad (|c_j| > 1),$$

and assume that $p \geq 1$, $s \geq p + 1$, $s \geq p + q$. Put $f = g^{-1}h^{-1}a$.

Notice that q is allowed to be zero (in which case $h(z) = 1$); if $q = 0$ and $g(z) = z^p$, then (f_{i-j}) is a banded matrix. The restriction to $p \geq 1$ and $s \geq p + 1$ is no loss of generality: if $p = 0$ or if $s = p$ (and hence, because $s \geq p + q$, also $q = 0$), then (f_{i-j}) is block triangular. Finally, if $s < p + q$ but a_0 is invertible, one may replace $f(z)$ by $f(1/z)$ (which does not affect $D_n(f)$) to obtain a matrix function satisfying $s \geq p+q$.

3.6. The Smith canonical form. (See e.g. [16].) Given a matrix polynomial $a(z)$ as in 3.5, there exist (analytic) matrix polynomials $y(z)$ and $w(z)$ of the size $r \times r$ such that the determinants $\det y(z)$ and $\det w(z)$ are nonzero and independent of z and

$$y(z)a(z)w(z) = \operatorname{diag}\Big(\prod_{j=1}^{R} (z - t_j)^{m_{j1}}, \ldots, \prod_{j=1}^{R} (z - t_j)^{m_{jr}} \Big)$$

with $0 \leq m_{j1} \leq \ldots \leq m_{jr}$ for $j = 1, \ldots, R$. Obviously, $m_{j1} + \ldots + m_{jr} = m_j$. The occurring diagonal matrix polynomial is referred to as the Smith canonical form of $a(z)$.

The exponents m_{jk} can be determined as follows. Let $e_k(z)$ ($k = 1, \ldots, r - 1$) denote the greatest common divisor of all minors of the size $k \times k$ of $a(z)$ and put $e_0(z) = 1$ and $e_r(z) = \det a(z)$. Then $\prod_{j=1}^{R} (z - t_j)^{m_{jk}}$ is a constant multiple of the polynomial $e_k(z)/e_{k-1}(z)$.

3.7. Gorodetsky's formula. Let $f = g^{-1}h^{-1}a$ be as in 3.5 and y, w, m_{jk} be as in 3.6. Define rows $H(z)$ and $G(z)$ by

$$H(z) = h(z)(1, z, \ldots, z^{p-1}), \quad G(z) = g(z)(1, z, \ldots, z^{s-p-1}),$$

and for $k = 1, \dots, r$, let $F_k(z)$ stand for the row

$$F_k(z) = \Big(y_{k1}(z)H(z), \dots, y_{kr}(z)H(z), y_{k1}(z)z^n G(z), \dots, y_{kr}(z)z^n G(z)\Big);$$

here $y_{k\ell}(z)$ is the $k\ell$ entry of the matrix polynomial $y(z)$. Note that the length of $F_k(z)$ is $rp + r(s-p) = rs$. Then put $F_k^{(\ell)}(z) = (d/dz)^\ell F_k(z)$ and let $M_n(f)$ denote the matrix whose rs rows are

$$F_k^{(\ell)}(t_j) \quad (j = 1, \dots, R; \quad k = 1, \dots, r; \quad \ell = 0, \dots, m_{jk} - 1),$$

arranged so that (j, k, ℓ) is in lexicographic order. If $m_{jk} = 0$, then $M_n(f)$ does not contain rows of the form $F_k^{(\ell)}(t_j)$.

Theorem. *We have* $\det M_0(f) \neq 0$ *and*

$$D_n(f) = (-1)^{r(s-p)n} (\det a_s)^n \frac{\det M_n(f)}{\det M_0(f)} \quad \textit{for all} \quad n \geq 1.$$

This theorem appears in [17],[18]. As to my knowledge, Gorodetsky has never published a proof (the proof outline given in 3.3 for the case of simple zeros is much more than what is contained in [17], [18]). In the scalar case a theorem similar to the one stated above was found independently by Trench [28].

3.8. Asymptotic formulas. The asymptotic behavior of the determinants $\det M_n(f)$ (and thus, by 3.7, of $D_n(f)$) as n approaches infinity is studied in [6]. There it is shown that $\log D_n(f)$ is always asymptotically equal to $A(f)n + B(f)\log n + O(1)$ as $n \to \infty$, and the constants $A(f)$ and $B(f)$ are identified. To determine $A(f)$ one needs only to know the moduli $|t_1|, \dots, |t_R|$ and the multiplicities $m_1, \dots, m_R$ of the zeros of $\det f$, whereas the identification of the coefficient $B(f)$ requires the knowledge of all the exponents m_{jk} of the Smith canonical form of the numerator of f. If, in addition, the matrix polynomial y appearing in 3.6 is available, then the $O(1)$ can be made more precise.

4 Quasitriangular Toeplitz matrices

4.1. Renormalization. Our subject now is the determinants of quasitriangular (block) Toeplitz matrices of the form

$$\begin{pmatrix} a_p & \dots & a_0 & 0 & \dots & 0 \\ a_{p+1} & \dots & a_1 & a_0 & \dots & 0 \\ \vdots & \ddots & & \ddots & \ddots & \vdots \\ a_{n-1} & & \ddots & & \ddots & a_0 \\ a_n & & & \ddots & & a_1 \\ \vdots & & & & \ddots & \vdots \\ a_{p+n-1} & \dots & \dots & \dots & \dots & a_p \end{pmatrix}, \tag{26}$$

where $a_0, a_1, \ldots$ is a given sequence of $r \times r$ matrices and a_0 is assumed to be invertible.

If the matrix function

$$a(z) = a_0 + a_1 z + a_2 z^2 + \ldots \tag{27}$$

is analytic in a disk $\{|z| < 1 + \varepsilon\}$, then

$$a_0 z^{-p} + \ldots + a_{p-1} z^{-1} + a_p + a_{p+1} z + \ldots$$

is the Laurent series of $z^{-p} a(z)$ in $\{0 < |z| < 1 + \varepsilon\}$ and so the determinant of (26) may be written as $D_n(z^{-p} a(z))$.

Throughout this section, however, we shall merely assume that $a(z)$ is analytic in some open neighborhood of $z = 0$. This implies that if $w > 0$ is a sufficiently small real number, then the matrix function $a_w(z) := a(wz)$ is analytic and invertible in a disk $\{|z| < 1 + \varepsilon\}$ (recall that a_0 is supposed to be invertible). Because

$$\det(f_{i-j})_{i,j=1}^n = \det(w^{i-j} f_{i-j})_{i,j=1}^n$$

for every $w \neq 0$ (this is the "renormalization trick", which was possibly first employed by Schmidt and Spitzer [24]), we see that the determinant of (26) equals $D_n((wz)^{-p} a(wz))$. Thus, the assumption that $a(z)$ be analytic and invertible in a disk larger than $\mathbf{D}$ is actually no loss of generality in comparison with the requirement that $a(z)$ be analytic in an open neighborhood of $z = 0$ and that a_0 be invertible.

Instead of assuming that, without loss of generality, $a(z)$ is analytic and invertible in some disk $\{|z| < 1 + \varepsilon\}$, we redefine $D_n(z^{-p} a(z))$. Namely, given a matrix function $a(z)$ of the form (27) which is analytic and invertible in an open neighborhood of $z = 0$, we denote in this and the next section by $D_n(z^{-p} a(z)$ the determinant of (26), that is, of the matrix $(a_{p+i-j})_{i,j=1}^n$.

4.2. The Baxter-Schmidt-Gorodetsky formula. Let the matrix function $a(z)$ given by (27) be analytic and invertible in an open neighborhood of $z = 0$ and let $b(z) := a^{-1}(z)$ in this neighborhood of $z = 0$.

Theorem. *We have*

$$D_n(z^{-p} a(z)) = (-1)^{prn} (\det a_0)^{n+p} D_p(z^{-n} b(z)) \quad \text{for all} \quad n \geq 1.$$

In the scalar case, this beautiful theorem was established by Baxter and Schmidt [5] at the beginning of the sixties. The matrix case version was published (and accompanied by a full proof) by Gorodetsky [19] only in this year. The following proof is different from the one of [19], it is an extraction of a proof by Tismenetsky [26] for banded block Toeplitz matrices and a remark of an (anonymous) referee of [26], cited on page 173 of that paper. This proof works for $n \geq p$ only.

Proof. Let $b(z) = b_0 + b_1 z + b_2 z^2 + \ldots$. The product of matrix (26) and the matrix

$$\begin{pmatrix} b_0 & 0 & & & & & & \\ b_1 & b_0 & 0 & & & & & \\ \vdots & \vdots & \vdots & \ddots & & & & \\ \vdots & \vdots & \vdots & \ddots & b_0 & 0 & & \\ \vdots & \vdots & \vdots & & \vdots & I & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & b_{n-2} & b_{n-3} & \ldots & b_{n-p} & 0 & \ldots & I \end{pmatrix}$$

is a block matrix

$$\begin{pmatrix} 0 & D \\ -R & * \end{pmatrix}$$

in which R is the $p \times p$ matrix whose $1\text{st}, 2\text{nd}, \ldots, p\text{th}$ columns are

$$\begin{aligned} &(a_0 b_n, \ldots, a_{p-1} b_n + \ldots + a_0 b_{p+n-1})^\top, \\ &(a_0 b_{n-1}, \ldots, a_{p-1} b_{n-1} + \ldots + a_0 b_{p+n-2})^\top, \\ &\ldots, \\ &(a_0 b_{n-p+1}, \ldots, a_{p-1} b_{n-p+1} + \ldots + a_0 b_n)^\top, \end{aligned}$$

respectively, and D is a lower-triangular $(n-p) \times (n-p)$ matrix all diagonal entries of which equal a_0. Hence

$$D_n(z^{-p} a(z))(\det b_0)^p = (-1)^{(n-p)pn} (\det R)(\det a_0)^{n-p},$$

and since

$$R = \begin{pmatrix} a_0 & & & \\ a_1 & a_0 & & \\ \vdots & \vdots & \ddots & \\ a_{p-1} & a_{p-2} & \ldots & a_0 \end{pmatrix} \begin{pmatrix} b_n & \ldots & b_{n-p+1} \\ \vdots & \ddots & \vdots \\ b_{n+p-1} & \ldots & b_n \end{pmatrix},$$

we obtain that

$$\det R = (-1)^{pn} (\det a_0)^p D_p(z^{-n} b(z)),$$

which implies the asserted formula at once. ■

4.3. Generalizations. We remark that Gorodetsky [19] did not only prove the preceding theorem, but even found an exact formula for $D_n(f)$ in the case where $f(z) = g^{-1}(z) z^m a(z)$, the function $g(z)$ is a scalar polynomial without zeros on $\mathbf{T}$, m is an integer, and $a(z)$ is an analytic and invertible (!) matrix function in some disk $\{|z| < 1 + \varepsilon\}$.

Note that the (block) Toeplitz matrices generated by such more general matrix functions are in general not quasitriangular. However, we wish to emphasize that the requirement that $a(z)$ be invertible in the closed unit disk is a serious restriction; it cannot be removed by the renormalization trick in the presence of the function $g(z)$.

5 Toeplitz band matrices

5.1. Consequences of the Baxter-Schmidt-Gorodetsky formula. We now consider the determinants $D_n(z^{-p}a(z))$ of the banded block Toeplitz matrices

$$\begin{pmatrix} a_p & \dots & \dots & a_0 & & & \\ \vdots & \ddots & & & \ddots & & \\ \vdots & & \ddots & & & \ddots & a_0 \\ \vdots & & & \ddots & & & \vdots \\ a_s & & & & \ddots & & \vdots \\ & \ddots & & & & \ddots & \vdots \\ & & a_s & \dots & \dots & & a_p \end{pmatrix}, \tag{28}$$

where a_k are $r \times r$ matrices and a_0 is supposed to be invertible. We also assume that $1 \le p < s$. The matrix function

$$a(z) = a_0 + a_1 z + \ldots + a_s z^s$$

is then invertible in a neighborhood $\{|z| < \varepsilon\}$ of the origin and we may define

$$a^{-1}(z) = b(z) = b_0 + b_1 z + b_2 z^2 + \ldots$$

in this neighborhood of zero. From the theorem in 4.2 we infer that

$$D_n(z^{-p}a(z)) = (-1)^{prn}(\det a_0)^{n+p} D_p(z^{-n}b(z)) \tag{29}$$

for all $n \ge 1$. Notice that the size of

$$D_p(z^{-n}b(z)) = \det \begin{pmatrix} b_n & \dots & b_{n-p+1} \\ \vdots & \ddots & \vdots \\ b_{n+p-1} & \dots & b_n \end{pmatrix} \tag{30}$$

is independent of n.

The Taylor coefficients b_j can be expressed in the form

$$b_j = \frac{1}{2\pi i} \int_{|z|=\varepsilon/2} a^{-1}(z) z^{-j-1} dz. \tag{31}$$

Realizing $a^{-1}(z)$ in the form

$$a^{-1}(z) = a_0^{-1}[I + zC(I - zA)^{-1}B] = a_0^{-1}\Big(I + \sum_{j=1}^{\infty} z^j C A^{j-1} B\Big),$$

we also obtain that $b_0 = a_0^{-1}$ and

$$b_j = a_0^{-1} C A^{j-1} B \tag{32}$$

for all $j \geq 1$. The two formulas resulting from combining (29), (30) with (31) on the one hand hand and with (32) on the other are Theorem 2 of Tismenetsky's paper [26] (the exponent $(n-p)r^2+p$ of -1 given there is incorrect). In [26] one can also find expressions for (30) in terms of so-called solvents.

We remark that Tismenetsky's article was written some years before Gorodetsky's work [19], but that the proof given in 4.2 makes heavy use of the approach of Tismenetsky. But Tismenetsky has still another formula ...

5.2. Tismenetsky's formula. Let $a(z)$ be as in the preceding subsection. There we derived an exact formula for the determinant $D_n(z^{-p}a(z))$ in terms of a $pr \times pr$ block Toeplitz matrix. Our aim here is to express $D_n(z^{-p}a(z))$ via an $sr \times sr$ matrix. Although $sr > pr$, the formula that will established here has some computational advantages.

The matrix function $(a_s + a_{s-1}z + \ldots + a_0 z^s)^{-1}$ vanishes at infinity. Hence we can find matrices $C \in \mathbf{C}^{r \times rs}$, $A \in \mathbf{C}^{rs \times rs}$, $B \in \mathbf{C}^{rs \times r}$ such that

$$(a_s + a_{s-1}z + \ldots + a_0 z^s)^{-1} = C(zI - A)^{-1}B.$$

It follows (see e.g. [16]) that

$$a_s C + a_{s-1}CA + \ldots + a_0 CA^s = 0. \tag{33}$$

Put $q = s - p + 1$ and denote by V_n the $rs \times rs$ matrix

$$V_n = \begin{pmatrix} C \\ \vdots \\ CA^{q-2} \\ CA^{q-1+n} \\ \vdots \\ CA^{s-1+n} \end{pmatrix}.$$

The following theorem as well as the proof that will be given below are Tismenetsky's [26].

Theorem. *We have* $\det V_0 \neq 0$ *and*

$$D_n(z^{-p}a(z)) = (-1)^{qrn}(\det a_0)^n \frac{\det V_n}{\det V_0} \quad \textit{for all} \quad n \geq s+1.$$

Proof outline. Let F_n stand for the $rn \times rn$ matrix

$$F_n = \begin{pmatrix} S & \begin{matrix} 0 \\ I \end{matrix} \end{pmatrix}$$

where S is the $rn \times rs$ matrix

$$S = \begin{pmatrix} CA^{q-1} \\ CA^{q} \\ \vdots \\ CA^{q-1+n-1} \end{pmatrix},$$

0 is the $rs \times r(n-s)$ zero matrix, and I is the $r(n-s) \times r(n-s)$ identity matrix. Multiply (28) from the right by F_n. Using (33) we see that the resulting matrix G_n is of the form

$$G_n = \begin{pmatrix} U_1 & 0_1 \\ 0_2 & T \\ U_2 & * \end{pmatrix},$$

where $0_1 \in \mathbf{C}^{r(q-1)\times r(n-s)}$ and $0_2 \in \mathbf{C}^{r(n-s)\times rs}$ are the zero matrices, the matrices $U_1 \in \mathbf{C}^{r(q-1)\times rs}$ and $U_2 \in \mathbf{C}^{rp\times rs}$ are given by

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = -\begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} V_n,$$

$$W_1 = \begin{pmatrix} a_s & \dots & a_{q-2} \\ & \ddots & \vdots \\ & & a_s \end{pmatrix}, \quad W_2 = \begin{pmatrix} a_0 & & \\ \vdots & \ddots & \\ a_{p-1} & \dots & a_0 \end{pmatrix},$$

and $T \in \mathbf{C}^{r(n-s)\times r(n-s)}$ is lower triangular with all diagonal entries equal to a_0. Since

$$D_n(z^{-p}a(z))(\det F_n) = \det G_n$$

and

$$\det F_n = \det \begin{pmatrix} CA^{q-1} \\ \vdots \\ CA^{q-1+s-1} \end{pmatrix} = (\det V_0)(\det A)^{q-1},$$

$$\det G_n = (-1)^{(n-s)rq}(\det a_0)^{n-s} \det \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},$$

$$\det \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = (-1)^{rs}(\det a_s)^{q-1}(\det a_0)^p(\det V_n),$$

$$\det A = (-1)^{rs}\frac{\det a_s}{\det a_0}$$

(the last formula following from the similarity theorem for minimal realizations along with the fact that A can be chosen as a companion matrix), we arrive at the desired formula. ■

5.3. Trench's formula. Now let $a(z)$ be a scalar polynomial with pairwise distinct zeros $t_1, \dots, t_R$ of the multiplicities $m_1, \dots, m_R$:

$$a(z) = a_0 + \dots + a_s z^s = a_s \prod_{j=1}^{R}(z - t_j)^{m_j}.$$

Assume $a_0 \neq 0$ and $a_s \neq 0$. Given integers n and p such that $n > s$ and $1 \leq p < s$, put $q = s - p + 1$ and denote by $v(z)$ the column

$$v(z) = (1, z, \dots, z^{q-2}, z^{q-1+n}, \dots, z^{s-1+n})^\top.$$

Let $v^{(j)}(z)$ be the jth derivative of $v(z)$ and denote by V_n the $s \times s$ matrix whose first m_1 columns are $v(t_1), \ldots, v^{(m_1-1)}(t_1)$, whose next m_2 columns are $v(t_2), \ldots, v^{(m_2-1)}(t_2)$, etc.

Theorem. *We have* $\det V_0 \neq 0$ *and*

$$D_n(z^{-p}a(z)) = (-1)^{qn} a_0^n \frac{\det V_n}{\det V_0} \quad \textit{for all} \quad n \geq 1.$$

This theorem was established by Trench [27] for $n \geq s+1$ and subsequently by Berg [4] for all $n \geq 1$. Tismenetsky [26] has shown how this theorem (for $n \geq s+1$) can be derived from his theorem in 5.2: it suffices to verify that (33) is satisfied by

$$C = (Q_1, \ldots, Q_R), \quad A = \operatorname{diag}(J_1, \ldots, J_R),$$

$$Q_j = \underbrace{(1, 0, \ldots, 0)}_{m_j}, \quad J_j = \begin{pmatrix} t_j & 1 & & & \\ & t_j & 1 & & \\ & & \ddots & \ddots & \\ & & & t_j & 1 \\ & & & & t_j \end{pmatrix} \in \mathbf{C}^{m_j \times m_j}.$$

Clearly, there is great similarity between the Trench-Tismenetsky formulas and the formulas by Gorodetsky quoted in 3.7.

6 The BGKvS formula for Wiener-Hopf determinants

6.1. Preliminaries. Henceforth let f be a rational matrix function which has no poles on $\mathbf{R}$ and equals I at infinity. Define k, $W_\tau(k)$, $\widetilde{D}_\tau(f)$, and $D_\tau(f)$ as in Section 1.

Let $f(z) = I + C(zI - A)^{-1}B$ be a realization of the form (1) and put $A^\times = A - BC$. We denote by U_+ any open, bounded, connected subset of the upper complex half-plane containing all eigenvalues of A (= poles of f) in the upper complex half-plane and we let P stand for the Riesz projection $\chi_{U_+}(A)$. We then have

$$k(t) = \begin{cases} iCe^{-itA}(I-P)B & \text{for} \quad t > 0, \\ -iCe^{-itA}PB & \text{for} \quad t < 0. \end{cases}$$

The following conditions are easily seen to be equivalent:

(i) $W_\tau(k)$ is a trace class operator;
(ii) $\hat{k} = (I - f)|\mathbf{R}$ belongs to $L^1_{r\times r}(\mathbf{R})$;
(iii) $k(0-0) = k(0+0)$;
(iv) $CB = 0$.

6.2. The BGKvS formula. This is the following expressions for the determinants $\widetilde{D}_\tau(f)$ and $D_\tau(f)$.

Theorem. *We have*

$$\begin{aligned}\widetilde{D}_\tau(f) &= e^{\tau g(f)}(\det Pe^{iA}P|\mathrm{Im}\,P)^\tau(\det Pe^{-i\tau A^\times}P|\mathrm{Im}\,P)\\ &= e^{\tau g(f)}\det(I-P+Pe^{i\tau A}e^{-i\tau A^\times})\end{aligned}$$

for all $\tau>0$ *with*

$$g(f) = -\mathrm{tr}\,\frac{1}{2\pi}\int_{\partial U_+} f(z)dz.$$

If $W_\tau(k)$ *is of trace class, then*

$$D_\tau(f) = (\det Pe^{iA}P|\mathrm{Im}\,P)^\tau(\det Pe^{-i\tau A^\times}P|\mathrm{Im}\,P)$$

for all $\tau>0$.

In the trace class case this theorem follows from the results of [1], [2], a proof for the Hilbert-Schmidt case is in [14]. The proof proceeds along the lines of the proof given in 2.5. Viz, one can show that

$$I-P+Pe^{i\tau A}e^{-i\tau A^\times}$$

is an indicator of $I-W_\tau(k)$, which implies that

$$\widetilde{D}_\tau(f) = e^{\mathrm{tr}\,K}\det(I-P+Pe^{i\tau A}e^{-i\tau A^\times})$$

where $K: L^2_r(0,\tau)\to L^2_r(0,\tau)$ is the finite rank operator

$$(Kf)(t) = -\int_0^\tau iCe^{i(t-s)A}PB\,f(s)ds \quad (0<t<\tau),$$

and the trace of K can be shown to be $-\tau g(f)$. If $W_\tau(k)$ is a trace class operator, then simply

$$D_\tau(f) = \det(I-P+Pe^{i\tau A}e^{-i\tau A^\times}).$$

6.3. The Achiezer-Kac-Mikaelyan formula. Let f be as in 6.1. In addition, assume now that

$$\det f(x)\neq 0 \ \text{ for all } \ x\in\mathbf{R}, \quad \arg\det f(x)\Big|_{x=-\infty}^{\infty} = 0.$$

Theorems of the Achiezer-Kac-Mikaelyan type say that then

$$\lim_{\tau\to\infty}\frac{\widetilde{D}_\tau(f)}{\widetilde{G}(f)^\tau} = E(f),$$

where $\widetilde{G}(f)\neq 0$ and $E(f)$ are certain constants (see [20], [21], [23], [12], [31], [11] and the references in these works). Under our assumptions, $\det f$ possesses a continuous logarithm $\log\det f$ on the real line $\mathbf{R}$. We have (see [31])

$$\widetilde{G}(f) = \exp\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}[\log\det f(x) + \mathrm{tr}\,(I-f(x))]dx\right)$$

and

$$E(f) = \det W(f)W(f^{-1}), \tag{34}$$

where $W(f)$ and $W(f^{-1})$ stand for the Wiener-Hopf operators on the space $L^2_r(0,\infty)$ with the symbols f and f^{-1}, respectively. The product $W(f)W(f^{-1})$ can be shown to differ from I by a trace class operator only. In the scalar case ($r = 1$) we also have

$$E(f) = \exp \int_{-\infty}^{\infty} xs(x)s(-x)dx$$

where s is defined by $\hat{s}(x) = \log f(x)$ for $x \in \mathbf{R}$. If $W_\tau(k)$ is of trace class, then

$$\lim_{\tau\to\infty} \frac{D_\tau(f)}{G(f)^\tau} = E(f)$$

where $E(f)$ is as above and $G(f) \neq 0$ can be given by

$$G(f) = \exp\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \log\det f(x)dx\right).$$

For other identifications of $E(f)$ see [23] and [13]. We remark that Dym and Ta'assan [13] provide an approach to the subject that amalgamates the Szegö-Widom and Achiezer-Kac-Mikaelyan theorems into a single (abstract) theorem.

6.4. The Gohberg-Kaashoek-van Schagen version. Assume f is as in 6.3. Let U be an open, bounded, connected subset of the complex plane that contains all poles of f (= eigenvalues of A) and let $U_+^\times$ be an open, bounded, connected subset of the upper half-plane containing all poles of f^{-1} (= eigenvalues of $A^\times$) in the upper half-plane. We then have

$$\begin{aligned}&\frac{1}{4\pi}\int_{\partial U} \operatorname{tr} f(z)\,dz = \frac{i}{2}\operatorname{tr}\left(C\,\frac{1}{2\pi i}\int_{\partial U}(zI-A)^{-1}dz\,B\right)\\ &= \frac{i}{2}\operatorname{tr} CB = \frac{i}{2}\operatorname{tr} BC = \frac{i}{2}(\operatorname{tr} A - \operatorname{tr} A^\times),\end{aligned}$$

and this quantity vanishes whenever $BC = 0$, i.e., whenever $W_\tau(k)$ is of trace class. Finally, put $P^\times = \chi_{U_+^\times}(A^\times)$.

By arguments similar to those of Section 2.7, Gohberg, Kaashoek, and van Schagen [14] derived the results of 6.3 from their formula cited in 6.2. They obtained that

$$\begin{aligned}G(f) &= e^{g(f)}\exp\left(\frac{1}{2\pi}\,\text{p.v.}\int_{-\infty}^{\infty}\log\det f(x)dx + \frac{1}{4\pi}\int_{\partial U}\operatorname{tr} f(z)\,dz\right)\\ &= e^{g(f)}\frac{\det Pe^{iA}P|\operatorname{Im} P}{\det P^\times e^{iA^\times}P^\times|\operatorname{Im} P^\times},\end{aligned}$$

and their expression for $E(f)$ is

$$E(f) = \det[(I-P)(I-P^\times) + PP^\times] = \det(I - P - P^\times). \tag{35}$$

Widom [31] gave a direct proof of the fact that the right-hand sides of (34) and (35) coincide.

7 Wiener-Hopf analogues of formulas by Gorodetsky and Day

7.1. General assumptions. Every rational matrix function f which has no poles on $\mathbf{R}$ and is I at infinity can be written in the form $f = g^{-1}h^{-1}a$ where

$$a(z) = z^s I + z^{s-1}a_{s-1} + \ldots + a_0 \quad (a_k \in \mathbf{C}^{r\times r}),$$

$$h(z) = \prod_{j=1}^{q}(z + ic_j) \quad (\operatorname{Re} c_j > 0), \qquad g(z) = \prod_{j=1}^{p}(z - id_j) \quad (\operatorname{Re} d_j > 0),$$

and $s = p + q$. If $p = 0$ or $q = 0$, then $W_\tau(k)$ is of Volterra type and therefore $\widetilde{D}_\tau(f) = 1$ for all $\tau > 0$. Thus, we consistently assume that $p \geq 1$ and $q \geq 1$.

7.2. The simple zeros case. Let $f = g^{-1}h^{-1}a$ be as in 7.1. In addition, suppose now that $\det a(z)$ has rs pairwise distinct zeros $t_1, \ldots, t_{rs}$ in $\mathbf{C}$. For $j = 1, \ldots, rs$, let H_j and G_j denote the rows

$$H_j = h(t_j)(1, t_j, \ldots, t_j^{p-1}), \quad G_j = g(t_j)(1, t_j, \ldots, t_j^{q-1}),$$

and let $y_j = (y_{j1}, \ldots, y_{jr})$ be any nonzero row-vectors such that $y_j a(t_j) = 0$. Given $\tau \geq 0$, we denote by $N_\tau(f)$ the $rs \times rs$ matrix whose jth row equals

$$(y_{j1}H_j, \ldots, y_{jr}H_j, y_{j1}e^{it_j\tau}G_j, \ldots, y_{jr}e^{it_j\tau}G_j).$$

Define finally $\delta(f)$ by

$$\delta(f) = \operatorname{tr} k(0+0) - r\sum_{j=1}^{q} c_j = \operatorname{tr} k(0-0) - r\sum_{j=1}^{p} d_j.$$

Theorem. *We have* $\det N_0(f) \neq 0$ *and*

$$\widetilde{D}_\tau(f) = e^{\tau\delta(f)}\frac{\det N_\tau(f)}{\det N_0(f)} \quad \textit{for all} \quad \tau > 0.$$

This theorem was established in [8] and is a Wiener-Hopf analogue of Gorodetsky's formula in 3.3. The article [8] contains a complete proof. The proof is rather long, and so we confine ourselves to merely pointing out some of its main ideas.

The first step consists in "discretizing" the determinants $\widetilde{D}_\tau(f)$. More precisely, one can show that

$$\widetilde{D}_\tau(f) = \det{}_2(I - W_\tau(k)) = \lim_{n\to\infty} D_n(f_{n,\tau})$$

where

$$f_{n,\tau}(z) = I - \frac{\tau}{n}\sum_{j=1}^{\infty} k\left(j\,\frac{\tau}{n}\right) z^j - \frac{\tau}{n}\sum_{j=1}^{\infty} k\left(-j\,\frac{\tau}{n}\right) z^{-j}.$$

Thus, $\widetilde{D}_\tau(f)$ is represented as the limit of a sequence of rationally generated block Toeplitz determinants both the size n and the generating function $f_{n,\tau}$ of which depend on n. These determinants are computed using the theorem in 3.3, which reduces the problem to finding the limit of

$$\frac{\det M_n(f_{n,\tau})}{\det M_0(f_{n,\tau})}$$

as n goes to infinity. This limit passage requires precise information about the zeros of $f_{n,\tau}$. Namely, one must verify that the zeros of $f_{n,\tau}$ are of the form

$$1 + it_j \frac{\tau}{n} + O\left(\frac{1}{n^2}\right).$$

Showing this is the difficult part of the proof and is also the point where the proof makes heavy use of the circumstance that all zeros of $\det a$ are simple.

7.3. Wiener-Hopf analogues of Gorodetsky's formula. I do presently not know any sufficiently general Wiener-Hopf analogue of the formula contained in 3.7. A few results in this direction were established in [8] by first perturbing $a(z)$ so as to obtain a matrix function whose determinant has only simple zeros, then applying the theorem of 7.2, and after that removing the perturbation. Here is a sample result achieved in this way.

We first need two notations: given any two scalar polynomials

$$u(z) = u_0 + u_1 z + \ldots + u_m z^m, \quad v(z) = v_0 + v_1 z + \ldots + v_k z^k,$$

we denote by $\mathrm{Res}\,(u, v)$ their resultant,

$$\mathrm{Res}\,(u,v) = \det \begin{pmatrix} u_0 & \ldots & \ldots & u_m & & \\ & \ddots & & & \ddots & \\ & & u_0 & \ldots & \ldots & u_m \\ v_0 & \ldots & v_k & & & \\ & \ddots & & \ddots & & \\ & & \ddots & & \ddots & \\ & & & v_0 & \ldots & v_k \end{pmatrix},$$

which has k rows with u's and m rows with v's, and by $R_{j,j}(u, v)$ the $2j \times 2j$ determinant

$$R_{j,j}(u,v) = \det \begin{pmatrix} u_0 & u_1 & \ldots & \ldots & \ldots \\ & \ddots & \ddots & & \\ & & u_0 & u_1 & \ldots \\ v_0 & v_1 & \ldots & \ldots & \ldots \\ & \ddots & \ddots & & \\ & & v_0 & v_1 & \ldots \end{pmatrix},$$

having j rows with u's and j rows with v's.

Let now $f = g^{-1}h^{-1}a$ be as in 7.1. Assume $r = 2$ and $\det a(z) = z^{2s}$. Denote by a_{11} and a_{12} the $1,1$ and $1,2$ entries of a and suppose a_{11} and a_{12} have no common divisor. This guarantees that $\mathrm{Res}\,(a_{11}, a_{12}) \neq 0$.

Theorem. *We have the asymptotic formula*

$$\widetilde{D}_\tau(f) = e^{\tau\delta(f)} \Big[\alpha(f)\beta(f)\,\tau^{4pq} + O(\tau^{4pq-1})\Big] \quad as \quad \tau \to \infty,$$

where

$$\alpha(f) = (-1)^{pq}\Big(\prod_{j=1}^{q} c_j^{2p}\Big)\Big(\prod_{j=1}^{p} d_j^{2p}\Big)\frac{G(2p+1)G(2q+1)}{G(2p+2q+1)},$$

$$G(m) := (m-2)!\dots 2!1!0! \quad (m \geq 2), \qquad G(1) := 1,$$

$$\beta(f) = \frac{R_{p,p}(a_{12}, a_{11})\, R_{q,q}(a_{12}, a_{11})}{[\mathrm{Res}\,(g,h)]^2\, \mathrm{Res}\,(a_{12}, a_{11})}.$$

7.4. The Wiener-Hopf analogue of Day's formula. Again let $f = g^{-1}h^{-1}a$ be as in 7.1. Assume now that $r = 1$ and that $a(z)$ has s pairwise distinct zeros $t_1, \dots, t_s$.

Theorem. *We have*

$$\widetilde{D}_\tau(f) = e^{\tau\delta(f)} \sum_M W_M e^{w_M \tau} \quad for\ all \quad \tau > 0.$$

Here the sum is taken over all sets $M \subset \{1, \dots, s\}$ with exactly p elements, and letting $\overline{M} = \{1, \dots, s\} \setminus M$, $P = \{1, \dots, p\}$, $Q = \{1, \dots, q\}$, $i = \sqrt{-1}$, we have

$$w_M = \sum_{k \in \overline{M}} it_k,$$

$$W_M = \prod_{k\in\overline{M}, m\in P} (it_k + d_m) \prod_{\ell\in Q, j\in M} (c_\ell - it_j) \prod_{\ell\in Q, m\in P} (c_\ell + d_m)^{-1} \prod_{k\in\overline{M}, j\in M} (it_k - it_j)^{-1}.$$

This theorem was established in [7] and proved there by the argument sketched in 7.2. Of course, in the same way as Day's formula 3.4 may be derived from the theorem in 3.3 by using the Laplace expansion theorem, one may get the present formula from the theorem stated in 7.2.

7.5. The Achiezer-Kac case. Now let f be a scalar-valued function subject to the hypotheses of 6.3. A little thought reveals that then f can be written in the form $f_- f_+$ where

$$f_-(z) = \prod_{m=1}^{p} \frac{z - iv_m}{z - id_m}, \quad f_+(z) = \prod_{\ell=1}^{q} \frac{z + iu_\ell}{z + ic_\ell},$$

where $\mathrm{Re}\, v_m$, $\mathrm{Re}\, d_m$, $\mathrm{Re}\, u_\ell$, $\mathrm{Re}\, c_\ell$ are all positive. Notice that $f = f_- f_+$ is nothing but the (normalized) Wiener-Hopf factorization of f.

The constants $\widetilde{G}(f)$, $G(f)$, $E(f)$ appearing in the Achiezer-Kac-Mikaelyan formulas 6.3 can now be specified as follows:

$$\begin{aligned}
\widetilde{G}(f) &= \exp\Big(k(0+0)+\sum_{\ell=1}^{q}(c_\ell-u_\ell)\Big)=\exp\Big(k(0-0)+\sum_{m=1}^{p}(v_m-u_m)\Big),\\
G(f) &= \exp\sum_{\ell=1}^{q}(c_\ell-u_\ell)=\exp\sum_{m=1}^{p}(v_m-d_m),\\
E(f) &= \prod_{\ell=1}^{q}\frac{f_-(-ic_\ell)}{f_-(-iu_\ell)}=\prod_{m=1}^{p}\frac{f_+(id_m)}{f_+(iv_m)}=\prod_{\ell=1}^{q}\prod_{m=1}^{p}\frac{(c_\ell+v_m)(u_\ell+d_m)}{(c_\ell+d_m)(u_\ell+v_m)}.
\end{aligned}$$

A proof is in [7].

7.6. Asymptotic formulas. In [7], the formula of 7.4 was employed to obtain asymptotic formulas for Wiener-Hopf determinants generated by arbitrary (!) rational scalar-valued functions. Thus, although the formula in 7.4 is restricted to the case of simple zeros, precise descriptions of the asymptotic behavior of $\widetilde{D}_\tau(f)$ as $\tau\to\infty$ are available also for functions f with multiple zeros. Despite its high explicity, the result of [7] is too complex to be repeated here. We therefore limit ourselves to a few important examples. In what follows $x_\tau\sim y_\tau$ means that $y_\tau\neq 0$ for all sufficiently large τ and $x_\tau/y_\tau\to 1$ as $\tau\to\infty$.

A single zero. If $p\geq 1$ and $q\geq 1$ are arbitrary integers, then the following asymptotic formula holds:

$$\widetilde{D}_\tau\left[\frac{z^{p+q}}{(z-i)^p(z+i)^q}\right]\sim e^{\tau[k(0+0)-q]}\left(\frac{\tau}{2}\right)^{pq}\frac{G(p+1)G(q+1)}{G(p+q+1)},$$

where

$$k(0+0)=q-p+\frac{1}{2^q}\sum_{j=1}^{p}\binom{p+q}{p-j}\binom{q+j-2}{j-1}\left(-\frac{1}{2}\right)^{j-1},$$

$$G(m)=(m-2)!\dots 2!1!0!\quad (m\geq 2),\qquad G(1)=1.$$

If $p=q$, then $W_\tau(k)$ is of trace class and

$$D_\tau\left[\frac{z^{2p}}{(z^2+1)^p}\right]\sim e^{-p\tau}\left(\frac{\tau}{2}\right)^{p^2}\frac{G(p+1)^2}{G(2p+1)}.$$

I propose to verify that the last formula is also true for all complex numbers p with $-1/2<\operatorname{Re}p<1/2$. Once this had been shown the reasoning of [10] would result in a proof of the continual analogue of the Fisher-Hartwig conjecture for Toeplitz determinants.

Several zeros of modulus type. Let $x_1, \ldots, x_m$ be pairwise distinct real numbers and let $p_1, \ldots, p_m$ be positive integers. Suppose $b = 1 - \hat{v}$ is a rational function such that

$$b(\infty) = 1, \quad b(x) \neq 0 \text{ for all } x \in \mathbf{R}, \quad \arg b(x)\Big|_{x=-\infty}^{\infty} = 0,$$

and $W_\tau(v)$ is of trace class. Put

$$f(x) = \prod_{j=1}^{m} \left[\frac{(x - x_j)^2}{(x - x_j)^2 + 1} \right]^{p_j} b(x) =: 1 - \hat{k}(x)$$

for $x \in \mathbf{R}$. Then $W_\tau(k)$ is of trace class and

$$\begin{aligned} D_\tau(f) \quad \sim \quad & e^{-\tau \sum_{j=1}^m p_j} G(b)^\tau \left(\frac{\tau}{2}\right)^{\sum_{j=1}^m p_j^2} \prod_{j=1}^{m} \frac{G(p_j+1)^2}{G(2p_j+1)} \\ & \times E(b) \prod_{j=1}^{m} \left[\frac{b_-(x_j - i) b_+(x_j + i)}{b(x_j)} \right]^{p_j} \\ & \times \prod_{k<j} \left[\frac{((x_k - x_j)^2 + 1)^2}{((x_k - x_j)^2 + 4)(x_k - x_j)^2} \right]^{p_k p_j}, \end{aligned}$$

where $G(b)$, $E(b)$, b_- and b_+ are as in 7.5.

References

[1] H. Bart, I. Gohberg, M. A. Kaashoek: Convolution equations and linear systems. *Integral Equations and Operator Theory* **5** (1982), 283-340.

[2] H. Bart, I. Gohberg, M. A. Kaashoek: The coupling method for solving integral equations. *Operator Theory: Advances and Applications* **12** (1984), 39-73.

[3] H. Bart, I. Gohberg, M. A. Kaashoek: The state space method in problems of analysis. In: *Proc. First Intern. Conf. Industrial and Applied Math. (ICIAM 87)* (van der Burgh, Mattheij, eds.), Paris-La Villette 1987, 1-16.

[4] L. Berg: Über eine Idendität von W. F. Trench zwischen der Toeplitzschen und einer verallgemeinerten Vandermondeschen Determinante. *ZAMM* **66** (1986), 314-315.

[5] G. Baxter, P. Schmidt: Determinants of a certain class of non-Hermitian Toeplitz matrices. *Math. Scand.* **9** (1961), 122-128.

[6] A. Böttcher: Asymptotic formulas for rationally generated block Toeplitz determinants. *Seminar Analysis* **1987/88** (Prössdorf, Silbermann, eds.), Berlin 1988, 1-13.

[7] A. Böttcher: Wiener-Hopf determinants with rational symbols. *Math. Nachr.* **144** (1989), 39-64.

[8] A. Böttcher, S. Dzhirgalova: On Wiener-Hopf determinants with rational matrix symbols. *Seminar Analysis* **1988/89** (Prössdorf, Silbermann, eds.), Berlin 1989, 31-39.

[9] A. Böttcher, B. Silbermann: *Asymptotics and Invertibility of Toeplitz Matrices.* Akademie-Verlag, Berlin 1983.

[10] A. Böttcher, B. Silbermann: Toeplitz operators and determinants with Fisher-Hartwig symbols. *J. Funct. Analysis* **62** (1985), 178-214.

[11] A. Böttcher, B. Silbermann: *Analysis of Toeplitz Operators.* Akademie-Verlag, Berlin 1989 and Springer-Verlag, Berlin 1990.

[12] K. M. Day: Toeplitz matrices generated by the Laurent series expansion of an arbitrary rational function. *Trans. Amer. Math. Soc.* **206** (1975), 224-245.

[13] H. Dym, S. Ta'assan: An abstract version of a limit theorem of Szegö. *J. Funct. Analysis* **43** (1981), 294-312.

[14] I. Gohberg, M. A. Kaashoek, F. van Schagen: Szegö-Achiezer-Kac formulas in terms of realizations of the symbol. *J. Funct. Analysis* **74** (1987), 24-51.

[15] I. Gohberg, M. G. Krein: *Introduction to the Theory of Linear Nonselfadjoint Operators.* Transl. Math. Monographs **18**, Amer. Math. Soc., Providence, R. I. 1969.

[16] I. Gohberg, P. Lancaster, L. Rodman: *Matrix Polynomials.* Academic Press, New York 1982.

[17] M. Gorodetsky: Toeplitz determinants generated by rational functions. Deposited at Viniti 1981, RZhMat 1982: 3 A 444 [Russian].

[18] M. Gorodetsky: Toeplitz determinants generated by rational functions. In: *Integral and Differential Equations and Approximate Solutions* (Dybin, ed.), Elista 1985, 49-54 [Russian].

[19] M. Gorodetsky: On block Toeplitz matrices with analytic symbol. *Linear Algebra Appl.* **116** (1989), 41-51.

[20] I. I. Hirschman: On a formula of Kac and Achiezer. *J. Math. Mech.* **16** (1966), 167-196.

[21] I. I. Hirschman: Recent developments in the theory of finite Toeplitz operators. *Adv. Probability* **1** (Ney, ed.), Marcel Dekker, New York 1971, 103-167.

[22] T. Høholdt, J. Justesen: Determinants of a class of Toeplitz matrices. *Math. Scand.* **43** (1978), 250-258.

[23] L. V. Mikaelyan: On the multidimensional continual analogue of a theorem of G. Szegö. *Izv. Akad. Nauk Armyan. SSR, Ser. Mat.* **11** (1976), 275-286 [Russian].

[24] P. Schmidt, F. Spitzer: The Toeplitz matrices of an arbitrary Laurent polynomial. *Math. Scand.* **8** (1960), 15-38.

[25] B. Simon: Notes on infinite determinants of Hilbert space operators. *Adv. Math.* **24** (1977), 244-273.

[26] M. Tismenetsky: Determinant of block Toeplitz band matrices. *Linear Algebra Appl.* **85** (1987), 165-184.

[27] W. F. Trench: On the eigenvalue problem for Toeplitz band matrices. *Linear Algebra Appl.* **64** (1985), 199-214.

[28] W. F. Trench: Solution of systems with Toeplitz matrices generated by rational functions. *Linear Algebra Appl.* **74** (1986), 191-211.

[29] H. Widom: On the eigenvalues of certain Hermitian operators. *Trans. Amer. Math. Soc.* **88** (1958), 491-522.

[30] H. Widom: Asymptotic behavior of block Toeplitz matrices and determinants, II. *Adv. Math.* **21** (1976), 1-29.

[31] H. Widom: On Wiener-Hopf determinants. *Operator Theory: Advances and Applications* **41** (1989), 519-543.